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ABSTRACT

A conference proceedings on mathematics education research contained the following keynote addresses: "Doing and Constructing Mathematics in Screen-Space" (Mason); "Students' Understanding of Geometry: Theoretical Perspectives" (Pegg); "The Australian Research Council and its Role in Supporting Education Research" (Brennan); "Bringing Theory into the Classroom: The School as a Learning Environment for Teachers" (Skemp); and "Australian Space--Pushing Back the Frontiers" (Harris). Paper presentations included the following: "Images and Definitions of Functions in Australian Schools and Universities" (Arnold); "Technology Enriched Instruction in Year 9 Algebra" (Asp et al.); "The Structured Classroom Interactions Schedule (SCIS): A Validation Study" (Atweh et al.); "Students' Manipulation of Algebra Symbols and Their Awareness of the Corresponding Conceptual Relationships" (Bell, Malone); "Challenging Primary Student Teachers' Views about Mathematics Education" (Biddulph); "The Graphics Calculator in Tertiary Mathematics" (Boers, Jones); "Constructivism, Learning and Cognitive Representation: The Case of Fraction Ideas" (Booker); "Mathematics Education in First Year Pre-service Primary Teachers" (Buckingham); "Professional Development and the Secondary Mathematics Teacher: A Case Study" (Clarke et al.); "The Other Consequences of a Problem-Based Mathematics Curriculum" (Clarke et al.); "Multimodal Functioning in Novel Mathematical Problem Solving" (Collis et al.); "Letting Children Design the Curriculum" (Doig, Lindsey); "Some Pluses and Minuses of Radical Constructivism in Mathematics Education" (Ellerton, Clements); "Implications of Newman Research for the Issue of 'What is Basic in School Mathematics?'" (Ellerton, Clements); "Calculus Students' Sources of Conviction" (Frid); "Assessment in Mathematics--Some Alternatives" (Grimison); "From Procedural Analogy to Understanding" (Hall); "Mathematics, Computers and 'At-Risk' Preschoolers" (Hall, Elliott); "A Comparison of Pattern-Based and Equation-Solving Approaches to Algebra" (MacGregor, Stacey); "Bridging Mathematics Students: Attitudes, Autonomous Learning Behaviours, and Problem Solving" (Milne); "Children's Solutions to Partition Problems" (Mulligan); "Factors Pertinent to Children's Responsiveness in Spatial Problem-Solving Activities" (Owens); "Cognitive Difficulties in Early Algebra" (Quinlan); "Pre-service Primary Teachers' Attitudes to Teaching Mathematics: A Reappraisal of a Recently Developed Instrument" (Relich, Way); "Children's Approaches to Mathematical Problem Solving" (Siemon); "Using Open Questions for Teaching: A Classroom Experiment" (Sullivan); "An Analysis of Children's Understanding of Numeration" (Thomas); "Children's Understanding of the Independence of Random Generators" (Truran, J.); and "Abstract Thinking in Rates of Change and Derivative" (White, Mitchelmore). (MDH/AA)

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MERGA 15
Space: the first and final frontier
UNIVERSITY OF WESTERN SYDNEY

Fifteenth Annual Conference of
the Mathematics Education
Research Group of Australasia
(MERGA)

Space - The First and Final Frontier

Conference Proceedings

held at

Hawkaid Conference Centre
University of Western Sydney
4 - 8 July 1992

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Proceedings of the Fifteenth Annual Conference

Mathematics Education Research Group of Australasia (MERGA)

Editors:
Beth Southwell
Bob Perry
Kaye Owens

UWS Conference Centre
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PREFACE

This book of Conference Proceedings has been prepared as the result of the decision taken at the Annual General Meeting of the Fifteenth MERGA Conference, held at the Hawkaid Conference Centre, University of Western Sydney, 4-8 July, 1992, to print Proceedings of future conferences. Because this decision was only made at the Conference, this particular set of proceedings has been prepared after the event with a minimum of refereeing and editing. It is anticipated that in future, an appropriate editor and procedure will be put in place as part of the Conference planning, thus enabling the Editor to undertake more stringent refereeing and editing procedures.

The Conference itself attracted a record number of participants and a significant number of papers. The theme, "Space - the first and final frontier" was the reference point about which many participants based their contributions to the Conference. It welded successfully the predominant strands of mathematics education research with the historical and cultural setting, astronomy and Aboriginal culture.

The Conference is characterised by a number of significant events. Besides the valuable input of keynote speakers from England and Australia, poster sessions were built into the program. This is an area in which it is recommended that there be considerable development in the future. The review, edited by Atweh and Watson, *Research in mathematics education in Australasia, 1988-1991* was successfully launched.

For the first time, a full scale independent evaluation of a MERGA conference was carried out. This evaluation reported high praise for the conference program and the quality of the papers and presentations. While there were areas in which there needed to be improvements, the Planning Committee were satisfied with the result. The Committee was more than satisfied with the level of support given by participants and others involved in the organisation.

Beth Southwell
Bob Perry
Kay Owens

AIMS

The aims of the Mathematics Education Research Group of Australasia (MERGA) are:

To encourage and promote quality research in all areas related to the teaching and learning of mathematics at all levels, particularly in Australasia.

To provide permanent means for sharing of research results and concerns among all members through regular publications and conferences.

To seek means of implementing research findings at all decision levels to the teaching of mathematics and the preparation of mathematics teachers.

To maintain liaison with other organisations with similar interests in mathematics education or educational research.

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Keynote Addresses

DOING AND CONSTRUING MATHEMATICS IN SCREEN-SPACE

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Centre for Mathematics Education
Open University

A preliminary analysis is offered of problems connected with the construal of images from and on screens, whether physical and dumb, like OHP screens, electronic and dumb like TV screens, physical and intelligent like software on computer screens, or intelligent like mental "screens". My interest is in relationships between doing and construing in the space accessed through mental and physical screens. Throwing images onto screens is a popular pastime, but there is a difference between "looking at" a screen and "looking through" a screen, with concomitant educational value.

INTRODUCTION

These notes are complementary to my presentation, which will take advantage of the opportunity to work on static and dynamic images, and to experience directly some of the aspects of images which I find vexing.

As we move from a world dominated by passively "dumb" screens (text on paper, posters on walls, images projected on screens, television), to a world dominated by actively "intelligent" screens (manipulable text, images, and animations, with the power to process these and to express one's thinking in mixed and multi-media), it might be worthwhile to try to learn from experience with presenting mathematics on mental and currently available physical screens, so that we can take maximal advantage of what mixed-media and computational tools for analysis can offer. In this paper I want to look at relationships between doing and construing in the space accessed through mental and physical screens.

A distinction of interest to me is that between

> students *engaging in* activity, summarised as *doing*

and

> student *learning from* activity, summarised as *construing*

Getting students active is relatively easy. Arranging that students learn from activity is quite another matter. They usually learn something, but what they learn is liable to be quite different from what the teacher intends. Distinctions between task-as-envisaged and task-as-set by the teacher, and between the task-as-perceived and the activity in which students actually engage, have been made by many authors, and I do not wish to rehearse them here. For me it all comes down to the fundamental questions:

> What are the students attending to? Where is student attention in the presence of "screened mathematics"? What is the role of their mental screen?

I am particularly interested in the role of awareness, that is, the extent to which students learn effectively both with, and without, having their attention drawn explicitly to what is "being learned", because I find that the distinctions between *outer* and *inner* activity, *explicit* and *implicit* learning, *overt* and *covert* curriculum, particularly vexing when it comes to still and animated screen images. Text is bad enough for permitting students to construe what they see in a variety of ways; images are even more open to ambiguity, to alternative stressing and ignoring (Gattegno, 1990). Yet at the same time, images can be remarkably effective.

Computers have been described as *conception* stretchers (Dennett, 1990), and one of the most significant ways in which conception stretching takes place is in the access to images and to image processing provided by computational devices. Dennett also suggested that:

> conceiving of something complex ... is a matter of learning your way around in a "space" you must construct in your mind.

How that mental space is related to the screen is of abiding interest to me, and of increasing interest in the educational community generally, as evidenced by the rapid increase in imagery-related papers at PME recently, and a sudden increase in the number of books about imagery in a variety of disciplines.

In attending to mental imagery, I find that I wish to depart from the semiotic orthodoxy which treats signifiers and signified as a dichotomous pair with an implied background referent. For me, symbols are not the mathematics, nor do they signify some Platonic or other mathematical referent; rather, they are an expression of seeing, of awareness, an integral part of thought experience. Diagrams are not the mathematics, nor do they signify some Platonic form; rather, they are an integral part of thought experience, an artificial snapshot of time-evolving sense-making. Words are not the mathematics, nor do they signify some "think" which is the mathematics; rather, they are an expression of a fleetingly one-dimensional stressing of certain features of thinking in process. Words and images which purport to signify, need not necessarily signify a signified, because words and images are at best fuzzy projections from a complexly experienced high-dimensional process space into at most one, sometimes two, and occasionally three physical dimensions (Mason, 1992). Even directing attention to signification may be a disservice, diverting attention away from possibly inexpressible but never the less crucial action which is part of thinking.

IMAGES ON SCREENS

I use the term "mental screen" to refer to internal experience of whatever form. The use of the term "screen" in the context of mental experience is *not* intended to summon an image of a homunculus watching an inner projection screen; indeed one of the features of the mental "screen" lies precisely in the all-encompassing, fully participatory experience that is so hard to describe metaphorically. Goodman (1990) distinguished "images of memory and imagination" from "optical and sensory images" though he emphasises that neither need be pictorial. But such a distinction seems to blur when it comes to mathematical images. Imagined and constructed generalised images sit side by side sensory images of mathematical apparatus and diagrams. Lacan (1985) links imagination not based on sensation with the symbolic:

> ... there is nothing in the nature of the wheel that will describe the pattern of marks that any one of its points makes on each turn. There is no cycloid in the imaginary. The cycloid is a discovery of the symbolic. (p. 208)

Goodman (1990) joins the ranks of the imagery-dubious when he observes that:

> Just as responsibilities are not things, but arise grammatically from speech and are more correctly given as having responsibility for such and such, so mental images arise grammatically from speech when what is intended is facility with describing, reasoning, identifying etc. (p. 363)

Although I am in general agreement with Goodman's overall perspective, my own experience of describing, reasoning, recognising and identifying is so strongly associated with rich inner experiences that I find the noun *imagery* sits easily and unproblematically as a summarising label for the whole.

Sometimes it is hard to distinguish between the mental experience and the physical screen; the latter acts as a window, and the mental screen becomes the world seen through that window. Animation is particularly engrossing: what is seen is not simply the screen image, but something more substantial, more general, more personal; a world beyond the window is contacted. This is what I mean by *looking through* as opposed to merely *looking at*. Just as a picture-frame is essential for focussing attention but often goes unnoticed, so in the merging of the physical and mental screens, the consequent framing focuses attention but often goes undetected.

The very pre-articulate, all-pervasive nature of mental imagery makes it critically important in human life generally and in education particularly. Arieti even coined the term *endocept* to refer to "that vague sense, that dim and faint form of knowing" (Neville, 1989) often associated with imagery. I am my self fully convinced of the importance of imagery in teaching and learning mathematics, and do not propose to put the case here. Rather, I want to look specifically at ways in which mental imagery is or can be invoked. Electronic screens permit us to throw a variety of images in front of students; students can become very involved in consequent *doing*, that is, in watching. They can also engage in discussing what was seen, and this may contribute to further construal. Students can also become entranced by video images, and even refer to them directly and indirectly in peer discussions. Television images, like pop music, form a background tapestry to their lives.

From a constructivist perspective, what are we to make of children assimilating words to pop songs and television advertisements without any apparent effort, and yet struggling to remember a few words in some foreign language, a few trigonometric relationships, or a mathematical diagram? Can we really get away with saying that some images are necessarily easily construed and recalled?

EVOKING MENTAL IMAGERY

> Imagine a lemon, hanging on a tree. Watch it go from unripe green to ripe yellow, then imagine picking it.

What intentional acts were necessary, and what flowed without effort in that exercise? It is a common experience that a snatch of a tune can evoke much or all of the rest of the song,

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together with a variety of associations; we can hear in our heads but be unable to sing the tune out loud. So how *do* you go about accessing an image?

Describe the wall behind your bed at home.

What do you have to do to be able to carry out this exercise?

The point of these exercises is to remind you that you have frequent recourse to your imagistic powers, but despite that, you may have little direct awareness of how you invoke those powers. How then can you hope to evoke similar powers in students?

FROM STATIC TO DYNAMIC IMAGES

The encyclopaedists of the 18th century were taken with the newly released powers of printing combined with those of etching. The result of Diderot's compendium, and many others as well, was an exploration of the use of static diagrams referenced by the text and intended to augment rather than merely illustrate the text. At first all the diagrams had to be together in appendices. With technological improvements they could be included where desired in the text. Now we can initiate dynamic images from electronic text. But have we learned a great deal? Do we really appreciate the embedded bias of any collection of images, no matter how "accessible"?

Static Images

The power of effective imagery is unquestionable: the once common cliche

> a picture is worth a thousand words

is now fully integrated into our daily experience. We are close to being swamped by images in all media. We say:

> "If I just make this into a picture, a chart, a diagram, then they will see at once what I am trying to say."

I have a sense of this sentiment operating in almost every context: civil servants preparing reports for ministers; managers preparing business reports for senior managers, educational publishers preparing texts for schools, programmers developing educational software, and a host of others. But what is it that makes some images enduring and others transitory? There is a ubiquitous assumption that

> the right image thrown up on a physical screen will surely provide the students with an image that will inform their understanding.

This sentiment is no doubt derived from our assumptions about advertisements:

> the right image thrown up on a physical screen will surely provide the potential customer with an image that will inform their purchasing.

To this end, there has already been a significant increase in the use of cartoons for teacher mathematics and other subjects, and it has even reached the point that a presentation without a cartoon is considered unprofessional. We are also witnessing the reduction of complex ideas to a simple two-minute slot, a two-page spread in a text-book, or a series of brief work cards. These are products of the influence of television on society.

But there is more to pictures than casting them on a screen in front of people, and it certainly is not a matter of more is better. As one advertising executive is quoted (in Blonsky, 1985) as saying:

> Too often we hear - and ask - 'are we talking about the product, or are we doing imagery?' It's the wrong question. Everything is imagery ... We don't really remember facts, the figures, the classic copy-points of the strategy. What we do is get a net impression, a sense of what the brand is about. This is what we call imagery. (p. 507)

Despite the success of (some) advertisers, I remain convinced that to get much educational benefit, students need to be active in processing images; they need to *work on* the images, not just *look at* them. For example, they can be encouraged to reconstruct an *account of* what they have seen, and to construct coherence for themselves by *accounting for* it, in order to be able to reconstruct it again another time. The distinction is important, because accounting for something is so attractive that it gets mixed up with, and even displaces, what it is that is being accounted for. And if there is no agreement on the phenomenon, there is little sense in explanation. Advertisers who set their wares into a mini-plot seem to confirm this view.

The point about processing diagrams is that it is not just a matter of *looking at* them, but rather *looking through* them.

Imagine a general triangle. Now draw your image.

If you conclude that angles which differ by less than 10 degrees look much the same, there are in fact very few "general" triangles! It is not a matter of locating a single general triangle, but rather what you stress when you look at or imagine it. Drawing the image actually brings about different stress from just imagining. Bishop Berkeley is said by Dennett (1990) to have described conceiving of a general triangle as a matter of imagining some particular triangle and then noting parenthetically that the image had unintended features, and reminding oneself not to use the image in certain ways when employing it in the course of thinking. This is a crude form of looking through rather than at. Hilbert (Courant, 1981) referred to the use of the *generic example* in much the same way, and Mason and Pimm (1984) developed the notions of *seeing the general in the particular*, and *seeing the particular in the general* in the same vein. But as we know, getting students to see the general in the particular is not an easy matter.

Dynamic Images

Film in classrooms has never really taken off, largely because of the awkwardness of the medium. Video has helped somewhat, but is still used sparsely. Static images have developed significantly with a surfeit of illustrations, diagrams and colour trying to reach the attraction levels of magazines. Now commercial interests are strongly promoting multi-media hardware, with the implicit slogan:

> If it doesn't move it isn't as good as it could be.

> or, in a modern version of the picture cliche,
>
> a little animation is worth a thousand pictures.

We might well find ourselves assimilating this "value" without realising it, and thereby lose touch with ways of making significant use of both the static and the dynamic. Technological advances enculturate us to assimilating values with astonishing rapidity. For example, black and white film processing is now more expensive than colour (which has forced the former into the realm of do-it-yourself). Soon it may be more expensive to take stills than movies, and more expensive to send physical mail than electronic mail to anyone. Because it is possible, there is now a major industry in producing diagrams, animations, and simulations on video-disk, which allow viewers to "see" how a variety of physical and abstract things "work".

Is "to see" the same as "to understand"? The promoters appear to believe so. Is a well-formed animation sufficient to generate meaning and understanding? I take *meaning* to refer to a rich network of triggered connections with past experience, so that something seems *meaningful* when such a multiple triggering takes place. The connections are largely *intra*, that is, within the domain. Understanding refers to a rich similar network of accessible *inter* and *intra* connections, with emphasis on accessibility by the individual, and on both the *inter* and *intra* connections.

From a constructivist perspective, a text is certainly not sufficient in itself to generate meaning and understanding. But the success of advertising in particular, and television in general, makes this answer not quite so clear in regard to images.

I am sure that there is much more to be learned about using static images in teaching. It may even be that access to animations will teach us about the use of static pictures. But I want to caution against a wholesale movement into dynamic images without regard to lessons learned from the static.

Movement attracts attention by virtue of the genetic, adrenalin releasing, fight-flight response mechanisms which are triggered by our neural network. But movement can also be entrancing, attracting attention so completely that we are lost in the action. Children's response to television, and their concomitant demands for stimulation at school, provide but one manifestation. Miller (1990) notices that when a drive-in movie is viewed from long distance without access to the sound, attention is drawn to the rapidity of cuts and jumps from shot to shot, but these are not generally noticed when the sound is also available.

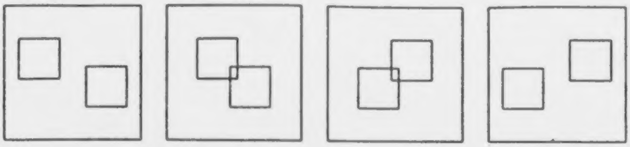
> When viewing a film from the correct distance, one becomes engrossed in the story and the continuity of meaning overwhelms the discontinuity of presentation ... what we lose when we view a film at a distance is the *meaning* of what is going on. (p. 190)

I have learned a great deal from working with static images such as posters, and also from working with animated film, with computer mice and tracker balls, and with dynamic mental imagery. I have worked at locating techniques to assist students to *work on* such visual images. I considered myself to be pro animation. Now I find myself questioning the push to a multitude of clever "visualisations", as computer scientists describe screen images, and the reason for my concern is that the engrossing feature of movement may act against the educational benefit.

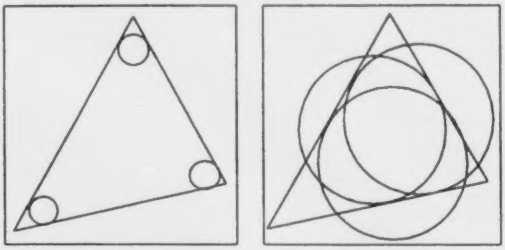
Static diagrams are notoriously unhelpful. Consider for example the difference between a picture, a diagram, and a sketch of something like of an engine. The picture is complete and total. All features receive the same stressing, so that the picture serves as a rest for the eyes as much as anything else. And pictures age so quickly. Diagrams also have a completeness which may render them inaccessible without considerable effort of reconstruction, whereas seeing a diagram built up spontaneously, or better, building it yourself, seems to be much more successful. Although an animation can provide the build up of a complex diagram, animation has its own entrancing features which contrast with participating in the building up of a diagram of one's own. It takes even more personal discipline to work at an animation than a static diagram, precisely because of the tendency to be swept up in the motion. It follows therefore that it is worthwhile developing working practices which focus attention on construal and reconstruction.

Fish and Scrivener (1990) draw attention to the importance of sketches, which are notable for their incompleteness. The evocative, incomplete sketch invites stressing and ignoring, active interpretation using both gestalt-evoked powers and conscious powers, in a way that complete diagrams and figures do not. Sketches employ the automatic gestalt-closure of perceptual completion. And there is a parallel with young children's preference for stories that are incomplete rather than complete.

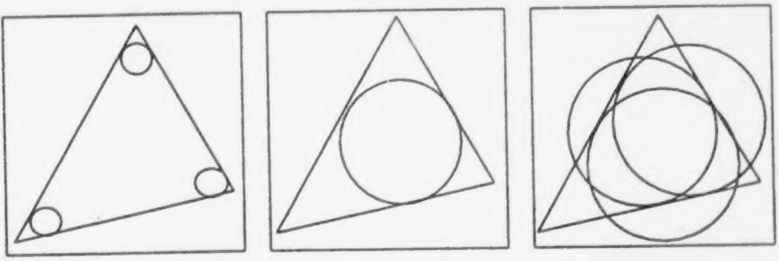
Part of the atomic quality of static diagrams is due to their being static, which signals something unchanging, undeveloping. Thinking of diagrams as snapshots of motion, of frames from a film sequence, calls upon inner powers to supply intermediate frames, and contributes to seeing through the diagram to a more fluid, general, abstract setting beyond. For example, the first sequence suggests movement in a variety of relativistic forms (does one square stay fixed, or does the frame?).



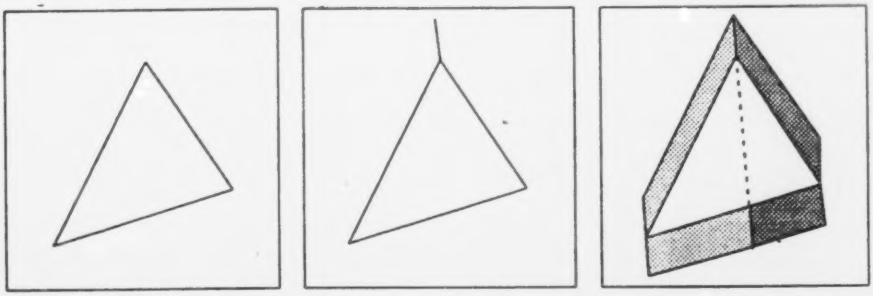
As a result of verbal descriptions by pupils of what they are seeing during their constructed movement, the next sequence of frames might lead to awareness of a particular intermediary position and thus to conjectures and explorations which would support meaning and understanding of a classical geometric result.



Interpolating an intermediate position focuses attention more directly on a known result, but depends for its support of understanding on trying to account for that frame rather than on "having constructed it".



Choosing how much to depict, how much to describe, and how much to leave to the viewer is very much a personal preference, explored in some detail in Beeney et al (1982) and is the subject of a thesis in preparation by Hewitt (see for example Hewitt, 1991). The following sequence is based on a poster designed by Zavadowski (1991).



The underlying theorem is attributed to Pappus. Zavadowski uses the metonymic title Pappus' House or Pappus' Tent to provide pupils re-access to the theorem through the label. Quite different experience of the theorem (the seeing) arises from use of a poster, a diagram sequence, a teacher's verbal instructions to imagine, and use of a CABRI Geometre configuration to move around with a mouse.

The notion that animations lie behind static diagrams is actually quite an old idea. The following quotations are taken from Olive Whicher (1971).

> When movement arises between viewer and the object, there ensures a manifold possibility of alteration in the form seen. (p. 52)
>
> We must practise seeing the construction in *movement*, realising that the situation we happen to have is only one among infinitely many. (p. 52)
>
> ... The time is at hand when through active scientific thought mankind must find the way in clear consciousness to the worlds which lie beyond the sense world, from out of which the sense world is born. (p. 71)

One of the features of physical screens is that it is very hard to depict the abstract and the metaphorical. Television is first and foremost a particularising medium, focusing attention on detail because of the coarse grain resolution. As such, it acts against access to Whicher's valued "other worlds", confining attention to the sense world. It is easier to look at a TV screen than to look through it. High density television and computer screens may overcome this propensity, but it may also be the case that it will be some time before people treat electronic screens the way they treat photographs and paintings.

In my experience, the most important part of using an animation or poster for teaching is the processing of those images, the internalising and construing. It is a matter of learning to be *looking through* screens rather than *looking at* them which I think is crucial, not just to mathematics, but perhaps to education generally.

TYPES OF IMAGES

An image can be:

> as clear as your awareness of your front door viewed from outside as you approach it with bags of groceries in each hand: you know whether the handle is on the left or the right;
>
> as transient as a fleeting sense of recognition as someone walks past you;
>
> as vivid as a scene which often comes to mind, such as a particular beach, or a particular kitchen;
>
> as indefinite as your sense of yourself and where you are going in life: like a beacon on a hill that is often obscured by trees, providing direction for actions but remaining illusive and out of reach; to speak about it is to tumble it into substantiality like images of chairs and lemons and triangles and thereby turn it into something else;
>
> as detailed as a re constructible generic picture "of" an instance of a generality, from or through which the general theorem can be read;
>
> as multiple as a song which won't go away but which comes with an image of a favourite place, while you simultaneously consider what you will have to eat from a menu.

How we construct or obtain images is an important question. For example, in trying to invoke students' mental powers, it helps to have some awareness of how images might be structured. Images can be viewed as *eidetic* or *constructed*. An eidetic image comes fully fashioned from something sensed in the material world: the taste of a particular lemon, the smell of a particular rose, the colours of a particular sunset. A constructed image is built up from other images, which themselves may have been constructed or eidetic. Are all images constructed from "atomic" eidetic images? or can images be formed in some other way?

Images rapidly become abstracted. The word sunset may not trigger any particular sunset, but more a sense of sunset-ness. If you are told that you are going to the cinema, you have a sense of cinema-ness. The language of frames (Minsky, 1975) and scripts (Schank and Abelson, 1977) provide two ways of speaking about the image-based expectations which are triggered. If asked to describe "a cinema", you might work from a particularly vivid image of a specific cinema, or you might find yourself working from a generalised or abstracted image, perhaps composed of fragments of images of actual cinemas. As mathematics teachers, we would presumably like students to have access to similarly abstracted or generalised images rather than being confined to images of particular triangles or particular functions. In other words, we want our students to look through the particular to the general.

Are eidetic images at the heart of all imagery? Neonates sometimes display the behaviour of dreaming, yet it is hard to believe that all their dream-imagery is eidetic, based entirely on prior sense impressions; some artists display images which are hard to relate to prior eidetic images, even through long chains of constructions; mathematicians seem to make progress when they break out of established patterns and fashion new images, new ways of thinking, despite what Newton said about standing on the shoulders of giants.

Gavriel Salomon (1979) used the term *supplanted images* to describe images taken from pictures seen on screens. These are eidetic images from some visual medium, and distinct from those constructed by the person themself. For example, he praised a number of the computer animations in the Open University mathematics television programs because they seemed to provide students with dynamic images related to mathematical ideas, images which are very hard to construct for yourself from a static diagram.

But to construe computer animations it is necessary to recognise and be familiar with the elements of that animation; in other words, to have reconstructed them for oneself. Even if supplantation is possible there are still questions of how to assist students in gaining access to them when appropriate, and being able to use them, to "read" and develop them; to use them as access to the general or abstract ideas "beyond".

Perhaps all images are memory, perhaps even *just* memory. Even if images were *only* a form of memory, imagery would be of great importance in education, for it indicates a power of the mind which is so great that it cannot be modelled adequately by current one-dimensionally time-dependent machines. Using peripheral vision, you can be aware of the whole of a picture while attending to particular detail simultaneously. But you can also pick out features without consciously scanning. You can pick out someone you know in a crowd, and shown a pair of nearly identical pictures, it is not hard to detect differences, without doing a pixel-by-pixel scan of each one. To model this sort of data-access requires massively-parallel computing.

But are images just memory? Eric Love reports that:

> I found it hard to make some mathematical diagrams move in my mind. Then having constructed them in CABRI Geometre, I find I can now do much more with the diagram. And yet I am not simply remembering pictures from the screen, because the actual images are different. (Said when referring to a two-point perspective configuration).

Eric's account suggests to me that experience of dynamic images, particularly when linked to the musculature through your own movement of tracker-ball or mouse, or imaged, can strengthen what is available for processing on the mental screen. As important in education as images themselves, is the employment of those images, for thinking, and for expressing thinking to others. It may be that appropriate work with computer programs can evoke and strengthen processing powers in and on our mental screens.

MATHEMATICAL IMAGES

> Imagine a chiliagon (a 1000-sided regular polygon). Now imagine a 999-gon. What is the difference?

Dennett (1990, p 297) observes that for Descartes it was possible to *conceive* of a chiliagon and of a 999-gon, but that it was not possible to *imagine* them differently, and in this way he distinguished between *conception* and *imagination*. Dennett goes on to suggest that "to imagine" is an act that can be accomplished, can be embarked upon deliberately, whereas "to conceive of" is not. I find it rather difficult to sustain this distinction, for my imaginations of the chiliagon and the 999-gon have much more substance than just mental postcards. I have access to the number of sides as well, and hence to a whole range of calculations, even though I could not identify the difference between pictures of the two. Furthermore, I can work deliberately at juxtaposing the visual, aural, and other sense-based attributes which with my "knowledge of" and "knowledge about", make up the totality of my image. Tall and Vinner (1981) introduced the term *concept image* to refer to this totality of connections and associations, and the idea is developed in Tall (1991).

If a technical mathematical term, like *function*, or *circle*, or *fraction* is employed, then the very hearing of the term is likely to trigger metonymically a range of potential responses, including visual images, incantations that accompany techniques, forms of questions and approaches, awareness of connections and related ideas, associated feelings, access to past experience particularly of success or failure, etc. Many of these remain potential rather than activated, nascent rather than manifested. They will come to life, that is be fired (in the sense of a frame) or activated (in the sense of a script) only when further triggering takes place.

It follows that as teachers it is important that we gain access to our own imagery, our awarenesses (Gattegno, 1990), and then to locate ways of accessing those. One major access route is through images which are constructed or displayed. For example:

> Imagine a pair of circles. Introduce the common chord of their intersection. Now introduce a third circle intersecting the other two. Introduce the two further common chords. The three common chords intersect in a common point.

The final assertion signals a *seeing*, (the root meaning of theorem). It could have been left out, and participants invited to "say what they see", it could have been offered in other ways. The entire task could be presented in a variety of ways:

verbally, with or without attempts to evoke images;

through a static diagram;

through a few static diagrams with the invitation to see them as frames from a film;

through an animation;

through software such as CABRI geometre that enables the user to make changes, to explore the scope of generality by seeking what changes and what stays the same as some other things are changed.

Each of these media, these teacher-provided intermediaries between content and student, require experience and discipline if they are to be exploited successfully. Young children have to learn to read text, and to relate pictures on pages to things from the material world, for example, learning that the back of a depicted object is not depicted on the back of the page. People need to learn to read paintings in different styles in order to interpret them fully. In the same way, it may be necessary to learn how to read diagrams, frame sequences, and animations. Although children eventually work out that there are no little people inside a television set, do they work out how to make the most of the programs and the pictures that they see?

INNER AND OUTER MEANING

The power of imagery has been recognised for thousands of years. One source of its power lies in its often pre-articulate nature, directing and formatting attention without conscious awareness (*viz* Freud and Jung). Another is in the way it speaks to us metaphorically, and triggers new thoughts metonymically.

Tahta (1980, 1981) distinguished *inner* and *outer* meaning of tasks, and to these I add *meta* meaning. *Outer* meanings have to do with explicit content such as known mathematical results described in terms of a mathematical label which purports to summarise a mathematical story. To exploit the outer meaning of a task, it makes sense to engage students in story telling by reconstructing what they have seen by giving brief-but-vivid accounts of fragments from the task activity, and weaving that into a story which accounts for those fragments. Throughout this story weaving, negotiation with others plays a crucial role.

Inner meaning refers to global awareness such as multiplicity of definitions or of perspective, the linking together of previously disparate elements into one continuous family, the perception of an infinite class of elements as a single entity, the choosing of constraints and the effect of those constraints, the stressing of different points of view that yield an invariant result, the simultaneous holding of several points of view by means of one invariant. There are also personal meanings which are triggered metonymically, such as Kieren's encounter with a girl for whom 5/6 was a very important fraction because it reminded her of her family, her mother being absent. Many students find metonymic resonance with apparently entirely abstract geometry, such as being disturbed by a line moving about a plane, sometimes intersecting a circle, sometimes lying outside it, or by reflecting a figure about a line which is not a line of symmetry, or by encountering algebraic computations that get very messy before they again simplify.

Such poignant metonymy is part of a very mathematical encounter. It cannot be avoided; nor need it be dwelt in. Some people complain that mathematics denies such inner associative feelings; others that mathematics provides a refuge from such feelings. Neither of these extreme positions need be the subject of complaint. Acknowledgment without stressing provides a middle ground. Being sensitive to emotion triggered unexpectedly, and demonstrating an accepting but not indulging behaviour can do much for adolescents struggling to locate themselves in an emotionally charged world. Being on the look-out for metaphoric inner content of mathematical tasks can provide a distancing-effect, a shift of attention, for teachers who find themselves otherwise bored by the elementary mathematics they are charged with teaching.

Meta meaning refers to the opportunities provided by any mathematical exercise to observe one's own behaviour and propensities, and thereby perhaps to increase sensitivity so as to inform action in the future.

Tahta pointed out that in order to contact the inner meaning of a task, it is necessary to participate fully, to engage in the outer task as if it were the (complete) task, and the same is true in order to contact meta meaning. If part of you is kept separate, then you are not participating fully.

Being separate or whole is hard to talk about accurately. There are states of partial participation in which part of you is separate from the action in a negative and disruptive manner, perhaps through trying to see how to do the minimum possible, to "just do the task as set", even to try to find things wrong with the task as set. There are also states of full participation in which attention is split: part of you is separate but watching, in a productive and positive manner. The difference often lies in the presence, or absence, of judgement. When part of you is judging, criticising, justifying actions, then full participation is impossible, and contact with inner meaning of tasks is unlikely. When part of you is observing without judging, then such truly full participation is likely to lead to contact with inner meaning as well as a sense of the outer task.

For example, it is essential when doing calculations, whether arithmetic or algebraic, to have an inner monitor which every so often asks whether this calculation is going in a useful direction, and which is awake to slips made by the automated-technique-running self. Such a monitor develops over time, and is supported by reflective practices as described in Schoenfeld (1985), Mason, Burton and Stacey (1984), or indeed Schon (1983) and many other writers.

For years I have noticed that it is possible to generate a collective state in the room which is quite different from the usual bustle of a classroom, particularly when working directly with mental images. I have associated this state with participants trusting me, subordinating their own goals temporarily, and my use of gentle voice tones and an imperative speech pattern as illustrated in the example earlier. While reading Neville (1989) I recognised similarities with a variety of trance-like states, such as that sometimes generated when small children are listening to a story, or when people are ensconced in front of the television. It is as if they have been transported, as if they are not really present in the room but rather in a world entered through a fusion of physical and mental screen. Direct work with and on mental imagery has sometimes generated similar states.

In education, the word *trance* probably has a predominantly negative connotation, denoting lack of awareness rather than heightened awareness; yet the term *entranced* has positive aspects as well, to do with being taken out of oneself, to reaching unusual places or states. Neville connects such states with an increase in suggestibility, but goes to great lengths to argue that such suggestibility is not capable of making people operate against their own interests or desires. My own view is that if something has the potential for good, then it also has the potential for evil. Much more investigation of such states is required. But it may be that the changes in voice tones, from the hard forward thrusting tones of the controlling teacher, to the soft, entrancing and enticing tones of someone describing a mental image, increases the likelihood that participants can connect with inner and meta meanings as well as outer meanings in associated tasks.

Attention, almost exclusively in many published schemes of work, on outer tasks, is perfectly understandable in a climate in which business is of supreme value. But it is possible to account for the purposes of a task without focussing on pupils getting something very specific. One way is to use the language of opportunity: a task may provide opportunity for pupils to encounter one or more mathematical ideas, and one or more aspects of mathematical thinking. If teachers are aware of a range of possibilities that might arise from a task, they are more likely to be able to follow the thinking of individual pupils than if each task is seen as having but one specific aim. The ability to gain a more general view of the opportunities offered in a mathematical task is enhanced by deepening awarenesses of what mathematics is about, of what constitutes mathematical thinking, and how different aspects can emerge in the classroom through attending to doing and construing.

DOING AND CONSTRUING OF DOING AND CONSTRUING

In trying to exploit inner and outer tasks, there are many pitfalls, summed up for me in the words *doing* and *construing*, or in the slogan

> From doing to construing.

For example, a task could be set in game format so that participants encounter the use of strategy rather than gut reaction. However, since games can be played even moderately successfully without being aware of strategy, it is not enough simply to be *doing* the game. Even if someone draws attention to the possibility of a strategy, it may not be within your ken at the time, as you may be completely caught up in the doing. Yet once you become aware of the possibility of attending to strategy, and begin to try it, your whole view changes.

When a teacher presents something on a screen to students, students' construal powers are immediately activated. Sperber and Wilson (1986) came to the conclusion after much study that sense-making is invoked via the cultural assumption that when people act as if what they are saying or doing is relevant to you, then your construal is triggered. Choosing to show something on a screen certainly carries such an implication. Maturana and Varela (1988) suggest similarly that

> observing an interaction *as if* meaning is attributed, determines the course of the interaction.

If students see a task as meaningful to themselves then the course of their interaction will be quite different from the case in which they do not.

My original title was going to be

> (Doing and Construing)^2

because I wanted to look at the four notions which arise from the expansion

(Doing and Construing) (Doing and Construing)

= Doing Doing and Doing Construing and Construing Doing and Construing Construing

For example, the expression *doing doing*, by its repetition, summons up in me the full absorption of the student (or indeed the teacher) in the activity, in the outer task. In this fully absorbed state, the most we can say is that experience is being accumulated, which may form the basis for later reflection. Mistaking the overt, outer task as the content and aim of a lesson is to be caught up in *doing doing*.

At the other end of the spectrum, *construing construing* suggests potentially excessive emphasis on self awareness. If outer awareness is constantly on being aware of process, then the effectiveness of reflection and construal is likely to be lost. It is too direct. Inner meaning requires some attention on outer doing. However, spending time occasionally examining how it is that one goes about making sense of some mathematics, or of an article in mathematics education, is much more likely to be productive. It signals a shift of attention from making sense, to becoming aware of making sense, with a view to preparing to do better in the future.

Doing construing signals to me the incredible speed with which process can become content. Construing is going on all the time. Unfortunately, in mathematics classrooms, the nature of that construing is sometimes centred on self-evaluation in terms of success and failure rather than on mathematical story telling to account for surprising situations. Images are being construed, but if the construal amounts to "this is too hard for me", or something similar, then the effect is negative rather than positive. Sense making may result in students deciding that that topic, that subject, that teacher, even being in school, are not for them. It may be to dwell in metonymic resonances and so to miss the "teacher content" entirely. And none of these rule out success in performance, at least on local tests!

Construing doing signals to me the sort of state which I wish to promote: explicit attention to accounting for what I am doing as an integral part of learning; establishing a mathematically conjecturing atmosphere in which those who are uncertain take the initiative to express their thoughts while those who are certain listen carefully and reflect back what they hear; establishing the habit of sharing and negotiating such stories with peers and others, through first agreeing what it is that is to be accounted for.

IN CONCLUSION

I am convinced that our visually-dynamic dominated culture, far from making education an easier task, makes even more demands for teacher and researcher time to be devoted to making use of static and dynamic images, and although there hasn't been room here to mention it, to processing, analysing, and re-expressing those images with a variety of mental and electronic tools. Animated images in particular may decrease the time needed to encounter an idea, but there may be a corresponding increase in the time needed for construal. In other words, electronic screens may enhance learning, but may not make it less time consuming. The attraction, the entrancement of moving images is potentially a force for real development in education, but it is also potentially a force working against the type of inner and outer task construal which we associate with learning mathematics.

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STUDENTS' UNDERSTANDING OF GEOMETRY: THEORETICAL PERSPECTIVES

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> Of all the decisions one must take in a curriculum development project with respect to choice of content, usually the most controversial and least defensible is the decision about Geometry.
>
> (CSMP staff, 1971, p. 281)

This comment, which questions the very place of Geometry in a modern syllabus is, itself, controversial. Today, with an emphasis on "value for money" education and the push for schooling to be relevant and have direct applicability to employment, the point being made is as current as it was in 1971. Nevertheless, Geometry continues to occupy a major part in the secondary and primary curriculum and this role, in NSW at least, has been expanded.

Research interest in Geometry has also grown considerably during the 80s. Further, there appears no sign of this trend abating in the 90s. One major catalyst for this sustained research effort can be traced to the ideas of the two dutch mathematics educators, Pierre van Hiele and Dina van Hiele-Geldof. Their work, which has come to be known as the van Hiele Theory, has helped shape and direct much of the research investigations associated with Geometry around the world.

This paper addresses students' understanding of Geometry by:

1. providing an overview of the van Hiele Theory;

2. considering research, from Australia and overseas, that reflects aspects of the Theory;

3. describing how the SOLO Taxonomy of Biggs and Collis can help clarify and extend ideas in the van Hiele Theory; and finally,

4. identifying future research themes.

Before starting on these central issues it is valuable to reflect on what 'theory' might mean to different researchers. This not only helps place in perspective why certain workers take particular paths but can also help explain why research can be interpreted in very different ways by various investigators.

One useful, but simplistic, view of research paradigms is to consider a single dimension with two poles. At one end there is the traditional scientific (or quantitative) paradigm and at the other end we have the phenomenological (or qualitative) paradigm. Researchers who ascribe to the quantitative end of the spectrum: look for context free generalisations, focus on individual variables and factors, and see a separation between facts and values. Social reality is seen to be objective and external to the individual. Most research designs are of a standardised nature and there is usually a strong reliance on statistical analysis to help quantify effects and causes. Obviously there are strengths and weaknesses with this philosophy of research. Work can be very precise, extraneous variables can be controlled through a variety of techniques, and hypotheses can be generated and tested. On the other hand the very notion of the need to quantify each aspect tends to highlight a mechanistic ethos which has the potential to denigrate human individuality. Also when quantification becomes an end in itself the results are often trivial and the sacrifices made to achieve measures can result in artificial situations which are unrelated to real life.

The qualitative paradigm is context based. The physical and social environment is seen to have a bearing on human behaviour. The emphasis is on the importance of subjective experience. As a result there is a more holistic approach to data collection. The research designs are usually flexible and there is a reliance on narratives or interviews. The strengths of such an approach are the identification of unexpected outcomes and relationships. Many subtleties and complexities hidden in the quantitative approach can be revealed. These strengths, however, are often at the expense of suitable measures of validity and reliability. Consequently the opportunity to generalise findings is reduced.

As mentioned earlier, this dichotomy represents a simplistic view of research as it is possible for a researcher to select elements from anywhere on the continuum to help answer specific questions. Nevertheless, it does help to explain some underlying philosophies of research and researchers.

My case highlights the deficiencies in the simple, one-dimensional continuum. I see myself down the scientific paradigm end and yet most of my recent research has involved working with students in schools on the topics they are being taught. There are many phenomenological elements to this work. However, central to my thinking is that there can be theories which explain student growth and that the testing and development of such theories should be a major aspect of any research. I would go so far as to say that testing of research questions derived from theories is of critical importance. Investigations undertaken in the absence of theory are almost certain to leave one as uniformed at the end as one was at the beginning.

For a theory to be useful it should be able to be tested and offer new questions that, in the absence of the theory, might not have been asked. It should also be able to satisfy at least three criteria:

1. Explanation, that is, offer reasons for what is observed;

2. Unification, that is, synthesise or link together previous work;

3. Prediction, that is, offer insights into new areas that have not been explored.

For me, in the early 80s, the van Hiele Theory satisfied these conditions and offered me the type of framework I was seeking. Now, while I feel I have moved on (or grown further) in this area I am cognisant of the legacy of the van Hiele Theory in developing my ideas.

THE VAN HIELE THEORY

The van Hiele Theory was developed in the 1950s when Pierre and Dina undertook companion PhDs (that is, they worked in a similar area of research, namely, Geometry but on different aspects) at the University of Utrecht. Their work evolved out of their experiences as teachers where they found secondary school Geometry difficult to teach.

Dina's work involved a teaching experiment aimed at developing students' understanding of Geometry. Pierre's thesis developed the notion of levels of thinking and the features that would lead to students exhibiting insight in Geometry.

Dina died in 1959 and Pierre has continued to develop the theory. His latest book *Structure and Insight* was published in 1986 and brings together the main themes which make up the Theory.

Van Hiele's ideas have much in common with those of Piaget in that they ascribe student understanding to a series of levels or stages. However, there are important differences between the two theories. For example, the van Hiele Theory:

i) places great importance on the role of language in moving through the levels;

ii) concentrates on learning rather than development, hence the focus is on how to help develop student understanding; and,

iii) postulates that ideas at a higher level result from the study of the structure at the lower level.

These features combine to help the theory stand apart.

In overview the theory has two main aspects which combine to provide a philosophy of mathematics education (not only of Geometry). The two key aspects are:

1. A series of levels that students grow through in acquiring competence and understanding;

2. Five teaching phases that assist students to move from one level to the next.

By taking into account both of these aspects, students can exhibit and develop what the van Hieles refer to as 'insight'. A student would indicate insight when he/she could act **adequately** with **intention** in a **new situation**. For this to happen it is clear that students must have ownership of their mathematics and be capable of addressing questions they have not been given previously .

This section considers the van Hiele Theory. In particular, the following questions are addressed:

1. What are the van Hiele levels of growth?

2. What features or properties underpin these levels?

What are the van Hiele levels of growth?

Van Hiele identified five levels of development. I will use the numbers 1 to 5 to identify the different levels and the word descriptions of Hoffer (1981).

Level 1 (Recognition)

Figures are judged by their appearance. A figure is recognised by its form or shape. The properties of a figure play no explicit role in the identification of a figure.

Level 2 (Analysis)
Figures are identified by their properties. The properties, however, are seen to be independent of one another. For example, the properties are not organised in such a way that students will realise that a square is a rectangle.

Level 3 (Ordering)
The properties of figures are no longer see as independent. There is seen to be an ordering of the properties with one property preceding or following another property. Relationships between different figures are also understood.

Level 4 (Deduction)
The place of deduction is understood. Necessary and sufficient conditions can be employed. Proofs can be carried out and not simply learned by rote. Definitions can be devised.

Level 5 (Rigour)
Comparison of various deductive systems can be undertaken. Different geometries can be explored based upon various systems of postulates.

Clearly, the levels represent a logical hierarchical arrangement with the substance of each level flowing from an analysis or investigation of the previous level. The recognition of a figure, a Level 1 feature, is an essential prerequisite for Level 2. The consideration of properties at Level 2 will eventually lead to Level 3 understanding where students see relationships between them, i.e., how one or two properties may lead to a third. In much the same way different figures can be seen to share common sets of properties. It is at this level that a square can be considered as a rectangle, rhombus, or parallelogram.

For Level 4 it is the relationships that become the focus of study. This leads to the development of theorems. In addition, there is now the opportunity to consider which relationships or properties are necessary and sufficient to describe a figure. From this, students can now generate definitions. Of course, Level 5 understanding evolves out of an analysis of the theorems and postulates at Level 4.

IMPORTANT FEATURES OR PROPERTIES OF THE LEVELS

Several properties are associated with the levels of thinking.

1. Students may be on different levels for different concepts. However, once one concept has been raised to a higher level, it will take less time for other concepts to reach that level.

2. To move a pupil from one level to the next, except maybe for exceptional students, requires direct instruction, exploration and reflection by the student. However, it takes time to move from one level to the next; teachers must be prepared to allow time for this growth to occur. Dina wrote of 50 lessons needed to move 12 year olds from Level 2 to Level 3.

3. A student cannot attain a higher level without first passing through the lower level(s). However, pupils can simulate higher levels by learning rules or definitions by rote or by applying routine algorithms that they do not understand.

4. Students need to confront a personal "crisis of thinking" in moving from one level to the next. They cannot be forced to think at a higher level. However certain teaching strategies can inhibit such growth and place boundaries on students' potential.

5. "Level reduction" occurs when structures at a higher level are re-interpreted at a lower level. This usually occurs by making the structures at the higher level visible, for example, the use of Z or F to help students 'understand' alternate and corresponding angles. The effect of this procedure, when it is teacher directed, can be counter productive as it can remove the stimulus for the students to attain a higher level.

6. Each level has its own language or linguistic symbols. People reasoning on different levels speak different languages and in general cannot understand one another. This language problem can occur even between students within the same classroom. Thus very serious communication problems exist between students on one level and their fellow students , teachers, text-books, and exercises on another level.

7. Each level has its own organisation of relationships. Teachers need to be aware that what may appear to be correct at one level may not be seen to be correct at a higher level. The most obvious example of this is that it is not until Level 3 that a square is seen to belong to the set of rectangles.

8. Concepts implicitly understood at one level become explicitly understood at the next.

9. The learning process is discontinuous. That is, a student having reached a given level remains at the level for a time as if maturing. Forcing a student to perform at a higher level or directing your teaching at a higher level will not succeed until the maturation process has occurred.

10. Rote learning or applying routine algorithms without understanding is no level.

RESEARCH ON THE VAN HIELE THEORY

Research on Levels

Most research on the van Hiele Theory has been directed at the notion of levels of understanding. While there has been enormous empirical support confirming the levels there have been problems. Usiskin (1982) used a multiple choice test across a number of topic areas in Geometry and found that 75% of students could be allocated to a level. This test has been used by over 100 other investigators (Usiskin and Senk, 1990). One typical example was by Chaiyasang (1987) who used the test on over 3000 students from Year 6 to 9 in eastern Thailand. He found that 90.1% of the students tested could be allocated a level.

However, I believe the test has a basic flaw. It assumes that students will be at the same level for different concepts. This is contrary to the van Hiele Theory and in direct opposition to Mayberry's (1981) research in which she found students were of different levels in different concepts. (This point has also been recently verified with Australian students by Lawrie (in press).) Other researchers (e.g., Crowely, 1990) have also questioned the test for its validity and reliability. The reason that such strong results are being identified in spite of the problems mentioned is an interesting research question in itself.

Other problems identified with the level descriptions are (i) the difficulty in testing Level 5, and (ii) the need for some level (Level 0) below the base level identified by van Hiele. In the former case Usiskin (1982) found little help in the writings of van Hiele in helping list behaviours that were suitable for Level 5. He concluded his study with the observation that "Level 5 either does not exist or is not testable" (p. 79).

The need for a level (or levels) below van Hiele's base level has been identified by several writers (Mayberry, 1981, 1983; Usiskin, 1982; Senk, 1989; Pegg and Davey, 1989). These studies have identified numbers of students who cannot satisfactorily meet van Hiele's Level 1 criteria. A possible reason for this anomaly could be that van Hilele's work is concerned primarily with secondary students. This being the case, then his base line would be appropriate in that instance. As researchers have extended his ideas and worked with younger children the inadequacies of his lowest level have become more apparent.

Finally, Pegg and Davey (1989) have suggested that it is valuable to consider van Hiele's Level 2 as two separate levels. The first of these is characterised by a student focusing on only one property. The second involves the identification of two or more properties. The reason for the split arises out of the large numbers of students, in Years 5 - 7, who are able to spontaneously provide only one property in descriptions - despite extensive probing on the part of the interviewer. This finding is consistent with predictions of the SOLO Taxonomy and is discussed later in the paper.

Research on Properties

Several of the properties or features listed previously, about the van Hiele Theory, have been addressed in research studies. The most significant overseas papers and reports of the 80s and 90s are by Usiskin (1982); Fuys, Geddes and Tischler (1985); Mayberry (1983); Burger and Shaughnessy (1986); Bobango (1987); Denis (1987); Senk (1989); Gutiérrez, Jaime and Fortuny (1991). The numbering below follows that established earlier:

1. **Different levels for different concepts**
   The work of both Mayberry (1981, 1983) and Burger and Shaughnessy (1986) has identified students to be on different levels for different concepts. Pegg and Davey (1989) found that, even on activities based on closely related figures such as a rhombus and a square, there were often differences. These differences can be explained in terms of student's familiarity with the shapes. However, the differences identified were not great. Hence it seems that while a student's level of understanding of various concepts might not be the same they are clearly not independent.

2. **Growth takes time**
   Dina van Hiele-Geldof (1984) indicated that it took her some 20 and 50 lessons to move a class to Level 2 and Level 3, respectively. While there has been no direct replication of her work, there is a strong indication that considerable time is required for student growth. Fuys *et al* (1985) found six to eight 45 minute sessions was

not sufficient time for the students in their study to grow to the next level. In fact a number of students showed little or no progress from their starting level.

3. Hierarchy

The research evidence on whether levels form a hierarchy has been uniformly positive. Mayberry (1981, 1983) using Guttman's scalogram analysis found that the levels do form a hierarchy (reproducibility = .97). These results were repeated in a study (Denis, 1987) using Mayberry items on a sample of Puerto Rican adolescents (reproducability = .99). De Villiers (1987) reporting on a study in South Africa cited a reproducibility coefficient of .90. The results of other writers, e.g., Fuys *et al* (1985), where some students under instruction at Level 1 are seen to be moving towards Level 2 and students at Level 2 are seen to be moving towards Level 3, can also be interpreted as verifying the hierarchical nature of the levels.

4. & 5. Crisis of thinking and level reduction

I could find no specific research related to either of these two areas. Both these areas are critical to teachers and curriculum planners. If a crisis of thinking is necessary for a student to move from one level to the next then the characteristics of such behaviour should be explored. Further, if level reduction techniques do result in long-term disadvantages for students and prevent them from facing a crisis of thinking then examples of such behaviour need to be identified.

6. Language and levels

As part of their study, Fuys *et al* (1985) considered several mathematical texts available in the US market. They found that: (i) the texts were primarily concerned with Level 1 and Level 4 activities; (ii) often three levels were addressed at once; and, (iii) if teachers do not make language an important aspect of teaching then they can set up a barrier at Level 2 for student growth. Other studies have also commented on the language used by students at various levels (e.g., Burger and Shaughnessy, 1986). Pegg and Davey (1989) noticed that certain words and word combinations had strong links with various levels. For example, students at Level 1 and low Level 2 used words such as "even" and "corner" instead of "equal" and "angle". Phrases such as "opposite sides equal" were found to be only associated with high Level 2 or Level 3 understanding.

7. Different perceptions at different levels

Most studies have found data to support the notion that students on different levels think differently. However, the findings associated with Level 3 thinking in some studies need to be challenged. It is not sufficient to say that a student is not at Level 3 if he/she does not believe a square is a rectangle. Class inclusion is not simply a part of a natural mathematical development. It is linked very closely to a teaching/learning process. It depends upon what has been established as properties. An interesting comparison is between a trapezium and a parallelogram. In this case, the acceptance of class inclusion depends on whether **only one pair** of equal sides is a property of a trapezium. If it is, then a parallelogram does not belong to the set of trapeziums. The main feature of Level 3 should not, in my view, be the acceptance of class inclusion but the willingness, ability and the perceived need to discuss the issue.

8. **Implicit - Explicit**

A major feature of the van Hiele Theory is that what is implied at one level is made explicit at the next. The hierarchy research has a bearing here. So too has the work of Vygotsky (1976). In particular, his notion of the zone of proximal development (ZPD) is relevant. In a practical situation this can be addressed in interview by the techniques of probing and prompting. A student's level is basically what they can offer spontaneously to some stimulus or question. To help clarify the issue, probing questions, i.e., ones that let the student know that you require more information, are valuable. They can take the form of "Can you tell me any more?" "Would you add further to your comments?" "Can you clarify your comments?". Prompting questions, on the other hand, enable the interviewer to provide the student with additional cues. After a time these cues cease to stimulate useful information. The differences between these two forms of questions is the area of thought which Vygotsky refers to as ZPD. I see it as a useful way of finding what implicit knowledge a student processes and what knowledge the student can be expected to acquire in the near distant future.

9. **Discreteness of levels**

This area of research has been commented on by several writers but has not been the focus of many investigations. There appears to be growing acceptance that the levels are more dynamic in nature than static. The research of the Gutiérrez team has extended this idea by conceiving the notion that students might be growing in several levels at the one time. This does not negate the hierarchy proposed by van Hiele. They see that some "critical mass" acquisition of a lower level is needed before growth into a higher level is obtained. Despite some problems with their design (see Lawrie (in press)) their work does open up the potential for a more detailed analysis of what it means to be 'at' a level. My own work identifies several features:

i) There are transition periods between the levels. This tends to show up growth as more of a continuum.

ii) Students entering a new level are most vulnerable and are prone to oscillate between the new and the old for some time.

iii) Within a given concept, and dependent on the teaching, students are relatively stable in their level of understanding at a given time.

iv) Students have a potential level. New concepts move relatively quickly to this level providing a plateau-type effect.

10. **Rote learning - no level**

While rote learning has been addressed extensively in the Psychology literature, I have identified no specific research directed at van Hiele's comment about rote learning. The role of rote learning in assisting or hindering new knowledge acquisitions is an important research question. Is rote learning at all valuable to understanding?

From the above, it is clear that the van Hiele Theory has been subjected to some research scrutiny in the past ten years. However, the investigations have addressed only limited aspects of the theory. Also, most research has been directed at global issues. Usiskin's

(1982) comment that "many mathematics educators are accepting and using this theory on the basis of characteristics of the theory rather than a testing of the individual components" (p. 7) is still relevant. The potential for more directed research which focuses on more specific features of the Theory is enormous.

THE SOLO TAXONOMY

Another framework that has evolved as a reaction to perceived short comings in Piaget's Theory is the SOLO Taxonomy of Biggs and Collis. Despite this common starting point, there are some differences between the formulations of the van Hiele Theory and the Solo Taxonomy. Primarily, van Hiele is about describing thinking processes while Biggs and Collis are interested in categorising the quality of an individual's response. In the latter case the analysis of the response made may lead to some understanding of the thinking processes, but it may not. Issues such as motivation, the form of the question, and prior knowledge in a specific area could all influence a person's answer. This could cloud or under-value the thinking processes used. Nevertheless, the two theoretical stances do share a number of common traits and useful information can be gleaned by using both perspectives when considering research findings. Before exploring this aspect it is valuable to review briefly, key aspects of the SOLO Taxonomy.

The SOLO Taxonomy uses two criteria when determining the level of a student's response to some question or problem. The first, referred to as the mode of functioning, is determined by the level of abstraction of the elements used. The second is the complexity of the structure within a given mode. Figure 1 summarises this information.

There are five modes and these can be compared to the stages outlined by Piaget. The modes are:

Sensori Motor (from birth). Here a person reacts to the physical environment. For the very young child it is this mode in which complex motor skills are acquired. In later life these skills evolve to be associated with various sports. This form of knowledge is described by Biggs and Collis as tacit knowledge, i.e., knowing how to carry out an action without necessarily being able to describe the act in detail.

Ikonic (from about 18 months). Here language emerges. A young child develops words and images which can stand for objects and events. In later life this form of functioning assists in the appreciation of art and music. Knowledge of this type is referred to as intuitive.

Concrete Symbolic (from about 6 years). Here symbol systems develop. A person expresses his/her thinking through the use of symbol systems such as written language and number systems. This mode represents a significant shift in the level of abstraction despite the links that tie the systems to the empirical world. This form of knowledge is referred to as declarative.

Formal (from about 16 years). Here abstract concepts such as principles or theories are considered. Students are no longer restricted to a concrete referent. As a result, the consideration of a range of possibilities and constraints is undertaken leading to the development of general cases.

Post Formal (from about 20 years). Here abstract concepts identified in the formal mode are challenged and questioned. Students are expected to advance knowledge in a discipline and would be seen to be associated with original research.

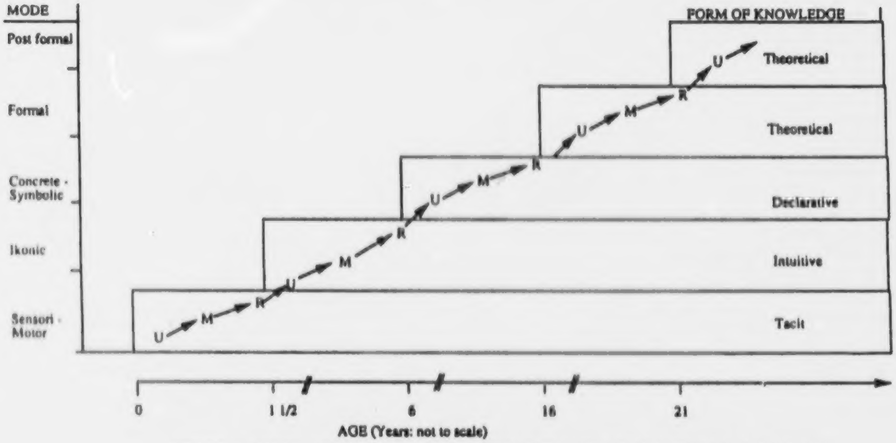

FIGURE 1: MODES, LEARNING CYCLES, AND FORMS OF KNOWLEDGE OF THE SOLO TAXONOMY

(adapted from Biggs and Collis Developmental Learning and the Quality of Intelligent Behaviour in H. Rowe (Ed) Intelligence: Reconceptualization and Measurement. Melbourne: ACER

The ages provided with each mode are those when the mode typically becomes available. Associated with each mode are a series of levels which provide a measure of the complexity of structure. These levels comprise basically three dimensions, namely:

1. How the response relates to cues or information in the question;
2. The amount of working memory or attention span required;
3. The need to (a) come to a conclusion (answer) referred to as closure and (b) be consistent and cope with contradiction (Biggs and Collis, 1982 pp.26-28).

There are five levels for each mode:

Prestructural. This level is below the mode of functioning required of the task. Examples would include a student refusing to answer a question, or providing a tautology. The amount of working memory used is small and there is a strong-felt need to reach a conclusion quickly and to ignore inconsistencies.

Unistructural. This is the first level within the "target" mode. It can be recognised by a student focusing on only one relevant aspect. The amount of working space used enables the student to select one cue and apply a relevant operation. The amount of working memory used is increased over the previous level but, because of selecting only one aspect, inconsistencies are common.

Multistructural. This is the second level within the "target" mode. It can be recognised by a student focusing on a number of relevant aspects but connections or relationships between the aspects are not made. The student closes when several aspects are used but inconsistencies can occur because of the perceived independence of the aspects.

Relational. This is the third and last level within the "target" mode. It is recognised by a student being able to generalise within a given or experienced context using related aspects. The working memory used needs to consider the elements and their inter-relationships. Closure is delayed until this is accomplished. The result is usually consistent within the specified context but may not be consistent across different contexts.

Extended Abstract. This level lies outside of the "target" mode and can be equated to the unistructural level of the next mode. It is recognised by students being able to generalise to situations not experienced. Inconsistencies are able to be resolved or tied to specific circumstances. One result of this is that answers to questions may seem to be left open. Clearly the working memory required for such a response is large.

THE VAN HIELE THEORY AND THE SOLO TAXONOMY

There have been three attempts in the research literature to consider the van Hiele Theory and the SOLO Taxonomy together. Olive (1991) chose to integrate the two theories by considering them in relation to Skemp's (1979) model of mathematical understanding. While his work was related to Geometry, the basis of his research was directed at ninth grade students' Logo programming skills.

The other two studies were by Jurdak (1989) and Pegg and Davey (1989). Pegg and Davey (1989) described a study in which the two theories were compared. The particular focus of the comparison was directed at Levels 1 to 3 in the van Hiele Theory. The findings of the study were at variance with those of Jurdak (1989) who also compared the two theories. The remainder of this paper seeks to explore the differences in the two papers. First, a comparison of the findings found in Pegg and Davey and Jurdak are discussed and second, reports on two further studies are included in which the results add further weight to support the original findings of Pegg and Davey.

Jurdak (1989) took an example from Burger and Shaughnessy (1986) which involved identifying quadrilaterals from a set of figures consisting of squares, parallelograms, and rhombuses. The students' answers were classified in terms of van Hiele levels. Jurdak took this information and reclassified it according to the SOLO Taxonomy. In summary, the results for the lower three van Hiele levels showed that: Level 1 (van Hiele) was equivalent to the unistructural level in the SOLO Taxonomy; Level 2 (van Hiele) was equivalent to the multistructural level; Level 3 (van Hiele) was equivalent to the relational level. The one-to-one correspondence identified means that for Level 1 to be equivalent to the unistructural level, the single aspect of focus was the shape of the figure "as a whole but not its properties" (p. 158).

Pegg and Davey (1989) approached the task of comparison differently. They tested and interviewed approximately two hundred and seventy students from Year 3 to Year 7 about their understanding of four basic 2-D shapes. When the students' responses were analysed they could be divided into three clear groupings. Many students could describe the figures using only global features. Such responses were:

> a rectangle is a long square,
>
> a parallelogram is like a rectangle that has been pushed to one side.

A second group of responses listed properties of the various figures. Examples included:

> a square has all four sides equal and the angles are all 90°,
>
> a parallelogram has four sides two of them are equal to each other the other two are equal to each other. If you run the lines forever it will never meet.

However, most significantly, a large number of responses which made up a third group were identified that only contained one property and this did not change under careful probing. Examples of this included:

> a square has four even sides,
>
> a rectangle: two sides are the same and the top and bottom are the same,
>
> a rhombus: it looks just like a diamond it has equal sides.

What became clear from the analysis was the large difference in quality between the two types of answers in these last two groups. Thus to consider the responses within only one mode (namely, the concrete symbolic), as Jurdak had done, was not doing full justice to the data. Instead, it seems accurate to categorise the descriptions which focused on global, visual criteria as belonging to the ikonic mode. When this is done, the descriptions which use only one property can be categorised as unistructural in the concrete symbolic mode. Of course those students whose responses were able to provide more than one property were classified as multistructural. It appeared from the interviews that, in general, the best that could be achieved by relatively young children was a multistructural response to the question that required descriptions to be given. The possibility that students whose responses were coded as multistructural might also be able to respond at the relational level (i.e. van Hiele's Level 3) could only be tested by asking additional questions.

There were also numerous examples of students who, having given one or a number of properties, felt the need to go further and provide some visually-based statement as if to support their answer. Examples of this included:

> a parallelogram is just like a rectangle tilted over. The lines are parallel and the angles are different to a rectangle. It is similar to a rhombus,
>
> a rhombus: has four sides of which are in pairs. Each side of a pair is facing the other so they are parallel. It looks like a square pushed over a bit to make a pair of lines slope. If you draw a diamond you have the right slope.

This tendency for students to use imagery to support their answer was a common feature. However, the van Hiele Theory makes no mention of the use of lower levels to support high levels. Van Hiele, implies that once a new level is attained it completely subsumes the previous level and that people can no longer use the lower level. The SOLO Taxonomy does not adopt this stance. Not only does it suggest that students can function within different modes on a particular task but the earlier modes continue to develop throughout a person's life. In the case of the examples above, the SOLO Taxonomy would categorise these responses as concrete symbolic with ikonic support.

In summary, the results showed that; (i) it is valuable to consider young students' descriptions of 2-D shapes within the framework offered by two modes of functioning, namely, ikonic and concrete-symbolic; (ii) Level 1 (van Hiele) represented a possible series of levels in the ikonic mode with the better answers which focused successfully on some global shape as being able to be classified at the relational level; (iii) there was no direct equivalent for the unistructural level (concrete symbolic mode) of the SOLO Taxonomy in the van Hiele theory. It is possible to surmise that van Hiele may have seen it as part of his Level 2; (iv) there appeared to be a direct equivalence between Level 2 and 3 in van Hiele's theory and the multistructural and relational levels in the concrete symbolic mode of the SOLO Taxonomy.

Two other studies undertaken by Davey and Pegg (1990, 1991) have a bearing on the results above. Both studies involved an intensive interview and testing programme. The first was directed at parallel concepts and the second at angle concepts. While each study explored many features, it is the descriptions provided by the students that will be the focus of the following discussion.

Students' descriptions of parallelism mirrored those identified above for 2-D figures. That is, three basic groupings of responses were identified. A number of young students gave simple descriptions based on some visual prototype. Examples included:

> looks like a sidewalk and a road,
>
> looks like a door.

These responses can be interpreted as Level 1 in van Hiele's terms or within the ikonic mode of the SOLO Taxonomy.

A second group of students focused on lines in a fairly general way. There appeared a gradual growth in their answers up to the point where the students were able to qualify their statement about straight lines by adding that "the lines do not meet". Examples of this growth included:

> lines on their side,
>
> 2 lines side by side, across, diagonal or straight,
>
> 2 straight lines that never meet and go on forever.

These responses are consistent with the classification of unistructural (concrete symbolic mode) in the SOLO Taxonomy as they focused on one property although there was a clear visual element to the answers.

The final group of descriptions was similar to the previous group except the answers were extended by including the additional fact that "the lines were equidistant". Examples from this group of responses included:

parallel means 2 straight lines apart that are the same distance between all the way,

two lines that do not meet and have the same distance in between.

This was seen to be the multistructural level (concrete symbolic mode) in the SOLO Taxonomy and Level 2 in the van Hiele Theory. To reach this answer a student had to first take into account the fact that the lines did not meet before they could focus on the equidistant property.

When students were asked about angles, four basic categories of responses were received. The first group of responses frequently used the word 'corner' and seemed to have no common understanding of corner or angle. Examples included:

a pointy triangle ... corners are sharp,

a corner is a point - an angle is the same like that in a cube.

The second group of responses brought in the idea of lines but in a very vague way. Examples included:

two things meeting together and it has to have a corner,

where a straight line ends and ends at the end of another line.

The third group focused on "where lines meet" and it was common for them to talk about special degree angles like 90°, 180° or 360°. Examples included:

a sharp bend in a line. The most common angle is 90° like the corner of a square,

Two straight lines meet together at a corner.

The fourth and final group of responses talked about the area or distance between two lines. Examples included:

distance between two lines,

length in degrees from one line to another.

The first group, with its strong visual focus and lack of specific detail, reflects levels in the ikonic mode or Level 1. The remaining three groups are, in order, the unistructural, multistructural and relational levels of the concrete symbolic mode or Level 2 and Level 3 of van Hiele's theory. It is important to realise that each of the groups of responses was associated in broad ways with various skills in angle problems. For example, it was not until group four above that students could consistently halve the size of a given angle or compare the size of different angles. These findings perhaps provide some insight into why students find angle concepts so difficult in that they need to be at the relational level (concrete symbolic mode) or Level 3 to be routinely successful.

Two other features deserve comment. First, the data of the studies described above also point to a further elaboration of the SOLO Taxonomy (see Pegg, 1992). Within the responses that focused on a single property, three distinct groups were identified. The first of these simply provided in their description some comment on lines or sides, e.g., "a square has four lines" or "a rectangle has four straight sides".

The second group attempted to qualify this description, as if they knew that just talking about lines/sides was insufficient. One example "a rectangle is similar to a square but the top and bottom are longer than the other two sides". The third group also focused on sides/lines but they were able to include the key characteristic, namely, equality, e.g, "a square has four even sides".

These groupings lead one to see an early cycle of growth, consistent with the descriptions of unistructural, multistructural and relational levels. The focus of this growth is towards identifying accurately a single property. This finding implies that there are two cycles (and possibly more) within the concrete symbolic mode.

Figure 2 below illustrates this diagrammatically. Here U_1, M_1, R_1, represents the first cycle and U_2, M_2, R_2 the second. The key feature being that R_1 and U_2 are equivalent. This means, the group of students' responses, coded initially as R_1, i.e., because they bring together all the aspects that make up the first property, can also be coded as U_2, i.e., a focus on one property.

Concrete Symbolic Mode

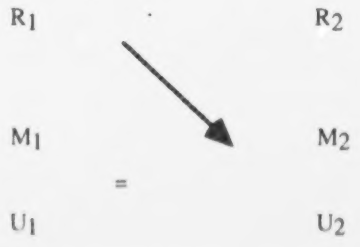

Figure 2: Relationships between cycles and levels within a mode.

This finding, about the possibilities of different cycles within a mode, offers new possibilities in understanding student growth in concepts.

Second, van Hiele (1986) has flagged a different interpretation for his levels that he believes might be of value to some people. In this he refers to: Level 1 - the First level as the **visual level**; Level 2 and Level 3 - the Second level as the **descriptive level**; and, Level 4 - the Third level as the **theoretical level**. However, he did not choose to explore this structure as it did not offer him the potential of his initial classification.

This suggestion is interesting for two reasons. It indicates that level descriptions are often context bound and that, depending on the purposes, the same data could well be interpreted in different ways. More importantly, however, is the significance of this different interpretation when a comparison is made with the SOLO Taxonomy. It would seem that what van Hiele has identified unknowingly by his three new levels are the ikonic, concrete symbolic, and formal modes of the SOLO Taxonomy.

FUTURE RESEARCH INITIATIVES

I have already commented, throughout the paper, on potential research questions. However, in addition to these, there do appear to be four main areas in which research is needed.

1. The ability range within Level 3 (van Hiele) or the relational level (concrete symbolic mode) seems to be very large in Geometry. Van Hiele talks of at least three aspects relevant to this level, namely: an ordering of properties; class inclusion of figures; and, simple deductions. Investigations are needed which explore relationships between these aspects and to what depth such understanding can be expected.

2. The work on the van Hiele Level 1 has been limited. By considering this level as a mode of functioning, i.e., the ikonic mode in SOLO terms, it opens up a field of investigation into young children's development of geometric concepts.

3. Similar to 2 above, but the focus is on Level 4 understanding. Again if this level is compared with the formal mode in SOLO it opens up the possibility of a more careful analysis of the way in which more capable students address advanced questions.

4. The notion of cycles of growth within the concrete symbolic mode has important implications to teaching. More information about these cycles, and the possible identification of further cycles, opens up the possibility that the SOLO Taxonomy may be an even more useful curriculum development tool.

CONCLUSION

In this paper I have attempted to explain, review and update an important research theme in the teaching and learning of Geometry. Clearly, it is not the only focus of research in Geometry. A quick look at the 'Research in Geometry and Measurement' Chapter in the recently published *Mathematics Education Research in Australasia: 1989-1991* will confirm this observation. Nevertheless, this area continues to occupy a large amount of research effort world wide. I find it a particularly exciting and challenging field of investigation. It is clearly a major frontier of research in mathematics education. One that has enormous potential to help us understand how students acquire concepts and one that can help us improve how and what we teach.

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THE AUSTRALIAN RESEARCH COUNCIL AND ITS ROLE IN SUPPORTING EDUCATION RESEARCH

M H BRENNAN
Chair, Australian Research Council

THE AUSTRALIAN RESEARCH COUNCIL

The Australian Research Council (ARC) was established in July 1988, as one of four advisory Councils of the National Board of Employment, Education and Training. Its functions are broadly defined in the Employment, Education and Training Act 1988 as being to advise the National Board on research priorities and the co-ordination of research policy and related matters, and to advise the Minister for Employment, Education and Training on the allocation of resources for research under a range of approved programs (see Appendix, Figure 1).

The Council comprises ten part-time members and a full-time Chair. Council members are drawn from a range of backgrounds, including higher education, government research organisations, and the private sector. Part-time members normally serve for three year terms. The current Chair, Professor Max Brennan, has a five year appointment (see Appendix, Figure 2).

Although the scope of the Council's responsibilities is broad, its policy advice must be given in the context of the advice given to the government by a variety of Commonwealth agencies and organisations. Similarly, its advice on the allocation of funds must be given in the context of the level and nature of government supported research in all sectors - government laboratories, industry, and higher education institutions.

> The Council's mission is to provide advice on research funding and research policy, and to promote the conduct of research and research training of the highest quality for the benefit of the Australian community. The Council has a special responsibility for research in the higher education sector, basic research and research training.
>
> (ARC Mission Statement, March 1992)

RESEARCH POLICY ADVICE

Council's role in providing advice on research policy usually occurs in response to a Ministerial reference seeking advice through the National Board of Employment, Education and Training.

Recent and current examples of such research policy matters include:

- aspects of the Australian Postgraduate Research Awards including number, stipends, priority area loadings, and mobility;

- internationalisation of higher education research and research training;

- expansion of the research resource base;
- the balance of government funding between disciplines of academic and academically related research and its relationship to national priorities;
- the provision of research infrastructure in higher education institutions (with the Higher Education Council and the Department of Employment, Education and Training).

THE ARC AS A FUNDING AGENCY

The ARC is the premier funding agency for higher education research. It also supports basic research in the private sector and in some other organisations, including State libraries and museums. The Council supports research in all fields except clinical medicine and dentistry, which are supported by the National Health and Medical Research Council; and it supports research in all higher education institutions except the Australian National University's Institute of Advanced Studies, which is funded directly for research through a block grant from government.

In all its programs the quality of the research and the researcher are of paramount importance. With the exception of some infrastructure funds distributed to institutions on a formula basis, all of the funding decisions are based on assessments made by highly qualified researchers - the "peer review" system.

The magnitude and impact of the ARC's programs on higher education institutions can be gauged from two comparisons. First, the operating grants of the institutions contain two elements based on the institution's performance in research and postgraduate research training. In 1992 these together generated a component of operating grants to institutions (excluding the Institute of Advanced Studies) of $483 million (December 1991 prices) out of a total of $3.5 billion. Although institutions are free to allocate funds between activities on whatever basis they choose, this figure provides a reasonable measure of the total expenditure from the operating grants on research and research training. It includes a fraction of the salaries of academic and other staff involved in research as well as research infrastructure and direct research costs.

Set against this total expenditure in 1992 of $483 million from operating grants, the ARC programs total $245.3 million.

A second comparison can be made with the other competitively awarded Commonwealth grants schemes. A convenient source of information on these schemes is the data compiled by the Department of Employment, Education and Training (DEET) for use as a basis for allocating the so-called "research quantum" to higher education institutions. This data base includes, for 1992, thirty eight schemes distributing a total of $180 million.

These two comparisons, with research and research training expenditure from operating grants and with other Commonwealth grants schemes, show the importance in money terms of the ARC's programs. The Council's challenge is to ensure that every research dollar is allocated and used as effectively as possible.

ARC FUNDING PROGRAMS

The ARC funding programs are listed below, together with preliminary figures for the 1992 (December 1991 prices) expenditures. A brief description of each of the programs can be found in the information booklet *Australian Research Council*, March 1992.

	($ million)
Postgraduate Awards	55.0
Fellowships	18.3
Research Grants	88.2
Collaborative Research Grants	2.2
Large Equipment Grants	3.3
Centres	20.5
Infrastructure Grants	53.7
Grants to Learned Academies	1.5
Evaluation	0.6
To be allocated	2.0
TOTAL:	**245.3**

It is interesting to note that, over the four years since its inception, the total sum available to support the Council's programs has trebled. Approximately half of this increase has come from a transfer of funds from university operating grants and half is "new money".

In the case of research grants, the increase has been closer to a factor two. Despite this substantial increase, the success rate for initial applications has decreased as a result of a much greater increase in the number of applications. The success rate for initials was 29% for the 1992 grant year. It will be down to 18-20% for 1993. This is an unacceptably low success rate.

The success rate can be increased by making more money available or by decreasing the number of applications. Both will be needed. The Council will prepare a case for more government funding based on the opportunity that is being missed to realise the full benefits of higher education and basic research. We will also need to encourage researchers to seek alternative sources of research funds (probably for strategic or applied research) or, for some academic staff, to engage in professional practice to a much greater extent than at present.

The overall objective of the adjustment I have outlined should be to ensure that every department in every higher education institution is "in touch with" the leading edge of research and professional practice in its subject area. In that way, each staff member (whether or not he, or she, is actually engaged in research) will see the subject as living and dynamic, and so will be able to share that vision and perception with the students. This is what distinguishes higher education from training, and from education at more junior levels.

THE BENEFITS OF RESEARCH SUPPORTED BY THE ARC

The research supported by the Council's programs ranges from pure basic research, through strategic research, to applied research. The emphasis, however, is at the basic research end of the spectrum. That emphasis is appropriate given that essentially all other

Commonwealth support for research, amounting to some $2.5 - 3 billion a year, is directed towards the strategic and applied end of the spectrum.

Nevertheless, we must ask whether the outcomes of such basic research are anything more than the (mere) advancement of knowledge. What are the tangible benefits from such a large expenditure of the taxpayer's money?

The Council has identified five benefits:

1. *Contributions to the Quality of Our Culture*
   Strong, independent universities are a hallmark of a democracy. They are both creatures of the culture and a force which shapes it. Research plays a vital role in determining the nature and extent of that influence.

2. *Graduates of High Quality*
   Research is the vehicle for education and training at the advanced undergraduate and postgraduate (research) student level, and it contributes, in less specific ways, to teaching at all levels in the higher education institutions. The ARC has the important responsibility of providing the highest quality research environment for such advanced education and training.

3. *Direct Applications of Research Results*
   This is the most obvious and direct benefit of basic research; the application often stems from several pieces of research, sometimes in apparently unconnected fields, perhaps undertaken over extended periods of time in different locations. It can lead to the commercialisation of a new product or process in manufacturing, mining or agriculture, or to an improvement in environmental management, health care or social welfare delivery.

4. *Increased Institutional Capacity for Consulting, Contract Research, and Other Service Activities*
   Through the acquisition or construction of sophisticated equipment, through the improvement of the library and other information systems, as well as through the advancement of the skills and knowledge of the staff and students, the capacity of institutions to offer consulting, contract research, and other service activities is enhanced. This increased institutional capacity represents a powerful resource for industry and the community in general.

5. *International Links*
   For a country like Australia, with a small population, a highly developed and respected research community can provide an important channel of information on research developments in other countries, and the country's influence in international forums is enhanced by its standing in research.

One of the challenges currently confronting the Council is to devise ways in which consideration of these potential benefits can be brought into its selection and evaluation processes while preserving the emphasis on excellence in the advancement of knowledge.

ARC FUNDING FOR EDUCATION RESEARCH

Given the funding context referred to above, the Council sees its primary role as a funding agency to be the support of basic research. Such research may be of quite an applied nature (as it is, for example, in one of the five Priority Areas - Scientific Instrumentation); but it does not extend to the kind of "action research" funded by the old Education Research and Development Committee (ERDC).

Indeed, it is relevant to note here that the Federal Government decision to terminate the ERDC program was taken in the belief (quite correct, in my view) that the employers, particularly the private school and State and Territory Government systems, should fund such research - in much the same way as we expect industry to fund the highly applied and development end of the Research and Development spectrum.

Turning to basic research in education, it is interesting to note the very significant increase in ARC support that has occurred since the Council's establishment in 1988. Generally speaking, this has been in response to an increase in the number of applications rather than being due to an increase in the success rate. For example, initial applications for grants in the area Education/Humanities rose from 20 in 1988 to an estimated* 48 in 1990.

The total picture for education research supported by the ARC is less clear since such research is often conducted under another heading - for example, Cognitive Processes or Learning, memory and Perception. Also, there have been several changes to the Council's research field classification system.

For the 1990 grant year, 20 grants were awarded for projects in fields specifically including Education in their titles. The total funding in this area was $458 969 (5.6% of the total of $7 938 901 for Humanities and Social Sciences).

The fields in this category are:

(a) Education/Humanities
(b) Educational Psychology
(c) Sociology of Education
(d) Education/Social Sciences

Eight projects in Cognitive Processes, with obvious education connections, lift the total to $646 270 (8.1% of the Humanities and Social Sciences total).

It should be noted, at this point, that these figures refer only to the grants program. Education research, like other research areas, is also supported through the Postgraduate Awards, Fellowships, Special Research and Key Centres and Research Infrastructure Programs. Currently, there are two Key Centres in education - the Centre for Aboriginal Studies and Education, University of South Australia, and the Centre for School Science and Mathematics, Especially for Women, at Curtin University.

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* The Figure includes a known number of applications for Large Grants and an estimated number of applications for Small Grants administered by the Institutions.

With the exception of education-oriented research in the Priority Area of Cognitive Science, education research in the broad area of the social sciences is now classified under the single heading "Education".

Under this general education heading, 55 out of 353 social sciences grants (Large and Small) were awarded for the 1991 grant year. The 55 grants totalled $944 000 in value, placing education third behind psychology and economics in the competition for social sciences grants funding. Clearly, education research has a significant place in ARC funded research.

For 1992, there were 27 Large Research Grants in Education with a total value of $659 600. This represents 12.5% of the total for social sciences.

THE FUTURE FOR EDUCATION RESEARCH IN AUSTRALIA

Within the higher education system the other significant source of research support in addition to the ARC is the Operating Grant for each institution. How much of this goes to support education research is not clear. (Like many others, I view the data collected by the Australian Bureau of Statistics on this matter as being highly suspect.)

One important issue for the future is the rapidly developing interest of staff from the former college sector in undertaking education research. Indeed, this is one of the sources of growth in the number of grants applications in this area as well as in many other areas of research.

One must ask how much basic research in education does Australia need and, also, how much can we afford. If, for example, the answers are "about the current level" then we must address the question of what non-teaching professional activities other than basic research should be undertaken by staff and (where appropriate) funded by the variety of sources available. Such activities might include (and here my comments are not specific to education) action research/development, consulting, contract research, and professional practice.

Without such non-teaching professional activity for the majority of staff, higher education institutions, most of which now carry the title "university", will be nothing more than super secondary schools.

APPENDIX

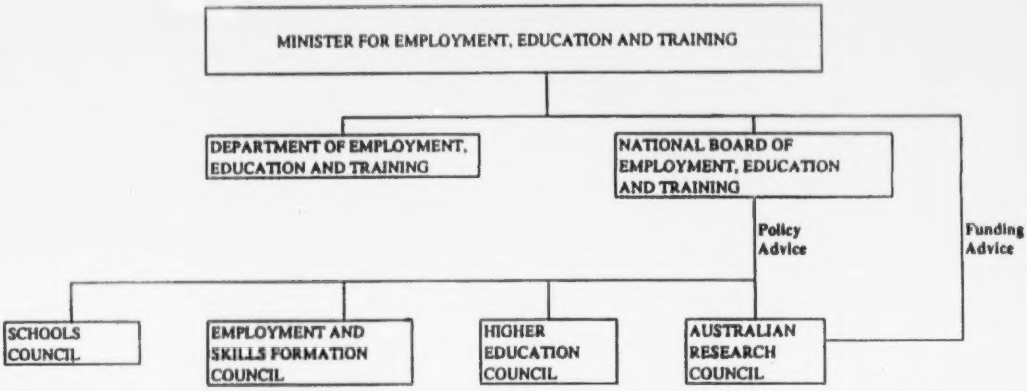
MINISTER FOR EMPLOYMENT, EDUCATION AND TRAINING
DEPARTMENT OF EMPLOYMENT, EDUCATION AND TRAINING
NATIONAL BOARD OF EMPLOYMENT, EDUCATION AND TRAINING
Policy Advice
Funding Advice
SCHOOLS COUNCIL
EMPLOYMENT AND SKILLS FORMATION COUNCIL
HIGHER EDUCATION COUNCIL
AUSTRALIAN RESEARCH COUNCIL

Figure 1: Councils of National Board of Employment, Education and Training.

Professor M H Brennan, AO	Chair
Professor R D Gibson	Vice-Chancellor Queensland University of Technology (Deputy Chair)
Professor G Brown	Deputy Vice-Chancellor (Research) The University of Adelaide
Dr B S Hickman	Delphicon Pty Ltd
Professor B Rawson	Department of Classics The Australian National University
Professor G Rigby	School of Electrical Engineering & Computer Sciences University of New South Wales
Dr M Sargent	Chief Executive Officer ACT Electricity & Water
Dr M Schapper	Vice President, Corporate Strategy CRA Limited
Professor P Sheehan	Academic Director of Research University of Queensland
Dr J W Stocker	Chief Executive, CSIRO
Professor R Symons	WAITE Agricultural Research Institute University of Adelaide

Figure 2: Membership of Australian Research Council.

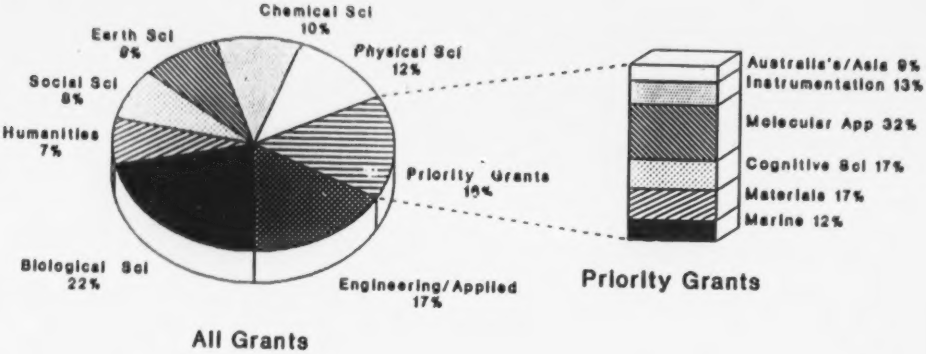

Figure 3: 1992 ARC Research Grants - Discipline and Priority Areas

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BRINGING THEORY INTO THE CLASSROOM: THE SCHOOL AS A LEARNING ENVIRONMENT FOR TEACHERS

RICHARD R SKEMP
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This paper is about an eight-year project for bridging the gap between theory and the classroom. As Dewey has written, "Theory is in the end ... the most practical of all things". The process of developing a theory is a lengthy one; but producing a practical embodiment is at least as lengthy; and we still have to get people to use it, and use it sensibly. This paper will give a pinhead introduction to the theory; a brief outline of the process by which a practical embodiment has been developed; and will describe how these classroom activities are currently being used not only for helping children to learn mathematics with understanding, but also for the professional development of teachers.

As Dewey (1929) has written, "Theory is in the end ... the most practical of all things". But the power of a theory remains only potential until it has been embodied in some form of teaching materials and activities. Our efforts in research, and the development and improvement of theory, will not help the children until we do this. A well-known American mathematics educator pointed out some time ago how wide is the gap between research in mathematics education, and the ways in which mathematics is taught in the great majority of classrooms. In the past twenty years, theory in mathematics education has come a long way. But how many text books do you know which embody an appropriate theory, and do so explicity and effectively? This was the line of thinking which led me to undertake an eight-year project for bridging the gap between theory and the classroom, and what has happened since.

In my country and others, since the early 1960s, much time and money have been spent in efforts to improve the teaching of mathematics in our nations' schools. Nevertheless, the problem is still with us. A major survey (Hart et al, 1981) has shown that after three years in secondary school, children understand important mathematical concepts very little better than when they left primary school: sometimes less. Concern at governmental level led to the setting up of a committee of investigation. During a preliminary survey which formed part of this, more than half of those approached *refused even to be interviewed* about their attitudes towards mathematics (Cockcroft, 1982). Such had been, for these adults, the result of ten years of mathematics as it had been taught to them in our schools; and as for most children, it is still being taught.

There is no reason to suppose that future efforts will be any more successful than those of the past, unless we can identify and remedy at least some of the causes of this large-scale failure. In the present paper, I shall suggest two causes, and outline the remedies which were developed in my *Primary Mathematics Project*.

The first cause I have already mentioned.

First Cause: projects based on mathematical models only, which do not take into account the psychology of intelligent learning. The major projects of the 1960's and 1970's were based on the assumption that the fault was in the mathematical curriculum, and centred their efforts on up-dating and otherwise improving this. "Modern Mathematics" was thought to be the answer. To show how little had been learnt from this experience, our UK government has now introduced yet another new curriculum, a national curriculum imposed by law. In contrast, my own work from the Psychology and Mathematics Project (1962-1969) onwards has been based on the belief that problems of learning and teaching mathematics are as much psychological as mathematical problems. Further development led to a new model of intelligence, foreshadowed in *The Psychology of Learning Mathematics* (Skemp, 1971), and developed at length in *Intelligence, Learning, and Action* (Skemp, 1979).

In the light of this model, mathematics can now be seen as one of the most powerful and adaptable mental tools which the intelligence of man has constructed for his own use, cooperatively over the centuries. Hence its importance in today's world of rapidly advancing science, technology, manufacture, and business. From this, it was predictable that unless we can find out how to teach mathematics in ways which bring children's intelligence into use, rather than the habit learning so extensively researched by the behaviourists, and applied by so many text book writers (knowingly or unwittingly), there was little likelihood of progress.

Second Cause: lack of success of pre-service education. We must face the fact that children get their first introduction to mathematics from primary teachers whose own knowledge of mathematics is for the most part rudimentary. Given this situation, the need for in-service education can be seen to be great, and the size of the task immense.

Recognising this need, many in-service courses have been and are still being offered by a variety of bodies such as universities, colleges of education, local authorities. With regret, it has to be accepted that these have not brought about the required changes.

I think there are several reasons for this. Among them are:

1. These courses are attended mainly by single teachers, the keen ones, who then return to their former school environment, where the long established teaching approaches and texts still dominate. In these circumstances, it is hard for a teacher to go it alone. Nevertheless, these are the ones at the leading edge of progress.

2. Teachers have not been provided with a full implementation of the new approaches which they may have been encouraged to use; only starting points. To develop all the curriculum materials which they need to put into action better methods of teaching (such as the greater use of practical materials which is so widely advocated) demands far more that is possible from a primary teacher, both in time and knowledge.

3. There is a third cause which lies beyond the scope of the present paper; that in many teachers, and in schools as a whole, there is a strong resistance to change.

On these assumptions, necessary conditions for putting things right include the following:

1. A model of intelligence which helps us to understand its function, and its relation to learning, rather than one which is centred on "I.Q." and its measurement.

2. Finding out how to apply this model to the learning and teaching of mathematics.

3. Developing and field testing methods and activities based on 2, sufficient for a feasability study.

4. Expansion of these into a full curriculum.

5. Finding out how to help teachers to use these methods with understanding; a model for in-service and pre-service education.

6. Dissemination.

I will take these one at a time.

A full exposition of the new model takes nearly two hundred pages (Skemp, 1979 and 1989b). I find I can offer a reasonable introduction if I give the whole of an hour to it, which is much longer than is available now. So what I'd like to do instead is to give some of the main foundations on which it is built, and also to relate these to the research methodology used at the time.

FIRST FOUNDATION: SCHEMATIC LEARNING

The first foundation was laid in my early paper *The Need for a Schematic Theory of Learning*, (Skemp, 1962) in which I demonstrated that material learnt schematically was, after four weeks, remembered seven times better than similar material which was presented in a way such that it could only be rote learnt. By "schematically" I mean that it was presented in such a way that all the new material could be related meaningfully to what had been learnt already, and in its turn, helped to give meaning to what was learnt next.

The methodology for this study was the standard one of the time, using a control group and experimental group, with all variables kept the same for the two groups except for a single experimental variable. Though subsequently I moved to a different methodology, coming back to it after 30 years I find that it still has merits, not least that it is replicable--as indeed it was, with variations, several times.

I'd like to say here that a very good book on schema theory, in the context of education in general, has been written by an Australian, Robert Howard, of the University of Newcastle. I recommend it if you don't know it already. (Howard, 1987)

DERIVED FROM THIS: CONCEPTUAL ANALYSIS AND CONCEPT MAPS

The experiment just mentioned used material devised specially for the purpose. To apply its findings to mathematics or to any other subject requires a process of conceptual analysis, in which what is to be learnt is analysed to find what are the concepts prerequisite for learning with understanding. These are then analysed in the same way, and so on back to the starting point, that is, whatever knowledge the learner already has.

The result of such an analysis may be shown in the form of a concept map. By way of example, here is one for subtraction at primary school level.

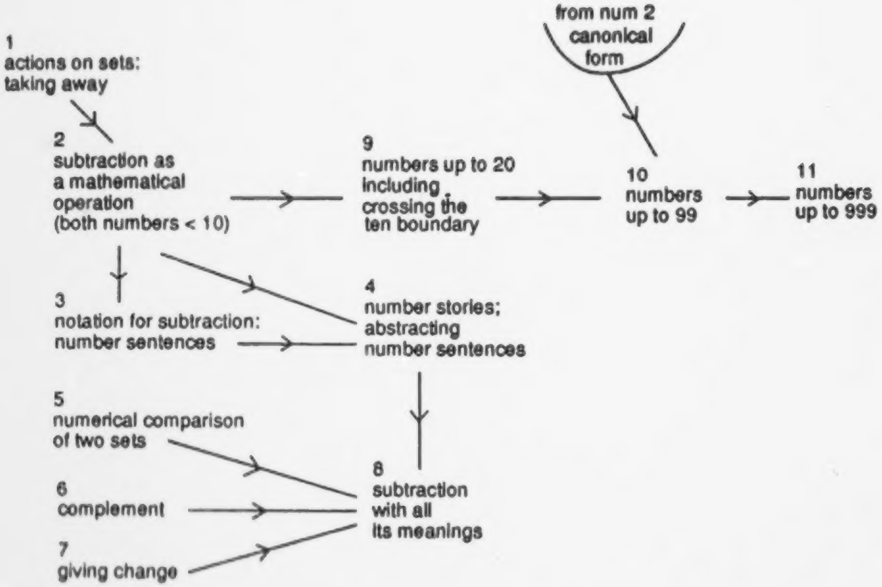
from num 2
canonical
form
1
actions on sets:
taking away
2
subtraction as
a mathematical
operation
(both numbers < 10)
9
numbers up to 20
including
crossing the
ten boundary
10
numbers
up to 99
11
numbers
up to 999
3
notation for subtraction:
number sentences
4
number stories;
abstracting
number sentences
5
numerical comparison
of two sets
6
complement
7
giving change
8
subtraction
with all
its meanings

Figure 1: Concept map for subtraction.

So far I've produced fourteen of these, for a core curriculum of primary mathematics (Skemp, 1989b). They could of course be combined into a single very large concept map, and indeed someone has done this, but myself I find it unwieldy, and an overload of information, so I have stayed with the separate ones.

SECOND DERIVATIVE: SCHEMAS AND ADAPTABILITY

In *The Psychology of Learning Mathematics* (21 years in print this year) I wrote "What is understanding, and by what means can we hope to bring it about? We certainly think we know whether we understand something or not; and most of us have a fairly deep-rooted belief that it matters. But just what happens when we understand, that doesn't happen when we don't understand, most of us have no idea".

And I went on to suggest that until we did, we would hardly be in a good position to bring about understanding in others. I wrote from experience, for it was as a teacher of mathematics that my interest in these problems first arose.

Some years later, at an ICME conference at Berkley, California, I asked a follow-up question of a more practical kind (Skemp, 1980). "What can we do when we understand, that we can't do when we don't understand?"

I offered three answers to the second question, in the particular context of mathematics. From these it became clearer why mathematics, one of the most abstract of all theories, is also one of the most practical. By understanding, we are better able to do three things:

1. To achieve our goals.

2. To co-operate with our fellow-beings.

3. To create in both the mental and the physical realms.

Today I can only deal with the original question, and the first of the above answers to the follow-up question. The following example provides a useful illustration to the general principles involved.

Some time ago I needed to find my way home from a school which I had been visiting for the first time with a maths adviser. He gave me the following directions for getting to the A45.

LEFT, LEFT, LEFT, RIGHT, HALF RIGHT, AND THEN LEFT ON THE A45

You can imagine what happened. I was soon lost, and it took me a long time to get home. a plan of this kind is closely tied to action, and if one of the actions is wrong, it's liable to throw out all which follow. In this case, if one gets off the path, one doesn't know what to do to get back on. The cognitive element is low. Habit learning is of this kind. It has low adaptablity.

In time for my next visit, my friend sent me a street map. Here is part of it. (The original also has street names).

Now I had structured knowledge, which is a much more useful and powerful kind.

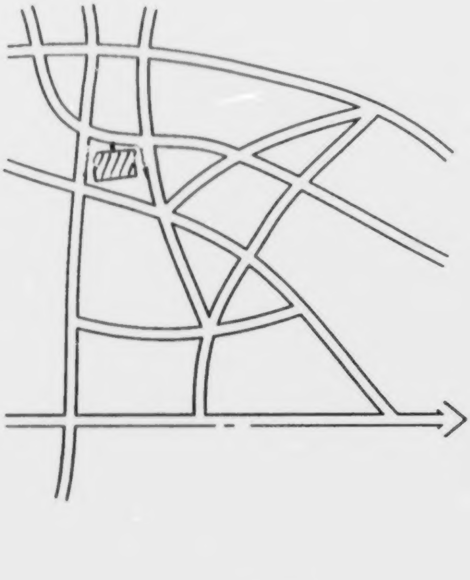

Figure 2

For a start, I could see where I had gone wrong last time. This it the route my friend described. But I didn't count the first left turn at the end of the school drive, so I ended up driving directly away from the A45, as you can see from the map.

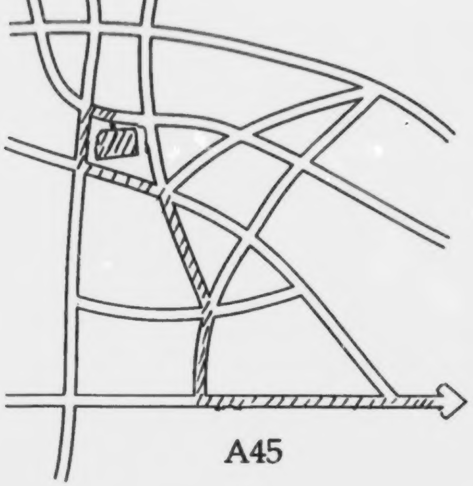

Figure 3

But even if I did go wrong, I could use the street names to find where I was, and then work out a new plan for getting home from there. And in general, this map contains the potential for working out how to get from any one location to any other location on the same map. so one could work out a way home which took in, say, a petrol filling station and a convenience store to pick up some milk. To do this by learning all the possible separate routes would entail memorising 3080 of these. This shows rather well the extra power which structured knowledge, that is to say schematic learning, makes available to us.

We are now in a position to answer the question with which we started; "What happens when we understand something, that doesn't happen when we don't understand?" And once again, the idea of a road map provides a good example from which we can generalise.

. H

Suppose we are here (making a point H on a blank page).. If this is all we know, then we are lost, as I was in the case I described.

If we have a map, and if we know where we are on it, this is quite a different matter. We can find our way wherever we want, provided that it is on the same map.

Suppose now that we are somewhere that isn't on the map. In this situation, we are again lost. But if in some way we are able to relate it to our map. We know where we are, and are back in control.

This principle is entirely general, whether we are talking about finding our way physically, or the abstract mental paths by which we find our way from awareness of a problem to our solution. Without understanding, we are mentally lost, just as if we were physically in unknown territory. And the emotions we feel are the same. Without understanding we feel frustrated and anxious. The achievement of understanding brings confidence and security.

It has to be the right map, of course, and the connections we make have to be the right ones too. With this in mind, we can now formulate our concept of understanding.

TO UNDERSTAND SOMETHING
IS TO RELATE IT CORRECTLY
TO AN APPROPRIATE SCHEMA

And we can see from this how understanding helps us to reach our goals - not just one, but a great variety according to our needs and wishes of the moment.

If schemas (knowledge structures) are so useful to have, it is natural to ask how can we acquire as many as possible, as quickly as possible. Unfortunately, mental structures of this kind cannot be communicated directly. Every learner has to construct them anew, in his or her own mind.

Fortunately, good teaching can greatly help intelligent learning *if we know how*. And indeed, effective help of this kind is a very necessary if in ten or fifteen years we want children to acquire knowledge which it has taken the best minds of mankind centuries to construct. Without this effective help, it usually doesn't happen.

The kind of teaching which supports intelligent learning involves not trying to take over the schema-constructing activity of the learner's own intelligence, but providing good learning situations which help it to do its own job. To do this requires first that teachers know how our intelligence sets about doing its task of constructing schemas in general, and mathematical schemas in particular.

BUILDING AND TESTING: THE THREE MODES

I use the term construction to mean a combination of building and testing. When constructing a wall, building consists of putting mortar on the wall, then another block or brick. We test for spacing, alignment, and verticality. Building a wall is a goal-directed activity, in which we build it a litle further, and then test that the change is in the right direction, namely towards the way we want it to be when completed.

In the activity of our intelligence in constructing schemas, we can distinguish three modes of building and three modes of testing.

SCHEMA CONSTRUCTION

BUILDING	MODE	TESTING
EXPERIENCE Our own encounters with the physical world	1	EXPERIMENT Expectations of events in the physical world
COMMUNICATION Other people share their knowledge with us	2	DISCUSSION Comparison of our own ideas with those of others
CREATIVITY Creating new ideas out of Those we already have: combining ideas, extrapolating patterns, intuition, imagination	3	INTERNAL CONSISTENCY Testing whether these new ideas are consistent with what we know already

The first mode of building knowledge structures is direct experience, from which a mental model is built yielding testable predictions. The second is social: sharing knowledge, and discussion, are major features of both academic and everyday life. So why not in school, too? In the third mode, our existing knowledge gives rise to new knowledge, eg by the extension of known patterns to new situations. These modes are more powerful in combination.

In conventional mathematics teaching, the main emphasis is on communication by the teacher of a method, after which the students practise using this method. Thus, only one (at most) of the six available means for intelligent learning is made available to the child. It is no wonder that so many of them have problems. The methods and materials I've been developing for the intelligent learning of mathematics are designed to bring into use all six of these.

All the foregoing modes for schema construction were at the stage of hypotheses, (albeit to me anyway and to quite a number of others, highly plausible hypotheses). They were tested during the next phase of the project. The methodology for this was that of the teaching experiment. This was pioneered in the United States by Steffe and others (Steffe, 1977) and is a combination of a teaching situation with the clinical interview as pioneered by Piaget. We found that activities based on the six modes of schema construction lent themselves specially well to this approach, being much more revealing of children's thinking and learning than conventional pencil-and-paper work.

The next two of the requirements listed on page 3 of this paper are:

2. Finding out how to apply the new model of intelligent learning to the learning and teaching of mathematics.

3. Developing and field testing methods and activities based on 2, sufficient for a feasability study.

These occupied the first two years of the project, and took the form of a close collaboration with a selected group of primary teachers. Our purpose was to find out how to apply the new model of intelligence to bring about intelligent teaching and learning of mathematics in the classroom. In this, we were together breaking new ground, both in theory and application.

A close collaboration was developed in which they learned from me about human intelligence, and improved their knowledge of mathematics, while I learned from them about the practicalities of the primary school classroom. My research associate and I tested and further developed the new approach while working in these teachers' own classrooms with their own children. We then discussed these results at our twice-weekly seminars together. They finished with MEd degrees, and I finished with the starting point for stage 4 of the Primary Mathematics Project.

This corresponded to the fourth requirement listed, the expansion of these activities into a full curriculum. It took the remaining six years of the project, and there isn't time today to describe this.

This next stage addressed itself to the second of the two problems described above, that of finding an effective approach to the in-service education of teachers. It was based on the assumptions that we needed to offer two kinds. One, short courses based in universities and other tertiary institutions, and the other, based in the schools themselves, rather than outside. This second kind would need if possible to involve at least half the teachers in the school, and to be low-intensity, over a period, rather than high-intensity and short-term. This last was related to the great amount which teachers need to learn, and the importance of giving them time to assimilate it at their own pace.

Originally this was a consultancy approach. It involved regular weekly visits to the school by myself and my research ssociate of that time. In this, much was learned both by the teachers (anyhow, some of them) and by myself. But even more important, a way was seen to overcome one of the main limitations of this approach, namely its high cost in the consultants' time. Clearly only a very limited number of schools could be helped in this way.

A major step forward became possible when we found that the newly developed materials themselves provide a basis for in-service self-education for teachers, independently of a consultant. Since each of the activities embodies the new theory of intelligence, in its practical application, we now find that by the use of these activities teachers can learn about the theory by seeing it in action. This relates their learning directly to their own classroom needs. This is further helped by the fact that the PMP activities externalise the children's thinking, and thereby enable teachers to use their own observations as a way of increasing their understanding of how children learn and think. The children learn mathematics better, and the teachers learn about the intelligent learning of mathematics by observing their own children in the process. We thus get "two for the price of one", time-wise. I like to think of it as a "happy hour" in the classroom.

Interestingly, and rather pleasingly, this also turned out to be an embodiment at teacher level of the model we were already applying for the children's learning. While using these activities, the activities of the children themselves became embodiments of the process of intelligent learning from which the teachers could learn about children's learning in the same way as the children were learning form embodiments of mathematical processes.

A new, self-help model for school based in-service education thus emerges.

1. The teachers first need to try out these activities with each other, as a preliminary to using them in the classroom. This also improves their own mathematical knowledge. Many of the existing school text books contain major conceptual gaps, not to mention mathematical inaccuracies.

2. They then learn about children's learning processes, by observing their own children using these activities. Experienced teachers have already said that they learn more from these observations than from all the books they read at college. This is mode 1 schema construction.

3. These experiences are then shared in weekly seminars with other teachers, and the ensuing discussion consolidates and expands teachers individual experiences. There also emerges, by consensus, a unified co-operative approach to the teaching of mathematics in the school concerned. This is mode 2 schema construction.

4. So far as mode 3 schema construction is concerned, the best areas for teachers' creativity are mainly in the classroom management and organisation of this new approach. The mathematical and psychological knowledge involved, and time required, are beyond what is available to most busy teachers. In my view they are entitled to have this done for them, just as we don't expect them to write the story books from which their children learn to read.

The self-help model has already shown itself successful in a number of schools.

DISSEMINATION

This takes us to the last of the requirements listed on page 3, which is dissemination. In this, a group of Canadians in the province of Alberta, led by Bruce and Marilyn Harrison, are leading the way. Over a period of four years more than 400 teachers have now taken short summer-session courses based on the theory and materials, and last year (my fifth consecutive visit to Alberta) we put on a second level course. This was to qualify those who had taken one or more of the first-level courses, and used them in their own teaching, to put on their own courses. We now have 24 consultants in the Intelligent Learning of Mathematics, of whom nine have already put on their own courses. They gave these courses in close liason with Bruce and Marilyn Harrison - who in turn continue to liaise closely with myself, with the help of fax machines in both our homes. (I'm thinking of putting in for a grant from somewhere to pay my resulting telephone bills!) Also in process of formation is a Canadian Network for the Intelligent Learning of Mathematics to enable teachers who have taken our courses to keep in touch, and provide support.

Now, thanks to the initiative and hard work of Beth Southwell and the financial support of the British Council, I have had the opportunity of coming to your country for the first time, and meeting with some of your mathematics educators, school teachers, and children in two of your schools. I hope that it has been of interest to you to hear some of what has been going on in the UK and Canada. It is certainly a great pleasure for me to come and exchange ideas with yourselves and others over here. Thank you for inviting me.

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AUSTRALIAN SPACE - PUSHING BACK THE FRONTIERS

PAM HARRIS

This keynote address was an audio-visual presentation using over 30 overhead projector and colour slide illustrations. It is therefore impossible to reproduce as a written paper in the usual form and I have not attempted to present it as an academic paper. What follows is a lightly edited transcript of the tape recorded talk, with an indication of the visuals shown and their source. Many of the illustrations can be found in just two publications - the speaker's recent book, ***Mathematics in a Cultural Context: Aboriginal Perspectives on Space, Time and Money*** *(Harris, 1991), and in* ***Dreamings: The Art of Aboriginal Australia*** *(Sutton, 1988).*

This transcript is intended only for those who attended the conference and should only be quoted as something that was heard, not as though it were a published paper.

INTRODUCTION

There is more than one context for this talk today. The most obvious context is that it is the last day of the MERGA conference and the last keynote address. The less obvious context, but one that we have talked about during the conference, is that it is International Space Year. This year was proposed by an American senator to commemorate the 500th anniversary of Christopher Columbus' voyage during which he "found" the Americas for the Europeans. It is also the 35th anniversary of the International Geophysical Year in which the Russians put the first satellite, **Sputnik**, into space, and that marked the beginning of the modern space age. Once scientists found they could get things into orbit, it was not long - only twelve years - before the Americans put men on the moon.

But, having decided on the topic of the talk and submitted an abstract, I discovered (from an Aboriginal person, not from anywhere else) that the MERGA conference coincided with National Aborigines Week, and my talk fell right in the middle of it. And National Aborigines Week has, I think, special significance this year, because at the end of last year the federal government of Australia passed an act which established a **Council for Aboriginal Reconciliation**. The aim of the reconciliation council is, for the next decade, right up to the year 2001 when we celebrate the centenary of Federation, to initiate and support a nationwide process of trying to bring together the indigenous people of Australia and the people who came and invaded the land and those who followed them; the indigenous people, the Aborigines and Torres Strait Islanders and those who descended from the first settlers and other later immigrants. One aspect of this process of reconciliation is a program of public awareness and public education to educate non-Aboriginal Australians about Aboriginal culture, Aboriginal priorities, disadvantages, dispossession, and the concerns they have today which stem from this original dispossession.

I have therefore chosen to talk less than originally planned about non-Aboriginal matters such as Captain Cook, and to concentrate more strongly on all matters Aboriginal. My aim is to raise awareness of some of the great differences between traditional Aboriginal ways of viewing and valuing the world, and the Western, "Eurocentric" ways of viewing and valuing the world which are inherent in school mathematics.

So, these are the three strands of the talk:

1. First, in acknowledgment of **International Space Year**, I am going to look at **navigation** over sea and land, contrasting the methods, skills and motivations of a British naval officer, Captain James Cook, finding his way across the sea, and the methods, skills and motivations of a group of Pintupi and Luritja people finding their way across the arid Western desert.
2. Then, with the MERGA interests in mind, I will give you some of the results of the research I did for the Mathematics in Aboriginal Schools Project, mainly from the section on shapes.
3. Finally, in celebration of **National Aborigines Week** and with reference to the process of reconciliation and the need for public education and greater awareness of Aboriginal culture and values, I am bringing in a discussion of the **Aboriginal art** of the central and Western Desert regions, these being the regions, and the people, with which I am most familiar.

NAVIGATION

Captain Cook - Navigator and Map-Maker

OHP - 1 Captain Cook in uniform holding map. (Source: Ward 1987:8)

Captain Cook was a naval officer who had done well for himself in charting the coast of Newfoundland. He did not come from very wealthy parents. He had struggled to get himself an education and had been good at mathematics.

OHP - 2 Captain Cook using instrument, working on map. (Source: Gill 1988:69)

He was a hard-working man. Here he is working on a map - he was noted for his map-making abilities. He also carried with him maps drawn by others who had explored the parts he was going to and relied, too, on the scientific knowledge developed in Europe. Scientific knowledge which contributed to better navigation was rapidly expanding at that time.

OHP - 3 A sextant. (Source: Gill 1988:92)

He carried with him what one historian called "an armory of instruments" (Cannon 1987:17). The sextant was one of them. The sextant, with its curved bar in front marked with degrees, helped him to establish latitude. It was an improvement on the earlier quadrant, enabling more precise readings on the sun and stars to establish position.

OHP - 4 An azimuth compass, similar to that used by Captain Cook. (Source: Gill 1988:101)

He also carried a relatively new instrument, the azimuth compass. The azimuth was an auxiliary device fitted across the top of the mariner's compass, which allowed him to get better bearings on distant objects such as stars, ships, and distant landfalls.

OHP - 5 An 18th century chronometer. (Source: Gill 1988:114)

But the most advanced instrument he carried with him - on his second voyage at least - was something that helped him to establish **longitude**. Establishing longitude had been a problem for a long time. Latitude could be established with fair accuracy by taking bearings on the sun and stars using instruments such as the sextant, but longitude could only be discovered by comparing ship's time (ie local time) with Greenwich mean time. Greenwich time could be worked out by a skilled navigator by making astronomical observations and doing a long series of mathematical equations which very few people could do, even if they had the time. In any case, by the time they had done all the equations the ship had shifted position and they were no longer comparing ship's time in that position with Greenwich time. The British parliament had been so concerned about the number of disasters at sea because of inability to get exact location, that they had established a Board of Longitude in about 1690 and offered a huge prize for anybody who could create a chronometer which could keep accurate time at sea. One of the problems was that the metal would expand and contract with the changes of temperature in different climates, and this problem had to be overcome. A man called John Harrison, who worked on the problem so long that he was nicknamed "Longitude Harrison", created a watch which did keep very accurate time, and this was a great help to Captain Cook, who became noted not only for the length and success of his voyages, but also for his extremely accurate mapping of the places he visited, including the east coast of Australia.

Captain Cook's motives were partly scientific - he was asked to go and check the transit of Venus from Tahiti. But he also carried sealed orders, which he only opened after observing the transit of Venus, that told him to go and look for the "Great South Land" that was believed to exist somewhere in the vast Pacific Ocean. The reasons the British wanted to find the southern continent were mainly related to what I would call "crass commercialism". All of the major European powers were looking for new commodities that they could trade, and because of the search for new commodities and the growing competition in international trade, they needed also to find places for colonies to support the trade. So Captain Cook, when he came to the east coast of Australia was motivated mainly by commodities and colonies; you might say, money and power.

ABORIGINES OF THE WESTERN DESERT - NAVIGATORS AND MAP-MAKERS

Captain Cook encountered Aboriginal people when he landed at Botany Bay and kept seeing them as he travelled up the coast. He gained some appreciation of how they lived and wrote in his diary that they did not appear to him to be "the most wretched people upon earth" as others had suggested, but in reality "far more happier than we Europeans; being wholly (sic) unacquainted not only with the superfluous but the necessary conveniences so much sought after in Europe" (Cannon 1987:11).

But I think he might have been surprised if he had been told that some Aboriginal people were very good at navigating too; in fact, in their own way, just as good as him - without all those instruments and all that scientific knowledge.

Two hundred years later, a man called David Lewis, an historian and medical doctor who had studied navigation in the Pacific Islands, written books about navigation, and who had himself almost circumnavigated Antarctica single-handed in a sailing boat, decided to see if he could discover how Aboriginal people navigate in the desert.

OHP - 6 Map of Australia showing regions of Lewis' journeys in 1972 and 1974. (Source: Lewis 1978: 76)

He went to the Oodnadatta area first in 1972 to test out whether it was feasible to study Aboriginal land navigation and he had such interesting results there that he decided to go ahead with a major excursion.

OHP - 8 Detail of the Wester Desert area 1973-74. (Source: Lewis 1978:76)

The map shows Yuendumu where I worked, where I lived for a total of 11 years. Lewis worked with people from Papunya, which is south of Yuendumu, and the nearest similarly large Aboriginal community, so that area is fairly familiar to me too. Lewis started from Papunya and went way out west past Lake Mackay on the Northern Territory/Western Australia border. Lake Mackay is Warlpiri country too. Warlpiri people live at Yuendumu (where I worked) and Pintupi people mostly live at Papunya and further west. Lewis' longest single trip was 1660 km, starting from Yayayi, west of Papunya, 300 km west of Alice Springs and going right across to the dried up wells of the old Canning stock route. Altogether he travelled a total distance of 7 800 km with his Aboriginal guides, including 1000 km of what he described as "completely trackless terrain". At the farthest point they were about 800 km west of the last human habitation in the Northern Territory.

OHP - 9 (colour picture) Lewis' Pintupi colleagues, Yapa Yapa Tjangala and Jeffrey Tjangala, on the Wati Kutjura track, Western Australia. (Source: Lewis 1978:77) [The picture is quite striking, with the two men standing on the end of a long piece of outjutting rock silhouetted against a clear blue sky. One is pointing with his arm fully extended.]

These Pintupi men were two of Lewis' guides on the trip that I'm talking about. They are on the Wati Kutjura track, which probably has no significance for you, but it does have a lot of significance for them. The word **wati** means "man", **kutjura** means "two", so they are on the dreaming track of two men, the **Wati kutjura** dreaming track. In a similar manner, perhaps, to how Captain Cook stood on the bow of his boat and looked out across the ocean to where he was going, these two men are looking out over their country. That is their place.

OHP - 10a (colour picture) Yapa Yapa and Jeffrey at Umari rock hole. (Source: Lewis 1978:80) [The men are standing on flat bare rock beside a sizeable water hole. The country beyond is spinifex and low scrub reaching to a low sandhill on the horizon.]

The early European settlers in Australia commented frequently on how Aboriginal people could find water in the most obscure places. Often they would be just walking with Aboriginal guides and suddenly a waterhole would appear in front of them in places where they had no idea that water might be found. Whereas Captain Cook was always looking for **land**, these Pintupi and Luritja people are always looking for **water**. This picture shows what is often described as the flat, featureless terrain where Aborigines often find water where white people would least expect it.

Umari (also called Yumari) is also one of the dreaming sites depicted in the art I will show later. Most of the waterholes - probably all of the major water sources - are sacred sites. As Lewis pointed out, the sacred significance of the waterholes was no doubt an important aid to survival in a country where water is "the ultimate physical necessity" (1978:77, 81).

OHP - 10b (colour picture) Jeffrey at Wibri rock hole. (Source: Lewis 1978:80) [Close-up of Jeffrey half lying on the rock surface with his head down to reach the water in a small hole in the rock.]

This picture again emphasises the importance of water. Jeffrey is drinking water from the rock hole. This scene is not at all unfamiliar today. The interesting thing about Lewis' work for me is that he was doing it at the same time as I was at Yuendumu learning how to teach mathematics to Warlpiri children. His travels are in the same timespan as my first period of work at Yuendumu. I've seen rockholes like that. There was one just close to Yuendumu (about 12 miles away) where we (the white staff) would sometimes go to watch the sunset. For us, it was just called The Rockhole.

OHP - 11 (colour picture) Rockhole at Muranji. (Source: Lewis 1978:78)

This rockhole is very important too. This is the rock hole in Kapi cave at Muranji. Lewis and his guides visited Muranji twice during their travels. They went once and then came back to it later. In between the two trips, one Luritja man called Big Peter Tjupurrula did a painting of the Muranji dreaming.

OHP - 12 Muranji Rockhole and Dreaming painted by Big Peter Tjupurrula. (Source: Harris 1991:31 from Lewis 1976a, p268)

This is a diagram of the painting that Big Peter did. He took it on the second trip to Muranji so that it could be explained and sung. If you are not familiar with the idea of an Aboriginal painting being "sung" then it is hard for me to explain. It literally means singing. Singing the dreaming of the painting. In my first year at Yuendumu one of my jobs was to organise Warlpiri cultural studies for the girls on Friday afternoons. What the women always did was to paint the bodies of the girls with the red and white ochre (or using the school powder paints), and they were always singing the dreamings while they were painting the bodies. (This is similar to Big Peter singing his painting. The designs and the singing go together to re-create the stories of the Dreaming.)

This Muranji dreaming is a story about an old woman who chased a little boy. It is also a map. It is an Aboriginal map contained in a dreaming painting.

OHP - 13a Sketch map of Muranji Rockhole and vicinity. (Source: Harris 1991:32 from Lewis 1976z, p 269)

OHP - 13b Big Peter's painting (OHP-12) and Lewis' sketch map (OHP-13a) side by side. [Both Big Peter's painting and the sketch map have the locations visited marked with matching numbers.]

If you compare Lewis' sketch map of the area and the topographical "map" contained in Big Peter's painting, you might think it seems a bit strange. The locations (represented by concentric circles) numbered 1, 2, 3, etc in sequence on the painting are all over the place on Lewis' white man's map. However, what you have to understand is that Aboriginal maps always have to be interpreted. You can only understand Big Peter's map if you know that, first of all, the central (most important) site is placed in the middle. You will see that also in the Aboriginal art I show later. So here, the central roundel (as the art people call it) is the main site, Kapi. That has to be explained by the person who did the painting. Then the rock holes and other features which they visited are presented in the sequence in which they visited them. That is why it looks nothing like Lewis' sketch map. In the painting, the main site is in the centre, the other sites are placed in the sequence of travel. The directions east and west are also right, says Lewis, if you take into account that all Big Peter is trying to indicate is easterly and westerly, the directions of the escarpment.

The most important point about the maps contained in the dreaming paintings - which help them to find their way in the desert - are that they always need inside interpretation. The landscape features represented have to be identified by the map maker himself, or else by someone who is really familiar with the dreaming and its topographical setting and also familiar with that painter's idiosyncrasies. And singing the painting is a really important part of the interpretation but also, as Lewis points out, it is a wonderful aid to memory and therefore a great aid to survival.

OHP - 14 Schematic map of suburban bus routes. (Source: Harris 1991:29)

If you're inclined to think that Big Peter's map is rather abstract, then you could compare it with this Brisbane bus map, or a Sydney train map, or the like. They are very abstract too. They show places in sequence and general direction, and that is all they show. That is also what the Aboriginal maps show. But it has been pointed out by more than one writer that Aboriginal maps are always something more than just a schematic map like this bus map, because the dreaming paintings create an image of the ancestral journeyings and events which are the basis of all their social structure and all their law and all their beliefs. Although a painting like Big Peter's is a schematic map showing roughly places and the connections between them, it is far more than that, whereas the Brisbane bus map is just a schematic map showing places and the connections between them and that is its only significance.

Lewis was really surprised, he said, because he went into investigating Aboriginal land navigation with all the ideas that he had about sea navigation. He thought they would be looking at the sun and the stars and they would get the directions from them. And perhaps the feel of the wind. But he said that all his preconceived ideas about navigation turned out to be wrong in relation to Pintupi living in the Western Desert. He noted with some surprise in his writing that:

> *the main references of the Aborigines proved to be the meandering tracks of the ancestral Dreamtime beings that form a network over the whole Western Desert* (Lewis, 1977:9).

Later, in the art section, I am going to show part of one of those networks.

Lewis said that the importance of sacred sites was brought out when he asked them to point out the direction of distant places (in various experiments with the people he was travelling with).

In 39 instances, accuracy varied between exactly right and 67% error - an enormous scatter. But every one of the big errors occurred when non-sacred places were being indicated. For the 6 far-away sacred sites, the error was never more than 10% and, in fact, averaged 2.8 degrees (Lewis 1977:11). (Maybe that has something to say about the importance of whether or not you are interested in what you are doing in school! Whether it is meaningful to you.)

Orientation and Mental Maps

Lewis also found that they were well oriented in compass terms. His companions, on questioning, make remarks like: "We knew north, south, east and west before the white man and his compass." "I can feel it in my head" (what direction I'm going).

One time Lewis, being puzzled about the means by which they could always find their way back directly to the place where they had started, began asking questions. On one occasion they had been out hunting and had been chasing a kangaroo for about half an hour, dodging all over the place. After killing the kangaroo, Jeffrey and Yapa Yapa just turned around and started heading back to the Landrover, which had been invisible since the first few minutes of the chase. (In Lewis 1977, p 11, the country they were in is described as "featureless and flat, with moderately open mulga and spear grass, devoid of sandhills, creek beds, escarpments, tall trees or other references". Visibility was "a hundred metres at most".)

Lewis was wondering how they knew where the Landrover was, so he began asking questions:

Q "how do you know we are heading straight towards the Landrover?"

A Jeffrey taps his forehead "Malu (kangaroo) swing round this way, then this" - indicating with sweeps of his arm. "We take short cut."

Q "Are you using the sun?"

A "No"

(Lewis 1977:11)

Lewis wondered whether Jeffrey had any points of reference. The only external reference Lewis could think of was their starting point; the sun was not being consulted; they were not using the "points of the compass" (which could have been located from looking at the sun), nor in this case did sacred sites come into the picture. So after his questioning and further incidents like that, Lewis concluded that the people he was with were orienting on some kind of dynamic "mental map" which was continually being up-dated in terms of time, distance and bearing, and more radically realigned at each major change of direction, so that when out hunting or wherever, they remained at all times aware of the precise direction of their starting point. (It seems to me that such a mental map - that kind of ability, if that's what it is - must be extremely useful in many areas of mathematics learning, if it could be tapped.)

Children's Orientation Skills

This leads me into a story that was given to me in response to my questions regarding the Space strand as part of the Mathematics in Aboriginal Schools Project. As I recall, it was from a teacher from Ali Curung, which is also in Warlpiri country, but further north along the Stuart Highway south of Tennant Creek. The teacher, who was new to the place I think, went out hunting with a group of children. After wandering around for a while, he got the feeling they were lost. Certainly he was a bit lost - not sure where they were or how they would get back. So he said to the children "What do you do when you get lost?" They didn't really understand the question at first. So he rephrased the question and managed to convey that he wanted to know what you do when you get lost. "Oh," said the children, "We just go home." It seemed as though being lost - in other words not knowing where you are or how to get home - was an idea they did not apply to themselves at all. Whether they could have given him any advice about what he should do if he were lost, I don't know, but I think he would have just got lost. Probably they, too, from a very early age were oriented in compass terms and able to know the precise direction of their starting location.

The knowledge of compass terms and orientation to the points of the compass of Warlpiri children in particular, where I lived, is illustrated in another story told to me by a teacher at Yuendumu. It happened outside the language room where I worked when I was teacher/linguist. An Aboriginal woman, Nungarrayi, was walking past carrying her baby and somebody called out, maybe from the window of the language room, to the mother and child. (Because in Warlpiri culture a way of greeting adults is to greet the child, you say something to the child and that's a way of acknowledging the mother.) So, the bystander called out to the child, but the child could not locate the direction of the voice. This was an eighteen month old baby. Then the person calling out said Kakarrani which means "to the east" and immediately the baby, only eighteen months old, turned its head and looked in the direction of the speaker. I talked to the linguist, May Laughren, about that and she said that is par for the course, not at all unusual.

The Warlpiri children's knowledge of compass terms was such that I preferred to use them in the classroom in some circumstances where left and right would normally be used. It is on the syllabus in the Space strand to teach young children to know left and right, but this is difficult for Warlpiri children. Really, left and right are not essential directions in your body. You know where your front is because that is the way you face. You know where your back is because that is behind you. You know where your head is and where your feet are because you can feel them in the orientation of your body. But left and right are not primary orientations bodily.

So in the classroom, if I was teaching them to write the letter "a" for example, I would use compass terms. (I knew enough in the context of the Yuendumu community to know that when I was standing at the blackboard, my right side was facing yatijarra, north, and my left side was on kurlirra side, the south.) So I would say, "Start yatijarra side, go up, then kurlirra side (that is to the south), then go down and then go back yatijarra way and go up and make a stick to finish the "a".

The point is that in Warlpiri language not only north, south, east and west, but also up and down are all in one united set of directional terms, not just north, south, east and west. The terms related to up and down are also included in the set of directional terms. So for a Warlpiri child left and right are really difficult; the compass terms are much more their way

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of doing things. There are many more stories in the books I showed (Harris 1989, 1991) about how Aboriginal people in the desert [and elsewhere] use compass terms even in the most confined spaces, like locating a picture out of several postcards on the front of the fridge or something like that. They'll say "to the east" or to the west or whatever.

MATHEMATICS IN ABORIGINAL SCHOOLS PROJECT - SPACE STRAND INVESTIGATION GUIDE

Background

Now I'm moving on to the research of the Mathematics in Aboriginal Schools Project. First I want to get you oriented to the location of my work.

OHP - 15	Australia divided into "remote" and "settled" regions.
OHP - 16	Major Aboriginal communities in "remote Australia".
OHP - 17	Schools with Aboriginal bilingual programs. (Source: Harris 1991:6, 7, 129)

I often feel I face a risk when I write things about Aboriginal people or my experiences living and working with them. Because some people say, "I work with Aboriginal people and they're nothing like that" or "the Aboriginal people I work with say that they don't do that", I want to make sure you understand the particular context of my work.

Economists divide Australia into "remote" and "settled" regions - this map (OHP-15) is from an economic survey (Altman and Nieuwenhuysen, 1979, *The Economic Status of Australian Aborigines*). Most of Australia's population lives around the south-east coast. So for census purposes and economic studies Australia, is divided into the settled regions and the remote. Most Australia and perhaps less than you might expect of Queensland is classified as "remote".

Within remote Australia, this map (OHP-16) shows the major Aboriginal communities. In the middle is Yuendumu where I worked, and it shows that it is on a reserve. West and south-west of that is the enormous North-west reserve going from the north-west of South Australia, over into Western Australia and up into the Northern Territory. The North-East Arnhem Land reserve is also shown. That is where Helen Watson (whose work you might have read) has been working, mostly at Yirrkala. That is a different scene from where I was working at Yuendumu. Working up in North-East Arnhem Land is a really different proposition from working with people in the central Australian and Western Desert areas. Some people who have worked in different areas have said to me they think the hardest place of all to work is the north-west of South Australia, and from my experience I would tend to agree with that.

There are a lot of Aboriginal people - although it is regrettable in another sense how few - who still have their first language and it is still strong and they are determined to keep it, even against the encroachment of English, and these languages are now being used in a lot of bilingual schools in the Northern Territory (OHP-17). I think it is important to acknowledge what the Northern Territory has done. When I was there we often grumbled about what little they did for bilingual education, but they've done an enormous amount compared to other parts of Australia and other parts of the world. In the middle is what we called the "Warlpiri triangle" - the three Warlpiri communities of Yuendumu, Willowra and Lajamanu that all had bilingual education programs. Warlpiri is the language that I worked in at Yuendumu.

My orientation when I did research was, and always has been, Aboriginal people who live in remote communities, fairly isolated, who still have an Aboriginal first language, which is very often a language used in a bilingual program in school.

So when I did the Space strand section of the Mathematics in Aboriginal Schools Project, I did the work on Space in preference to looking at Number. I had a choice. I included the study of Time and Money mainly because I had research data "left over" from the work I had done at Monash University (1976), from the measurement study (published in the NT 1980). I felt that Time and Money were just so important in the relations between Aboriginal and non-Aboriginal Australians. How often you hear white Australians bemoaning how Aboriginal people use their time and, worse still many think, how they use their money - for gambling, drinking and so on; the criticisms are endless. This is why I did major chapters on time and money. Then I had a choice about Number and Space. Because in the Northern Territory at the time I was doing the project, we had three major strands in the mathematics syllabus - Number was first in the syllabus; Space came in the middle, and Measurement. So I just put aside Number and decided that because Space is far more important to Aboriginal people I would concentrate on that first. I knew that what I was likely to find out if I looked at number concepts was what Aboriginal people don't have and don't do, and I like to avoid that kind of research which only has negative-sounding results. So I concentrated on Space.

By looking at the space sections of various syllabi, I made a list of headings of what things are usually included in the space section of a primary school maths syllabus, and thereby devised what I called a Space Strand Investigation Guide. (See Harris 1991, Appendix E) I put this out to linguists mainly, many of them missionary linguists, and to teacher/linguists employed in the bilingual programs and anyone else I thought interested and able to help. I had to keep stressing that it was a **guide** to investigating spatial concepts, not a questionnaire. It was a guide intended to help them to sit down with the Aboriginal language workers whom they worked with and find out what they could about spatial terminology in the language and the people's spatial concepts.

Shape

Not surprisingly what came back were messages to the effect that what I was asking about was nonsense to the Aboriginal people and they would tell me what they were interested in. Similar messages came from the very first time I sent out a questionnaire. One of the main comments coming back from the Space Strand Investigation Guide went like this, "In our language objects are classified by shape as to whether they 'sit', 'stand' or 'lie'. There is no word for 'to be'." (That is reminiscent of some of the American Indian languages where every noun is classified according to the shape of the object.) The comment about verbs in the Aboriginal languages came from places as far away as Noreen Pym, missionary linguist in the Iwaidja language on Croker Island, right up off the coast of Arnhem Land, and Faye Bell who had worked in the central desert at Areyonga in the Pitjantjatjara language.

OHP-18 Stance verbs in Australian languages. (Source: Harris 1991, pp 40-41)

Faye Bell said:

> The four verbs (nyinanyi "sitting", ngaranyi "standing", ngarinyi "lying", and pupanyi "leaning") describe the state of "being" for all things ... it would appear that relative proportions within a three dimensional frame of reference determine which verb describes the state of being.

What I find really interesting is that these highly trained linguists found it really difficult to use the verbs correctly. Two of them mentioned this. They said that some things that appeared to them to be lying turned out to be "standing", and something that appeared to them to be standing turned out to be "crouching" or "leaning". That sort of thing. (You have to use one of the verbs because there is no general verb "to be" as there is in English.) This emphasises to me the difficulties that must be encountered by Aboriginal children who have an Australian first language and have to do their schooling all in English. They cope easily with these classifications but must have a lot of problems with our English classifications but must have a lot of problems with our English classifications in other areas. It is the problem of trying to think in a different framework. If the experienced linguists can't do it easily, why should the 8-year old Warlpiri child in school be expected to do it without a lot of help? Why do we expect them to cope easily with another people's ways of classifying the world?

School Shapes

In the Space Strand Investigation Guide I asked specific questions about what I called "school shapes". My labelling of them as "school shapes" goes back to my early days of teaching at Yuendumu when I was required to use attribute blocks for teaching maths. I absolutely hated using the attribute blocks. It seemed to me quite nonsensical to sit there asking Warlpiri children "Give me a square, show me a circle, can you find a triangle?" and so on. I just thought that if we had to classify, we should classify something that was meaningful to them.

OHP-19 Plane and solid shape.
OHP-20 Circular, spherical shapes. (Source: Harris 1991:40-42)

In response to the Space Strand Investigation Guide questions about school shapes, one linguist said that an Iwaidja woman (on Croker Island off the coast of Arnhem Land) described the plane shapes drawn by the linguist as follows. A square was described as "flat, smooth"; a circle as "round" (however you say that in Iwaidja language); an oblong simply as "long", and a triangle as "having sides".

What I found intriguing was that from a linguist with Gurindji people in the desert a long way south-west of there came categories for solid shapes which were almost the same as those the Iwaidja woman gave for the plane shapes. They said that there seemed to be roughly three categories for solid shapes - flat, round, and long. Books, mangoes, and boxes are "flat", berries, eggs, and balls would come into the category of "round", and yams, swags, and tin cans would be considered "long". It was noted that most things that would be considered "long" also had a component of being somewhat cylindrical.

Overall, the replies about shape were very much along the lines that there are no descriptions (ie no vernacular vocabulary) for the shapes with straight lines. One may assume that this is partly because in nature itself there are very few straight line shapes like squares, cubes and triangles. But they did give descriptions for circular shapes. For example, in Gurindji the word *mirta* could stand for "circle" or "round thing". *Mirta* is the word for a round coil women used to place on their head to help carry water or food. The plural of that, *mirtamirta*, could be used for "circles", plural. But you would not know, except perhaps from the context, whether it referred to circles concentrically placed or circles scattered around separately.

WESTERN DESERT ART

One conclusion from the sections on plane and solid shapes in the Space Strand Investigation Guide is that the straight-line shapes of geometry are not usually described in Aboriginal languages, but there are terms for circular and spherical shapes. A look at the art of the Western Desert suggests perhaps a reason for the greater number of terms for circular shapes.

Slide 1 Figure 167 in Sutton 1988 "Ceremonial Ground at Kulkuta Anatjari Jampijinpa, Papunya. (The painting consists entirely of dotted concentric circles and the spaces between them.)

Circle is the main shape that is used in Western Desert art. The people who compiled the book of the Dreamings exhibition of Aboriginal art (the one that went to New York) analysed 300 samples of Western Desert art. The circle is the main icon used, followed by the circle and line motif. Why circle? Circle represents ceremony. One French anthropologist said that whereas in Western art the aesthetic focus might be on religion, and is a personal, individual expression, in Aboriginal art the aesthetic focus is ceremony. And that is a social thing, not an individual thing. The circle has all the connotations of a camp, a waterhole, a circle of people around a campfire, or the campfire itself. The circle has connotations of intimacy, of unit, and of an unhierarchical social order. These are all part of the symbolism of circle.

Slide 2 Figure 177 in Sutton 1988 "Jangala and Two Women at Ngurrapalangu" Uta Uta Jangala, Central Australia.

The circles in the paintings often represent topological features. In this one the circles (3 large concentric groups and 14 smaller ones) all represent caves. The bumps at the top are people - a Jangala, his sister Nangala, and a Napanagka woman who are in the Dreaming story. The other dotting and other shapes in the background are hills and vegetation in the area.

Slide 3 Figure 148 in Sutton 1988 "Kangaroo and Shield People Dreaming at Lake Mackay" Timmy Japangardi, Papunya. (Consists of many circles, almost all of them joined to each other by two or more lines with infilling between them. There are just a few unconnected circles.)

The unconnected circles in the middle of this one are trees, and the circles which are connected are campsite water soaks. At the very top, the little tracks are kangaroo tracks, as this is about a kangaroo ancestor and the connected circles and lines depict travels from place to place. This is an example of the circle-and-line motif which Nancy Munn, who studied Warlpiri iconography, called the "site-path" motif. The site-path motif makes up about 60% of all the symbolism in Warlpiri art.

Slide 4	Figure 178 in Sutton 1988 "Old Man Dreaming at Yumari" Uta Uta Jangala, Kintore.
Slide 5	Figure 179b in Sutton 1988 "Yumari" Uta Uta Jangala, Papunya.

These two paintings bring us back to Yumari (Umari) rock hole - the one I showed on the coloured overhead. They are both Yumari Dreamings. It is an Old Man Dreaming and was considered too secret for many details to be revealed in the catalogue. This also shows very strongly the site-path motif. You can see the network - the network that Lewis talked about that crisscrosses the country, connecting people to each other and to the land.

OHP-21	Selected Dreaming tracks in the Yuendumu region. (Source: Sutton 1988:122) (This relates to the following slides. The OHP remained on the screen while the slides were shown alongside.)

Slide 6	Figure 157 in Sutton 1988 "Wild Yam Dreaming at Yajarlu" Paddy Jupurrurla Nelson, Yuendumu.
Slide 7	Figure 138 in Sutton 1988 "Water Dreaming at Mikanji" Judy Nampijinpa Granites, Yuendumu.
Slide 8	Figure 160 in Sutton 1988 "Water Dreaming at Mikanji" Tilo Nangala, Yuendumu.
Slide 9	Figure 144 in Sutton 1988 "Burrowing Skink Dreaming at Parrikirlangu" Darby Jampijinpa, Yuendumu.

Explanation of Overhead Projection Transparencies

Now, to help you understand more about the Dreaming networks, I am showing a diagram of some of the tracks and some paintings that belong to those very same tracks. This diagram (OHP-21) comes from the book called *Dreamings: The art of Aboriginal Australia.* It shows selected Dreaming tracks in the area of Yuendumu, which is where I lived. So a lot of the people and the place names mentioned are familiar to me. Some of them are people I worked with. These are their homeland centres - their **home land**, their real home was in these places.

It shows one Dreaming track for Pamapardu (the Flying Ant) going from Wantunguru in the east towards the west. Then crossing over that is the Wild Yam (Yarla) Dreaming starting at Yamaparnta in the north and going through Yumurrpa to Warrikinpirri in the south. In that southern part you are just about into Pintupi country; in the north you are in Warlpiri country. Then there is the Blue Tongue Lizard Dreaming, the Initiated Youths (Maliyarra-maliyarra) Dreaming (the word written twice indicates plural) and the Yankirri (Emu) Dreaming which is quite important at Yuendumu, and those three come from the north-east to Warlukurlangu and on down to Ngarna in the south. It is this place, Warlukurlangu, that the Warlpiri artists chose for the name of their art group. They called themselves the Warlukurlangu Artists Association. That's why, because of that particular Dreaming site. Then, perhaps even more interesting, with more connections, is the Rain Dreaming. A lot of Rain Dreamings interconnect at Mikanji.

Explanation of Slides

The first two were Yumari rock hole, and the Old Man Dreaming. Now to Yajarlu, the Wild Yam Dreaming (Slide 6). This place, Yajarlu, is where I went camping overnight just before I left Yuendumu. So the place and the people whose homeland it is and the fact that you can find water there, are all very real to me, and I'm just a white visitor who's been there once. (How much more real and significant it is to them, who "belong" there.)

This painting is a bit unusual in the composition because the roundels are not connected and the line wanders around a bit (ie a contrast to the previous two in which circles were mostly connected with straight lines, the site-path motif). Again, the lines are the travels of ancestors and the circles are places where the mythical ancestors stopped.

(Slide 7) Those I have shown so far have been men's paintings. This is a woman's painting, done by Judy Nampijinpa. "Nampijinpa", "Jangala" and those other names I mention are tribal subsection names. There are 8 subsections at Yuendumu. (Women's names start with N and corresponding men's names start with J, eg Nangala and Jangala are sister and brother.) Judy inherited this Dreaming (a segment of a Rain Dreaming) from her father (and father's father) and she learned from her father and father's sisters the symbols she had to use, the ritual designs.

(Slide 8) Looking rather different, but still a Water Dreaming, is this one by Tilo Nangala. Nangala and Nampijinpa both inherit the same segment of this Dreaming track from their fathers. This helps illustrate the important connections associated with the Dreaming tracks. (Refer back to OHP-21) The Rain, or Water (Ngapa) Dreaming track connects Pintupi people from the south-west, up to Mikanji where several Rain Dreamings interconnect, to Warlpiri country as it goes north, and going westwards towards Mt Wedge it connects with Anmatyerre people. So it is not only families and subsections that are connected by the Dreamings, such as I've just shown with the paintings by Judy Nampijinpa and Tilo Nangala. The Dreaming tracks also connected neighbouring tribes and this was really important for their survival, because in times of drought when, say, the Pintupi waterholes were all dried up, they could approach their Dreaming partners in the connected tribal areas for access to other waterholes.

(Slide 9) This painting is also associated with one of the Dreaming tracks on the OHP - the Blue Tongue Lizard Dreaming but it also introduces the topic of symmetry. (The painting has roundels (of concentric circles) in the shape of a cross, ie three down the vertical axis and one at the mid-point of each side all connected by dotted lines. Four slightly smaller roundels roughly mark the corners (the two bottom ones are higher than expected, up the side) and then there are two roundels above either side of the vertical line and above the horizontal arms of the cross. These last two are not balanced by any similar motif in the lower half of the painting.

In Warlpiri art, there is a tendency always to be symmetrical. If there is one roundel (circle or set of concentric circles) it will be in the middle, if there are three they will be down the centre axis and maybe the one in the middle will be bigger, if there are more than that they will be at the corners of the rectangle or at the mid-points of the edges of the rectangle. This is a bit unusual in that it has two extra circles that aren't balanced on the bottom.

Slide 10 Figure 91 in Sutton 1988 "Witchetty Grub Dreaming" Theo Brown Jakamarra, Mbunghara. (There is a roundel of 5 concentric dotted circles in the middle and the two axes and the diagonals of the square are marked with a distinctive recurring motif of small roundels and crescent shapes with an unobtrusive overall background.)

This one, I think, is beautifully symmetrical. A good example of the "tendency towards symmetry". It is a Witchetty Grub Dreaming. The central circle is a ceremonial site. The Dreamings are related to not only land nurturing the people but also the responsibility of the people to nurture the land, so they have the ceremonies to ensure the continuation of the species and a good supply in the coming season. In this painting the small circles are the leaves on the tree on which the grubs feed, and the little crescent shapes are the grubs themselves.

Slide 11 "Witchetty Grub Ceremonies" Paddy Carroll Tjungurrayi. (Source: 1986 Aboriginal Art calendar)

I thought this one was completely symmetrical. But I showed it to some mathematicians at QUT and one of them surprised me by asking if there was any reason why it was not completely symmetrical? I said, "Where?" "There", he replied, "there's a mirror image most of the way (in the curly shapes either side of the centre line), but there are a couple of places near the bottom and at the top where it is not a mirror image". Well, I couldn't see it and they had to stand up and point it out to me on the screen, and I wonder whether the difference between me (with my rather limited mathematics background) looking at this picture and a professor of mathematics education looking at it is perhaps a little bit like the difference between an Aboriginal person and a mathematics person looking at it. You just don't see the same things.

It is interesting that it was noted in the Dreamings book that if an Aboriginal artist comments on another person's work - "That's a good painting you're doing!" - they are not talking about whether it is perfectly symmetrical, or the dots are all clear and neat and the same size, or that sort of thing (although that comes into it a bit). When an Aboriginal painter in the Western Desert area says to another Aboriginal painter "That's a good one" what he means is you (the artist) are faithfully and truly representing the Dreaming that has been handed down to you by your fathers. That's what is "good" to them.

CONCLUSION

What does all that show us?

What I was trying to do was to raise your awareness. As I said at the beginning, an aim of the Council for Aboriginal Reconciliation for the next 10 years is to raise the awareness of non-Aboriginal Australians of the culture, the values, and the disadvantages suffered by Aboriginal people. I hope I have at least done that. I often find it difficult - I certainly did when I wrote the Space Strand book (Harris 1989) and chapter (in Harris 1991) - to say what action should follow from what I've written. It's a difficult question.

But following this presentation, I think that there is some action that we, as MERGA people can take, because, you see, this year the theme of the conference fitted in nicely with the fact that it was International Space Year. And next year, as it happens, has been declared by the United Nations as International Year for the World's Indigenous People - next year, 1993. And next year MERGA is going to be held in Brisbane. Brisbane has a higher ratio of Aboriginal and Torres Strait Islanders living in it than any other Australian city. And it is in Queensland, the state of Australia that has the highest number of Aboriginal people of any state or territory in Australia. The projected theme for the conference is **"Contexts in mathematics education"**. Surely this presents an opportunity and a challenge:

> International year for Indigenous People,
> Brisbane,
> Queensland (which also has perhaps the blackest history of its treatment of indigenous people),
> and a theme of contexts in mathematics education.

Is there something we could do to include Aboriginal people and Aboriginal values? or emphasise them somehow at the next MERGA conference? Is there something we could do to pay special attention to the context of teaching in remote Aboriginal communities like those shown on the map of "remote Australia"? Could we pay special attention to the problems of non-Aboriginal teachers who go to those communities? Could we listen to Aboriginal people somehow in the preparation of the conference?

I've been influenced in the preparation of this talk by two Aboriginal people, two Aboriginal women. A month ago I went to a 2-day course called "An Aboriginal World" taken by a woman called Mary Graham, in Brisbane. I also have a copy of an interview with her about why she runs these courses on an Aboriginal world. Also I have a paper by Lilian Holt that she prepared for an international conference. Lilian is the principal of the Aboriginal Community College in Adelaide. In both these papers, the theme that comes through again and again and again is that Aboriginal people feel that they have a spiritual identity and White Australians do not; and that White Australians need one. Mary Graham quotes the oft-quoted old Aboriginal man who said "White man got no Dreaming" and she explains that what he meant is that they have not got any collective spiritual identity and, as Lewis discovered when he went out with Pintupi and Luritja in the Western Desert, and as anybody will discover if they really study the meaning of the Aboriginal art, some of which I've shown you, Aboriginal people do have a collective spiritual identity which is based in the land and connects them with the land and with each other.

So, going back to the beginning of my talk, Captain Cook pushed back the frontiers of the oceans to find Australia and draw maps of it, the first English settlers pushed back the frontiers of the inland to explore Australia and come to dominate it completely. Perhaps the last frontier to be pushed back is in the realm of the spiritual and also the frontier of communication between Aboriginal and non-Aboriginal people, being able to talk to each other. To help the communication, and in the spirit of the Council for Aboriginal Reconciliation, I will finish by quoting from Lilian Holt's paper.

The topic on which she was asked to speak was "Aboriginal Justice, Democracy, and Adult Education". Throughout the paper she expresses her concern for the missing dimension of spirituality in Australian life and education and she calls for a more holistic education. She wants a three dimensional education which includes the physical, the mental and the spiritual such as they try to have at the Aboriginal college in Adelaide. Along with other writers like Edward de Bono she raises the question of how to survive in an increasingly complex and technological society, which can even put men on the moon, a society where skills are so necessary and everybody knows you need them, but how to do that while at the same time enhancing your spirit and identity? Lilian expresses discomfort about national educational policies which are only geared towards numbers for employment and with an emphasis solely on production and profit and she says, "Sure, we need to advance, but to where and for what?"

I leave you with the closing lines of her speech:

> The qualitative in education has, for too long, been diminished and dismissed. The language of the spirit - passion and intuition - those unquantifiable aspects - may need to be included if we are to get to the core issues of true human education. The logically viable and the quantifiable will no longer take precedence, but will have equal partnership with all other areas.
>
> Catering to the physical, mental and spiritual is, I passionately believe, what true life-long education should be about. In the incorporation of all this, justice and democracy automatically flow, not just for indigenous people such as ... Aboriginal Australians, but for all of humanity. (Holt 1991:12)

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Paper Presentations

IMAGES AND DEFINITIONS OF FUNCTIONS IN AUSTRALIAN SCHOOLS AND UNIVERSITIES

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The study of functions is central to modern mathematics, and occupies the major part of the time spent on this subject in the senior school. The concept itself is rich and diverse, able to be conceived of using a variety of images, singly and in combination. This study explores ways in which Australian secondary and tertiary mathematics students think about functions in terms of both images and definitions. It further investigates the extent to which these factors influence their ability to discriminate between functions and non-functions, and to solve problems involving functions and their properties. The results have implications for both the teaching of functions, and the use of mathematical computer software which facilitates the representation and manipulation of functions in a variety of forms.

> A real-valued function F defined on a set D is a rule which assigns to each element x of D exactly one real number F(x).
>
> (Walters and Wehrhahn (1987, p.20) in Barnes, 1988; 120)

> A function is when you go out and have a good time, and relations - you visit them at Christmas.
>
> (Student respondent to Function Questionnaire, 1991)

What is a mathematical function? How do students think about functions? If you were to ask your best and brightest students, what response would you get? This paper describes a recent study of 383 high ability secondary and tertiary Mathematics students from Queensland, New South Wales, South Australia and Western Australia and suggests that you might well be disappointed if you expected that most of them could distinguish functions from non-functions, or even say clearly what a function is. While we may not expect secondary students to be able to recite the formal definition above by heart, the requirements of tertiary mathematics may well be finding many of our students ill-prepared in this regard. At the same time, there is a surprising level of consistency across schools, sexes and even States in the ways in which students think about functions.

IMAGES AND DEFINITIONS OF FUNCTIONS

Teachers of mathematics are well aware that functions are the "bread and butter" of our discipline in the secondary school and beyond. Senior students are expected to be familiar with a range of common functions, including the linear, quadratic, trigonometric, exponential and logarithmic functions; to sketch, manipulate, differentiate and integrate them. The study of functions occupies the major part of the time spent on mathematics in the senior school.

While the "formal definition" (as stated above) may appear somewhat daunting and rigid, the ways in which functions are conceived of and used prove quite interesting. The concept of "function", not surprisingly, is one which is mathematically rich, capable of being thought of using a number of distinct "images". Numerous studies over the past decade, both in Australia and overseas, have investigated the ways in which different groups of people think about and use functions (see Dreyfuss and Eisenberg, 1982; Tall and Vinner, 1989; Vinner, 1983; Vinner and Dreyfuss, 1989). Barnes (1988; 121), in interviews with secondary and tertiary students in New South Wales, identified six frequently occurring images of functions:

- A graph or curve
- A relationship between two variables
- A "function machine" (input-output device)
- A set of ordered pairs or table of values
- An algebraic formula or equation
- The symbol f(x)

She also distinguished the image of function as a *"mapping between two sets"*, which was not widely used by students, but was seen as useful in thinking about the concept.

The SOLO taxonomy (Biggs and Collis, 1989) distinguishes levels of understanding in the acquisition of concepts which may be relevant here. Students who focus quickly upon a single property or characteristic of a concept are said to be *unistructural*; those who recognise several properties as relevant, but do not link these together may be thought of as *multistructural;* those who are able to see the relationships between the various properties or representations of a concept are said to be *relational*. Some may go beyond this level, forming new connections and seeing applications of the concept in new situations; such learners are said to be operating at an *extended abstract* level. These levels (together with an initial *preoperational* level) are considered to cycle through each of the developmental modes - sensori-motor, ikonic, concrete-symbolic, formal and post-formal operations. Learners who describe functions using multiple images would be considered to be operating at a higher cognitive level than those who use only a single image. More recent developments in SOLO theory (Biggs and Collis, 1991, Collis and Biggs, 1991) suggest that individuals who operate in a *multimodal* way (able to draw upon earlier modes of thinking) will be more effective as problem solvers and critical thinkers. In the present study, students able to draw upon *ikonic* images of function to supplement their more usual concrete-symbolic way of thinking may well prove more capable at analysing functions than those who tend to approach such situations unimodally.

In learning a concept such as function, students begin with an active process (substituting numerical values into expressions) and gradually come to view such expressions as objects in their own right. Such objects may then be manipulated and analysed for their own sake, eventually becoming the basis for new processes in the formation of new higher order concepts (as in the composition of functions) (see Sfard, 1991). This process is consistent with that described by Bruner (in Bruner and Anglin, 1973), who sees cognitive growth as moving through stages of representation which he described as *enactive*, *iconic* and *symbolic*.

The evolution of the function concept throughout secondary schooling follows this pattern. Although students begin with the enactive ideas of "function machine" or numerical substitution in junior secondary, they move quickly to the study of functions as objects, which appears to remain the focus for all future study. The valuable perception of function as process may well be lost for many students by the time they reach their senior years. Whether they are fixed at the analytic concrete-symbolic mode or the global ikonic mode, students unable to utilise both ways of thinking may be disadvantaged in thinking about and using functions effectively.

Recently, the use of computer technology has begun to impact upon senior mathematics classrooms. To a large extent this has been in the form of *graph plotting* software (such as *CAPgraph*, *Algebra Graf(x)* and hand-held plotters such as the *CASIO-7000G*). Such tools allow students to quickly and easily change the representation of functions from algebraic to graphical form, and their use might be expected to result in an increasing familiarity with the graphical form of functions. More recently still has seen the emergence of true *multiple representation software* (such as *ANUgraph* and *CC3 - the Calculus Calculator*) which allow both the representation and manipulation of functions in algebraic, graphical and tabular forms. The effects of such technology upon the ways in which our students visualise and use mathematical functions are likely to be significant, but remain, as yet, unknown.

THE SAMPLE

The sample for this study consisted of 383 high-level mathematics students - 160 first-year Mathematics students from universities in Queensland and New South Wales, and 223 Year 12 students attempting either the Mathematics I and II courses (the highest level possible in Queensland, South Australia and Western Australia) or the high-level Three Unit course in New South Wales. Three schools from New South Wales (N1-3), seven schools from Queensland (Q1-7), four from South Australia (S1-4) and one school from Western Australia (W1) made up the school sample.

THE QUESTIONNAIRE

The questionnaire consisted of three sections and was modelled upon studies conducted by Barnes (1988), Vinner and Dreyfuss (1989) and Ayers, Davis, Dubinsky and Lewin (1988). The first section required students to rank a list of six images in terms of their preferred way of thinking about functions. Students were also asked to describe (in their own words) what they thought a function was, and to decide if there was any difference between functions and relations in Mathematics.

The second section involved students in distinguishing functions from non-functions, when presented in both algebraic and graphical forms. Students were asked to decide if the given expressions and graphs showed y as a function of x, and to explain how they decided. The examples chosen highlight several of the difficulties described by Markovits, Eylon and Bruckheimer (1988) in their study of students' knowledge of functions. Problems associated with constant functions (for many students a contradiction in terms), discontinuities and "piecewise" functions have been included.

The third section of the questionnaire required students to attempt solutions to a number of problems involving functions. Two of the problems were chosen so as to present the ideas in a relatively unfamiliar context - one involving composition of non-numerical functions, the other involving difference equations; the other problem was a more "mathematical" question, involving composition of functions. These are described in more detail elsewhere (Arnold, in press).

THE RESULTS

Function Images and Definitions

The pattern of images which students chose as their preferred way of thinking about functions was surprisingly consistent across schools, states and universities. With the exception of the New South Wales schools, the most frequent function image described was that of an algebraic formula. The graphical image was the next most common choice (in the case of the New South Wales schools, this pattern was reversed). Less frequent were the images of "rule or relationship between two variables" and "function machine". Few of the respondents thought of functions primarily as "tables of values" or as mappings. These patterns of images were found to be consistent across the schools in each state group, and across the two universities - no statistically significant differences were found for these groups related to their image preferences.

When asked to describe in their own words what they understood by a function, the picture assumes quite a different countenance (see Figure 1). What are termed here "function descriptors" were largely made up of the images already described, but the pattern of distribution is quite different. In all groups except the South Australian schools (who preferred a graphical description), functions were described most commonly as some type of rule or relationship. This corresponds to the common usage of the term outside of its specific mathematical sense.

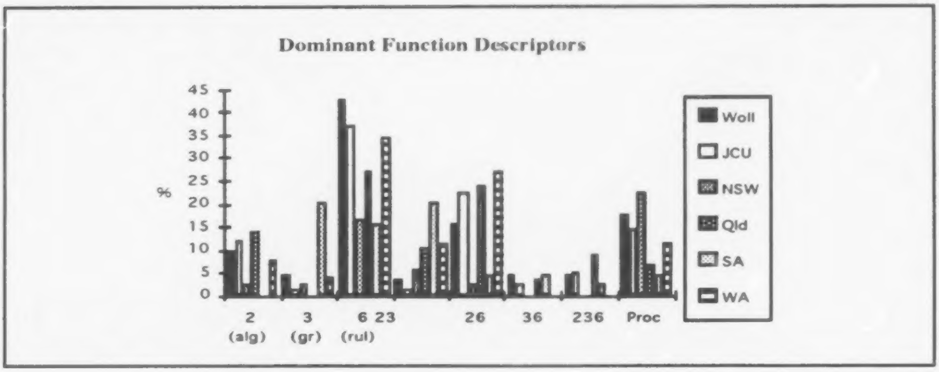

Figure 1: Dominant Function Descriptors.

Upon analysis of the entire sample's responses to the function definition question, ten different descriptors were found to account for over 95% of these answers. These are described in Table 1 with the code used to describe each in Figure 1. These descriptors, like the images above, proved surprisingly stable and consistent across the various groups which made up the sample (see Table 2). The two university groups exhibited identical orders for their most common descriptors, while across all groups the same five descriptors accounted for over 90% of responses.

Table 1: Most commonly occurring function descriptors (> 95%)

Function Descriptor	Code
1. Rule or relationship between variables	6
2. A rule which can be expressed algebraically	26
3. An algebraic formula	2
4. A formula which can be expressed as a graph	23
5. A process by which one number is changed to another	Proc (1)
6. A rule which can be expressed algebraically and graphically	236
7. A graph	3
8. A rule which can be graphed	36
9. A process which can be expressed using a formula	12
10. A rule which describes a process of change in variables.	16

How reliable might these descriptors be considered? No significant differences were found between the two universities, the seven Queensland schools or the four South Australian schools. The New South Wales schools were found to differ significantly [N = 36, df = 22, c^2 = 41.25, p = .0077], with one school (N2) exhibiting quite a different response pattern to the others, and a large number of non-attempts, particularly on the problems. This makes it difficult to comment further in this regard, although these schools were found to be consistent on sex, images and knowledge of the formal definition of function.

Table 2: Order of frequency of most common function descriptors (>90% of sample)

Woll.Uni	JCU	NSW school	Qld school	SA school	WA school
6	6	Process	6	3	6
26	26	6	26	23	26
2	2	12	2	6	23
236	236	23	23	Process	2
3	Process		236		
36			Process		

Further evidence for the stability of these function descriptors comes from comparison with their related images. Chi-squared analysis of contingency tables associating each of the three dominant images (algebraic formula, graph and rule) with function descriptors containing these images revealed significant differences. Those students whose function definition contained an algebraic element were significantly more likely to have chosen this as their preferred image [N

= 383, df = 1, $\chi^2 = 12.97$, $p = .0003$]; those who used "graph" in their definition matched significantly with those who preferred a graphical image [N = 383, df = 1, $\chi^2 = 23.78$, $p = .0001$], and similarly for the "rule or relationship" element [N = 383, df = 1, $\chi^2 = 7.95$, $p = .0048$].

These results suggest that the images and definitions which are described here are stable and potentially useful factors in describing students thinking about functions in senior secondary and early tertiary studies.

One descriptor stands out as different to the others, in that it is not composed of one or more of the three dominant images. This is the definition of "function as process", which was found to be particularly strong in the New South Wales schools (excluding N2), the two university groups and, to a lesser extent, the South Australian schools. Although not as dominant a descriptor as some of the others (only 47 of the total sample of 383 described functions in this way), it proves interesting for a number of reasons. The idea of function as process was found to elicit significant differences when contrasted with sex (it was male-dominated [N = 383, df = 1, $\chi^2 = 6.63$, $p = .01$]), with tertiary study (more than half of those who chose it came from the two university groups [N = 383, df = 1, $\chi^2 = 10.68$, $p = .002$] and with knowledge of the function definition [N = 383, df = 1, $\chi^2 = 8.77$, $p = .0031$]. It was also associated with significant success on one of the function problems, discussed below.

The final comment in this section concerns the occurrence of function definitions which utilise multiple images. Although the majority by far used single image definitions, double and triple image definitions were to be found in most groups, and more commonly in schools than universities. Once again, the relevance of multiple image definitions is discussed below in reference to the items involving function problems.

Knowledge of the Formal Definition of Function

While the images students use when thinking about functions may have been fairly stable across the various groups of this study, the same cannot be said about the knowledge of students regarding the nature of functions, and their abilities to distinguish functions from non-functions. For the purposes of this study, students were counted as "knowing" the function definition if they displayed knowledge of one or both of the following:

- The "algebraic" definition, that for each "x" there exists a unique "y" value;
- The "geometric" definition, often referred to as the "vertical line test", that a vertical line moved across the graph of a function should touch the curve at only one point for each point in the domain.

An early feature observed in processing the results from the Queensland schools was the lack of knowledge students displayed of either of these notions (Figure 2). Students from three of the seven schools showed no knowledge at all that a function in Mathematics was any more than "a relationship between things". When questioned regarding this, teachers generally saw no problem in this regard; while they used functions frequently, they did not see it as necessary for their students to be able to distinguish functions from non-functions.

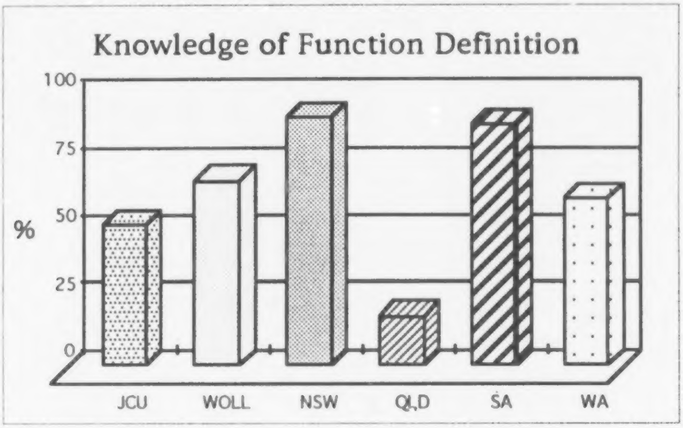

Figure 2: Percentage of State groups displaying knowledge of the Function Definition.

Of those who knew the formal definition, the majority overall used the algebraic form (46%), with 29% using the geometric form (the "vertical line test") and 25% displaying knowledge of both forms. This distribution, however, varied widely across the various groups. Males were found to be significantly more likely to use the algebraic form [N = 208, df = 1, $c^2 = 6.93$, $p = .0085$], while females were more likely to use the geometric form [N = 208, df = 1, $c^2 = 7.55$, $p = .006$]. The vertical line test was also found to be more prevalent among the school sample [N = 208, df = 2, $c^2 = 7.02$, $p = .0299$], and the algebraic form more commonly used by the university students [N = 208, df = 2, $c^2 = 6.58$, $p = .0373$].

Recognition of Functions and Non-Functions

It seems logical that, if students do not know the definition of a function, then they will have difficulty distinguishing examples from non-examples. This was found to be the case. The majority of students who did not know the definition decided on the basis of whether the example in question "looked familiar" (hence a circle was considered a function by many). Others looked for whether it "had an x and a y in it" (hence large numbers decided that y = 4 is not a function since there was no "x" term). Most often, the example was a function if it was perceived that, in some way, "y depended upon x".

In distinguishing those students who demonstrated knowledge of either the uniqueness property or the vertical line test from those who did not, there was found to be a clear advantage for the former in recognising functions and non-functions. Out of 8 questions, over 80% of those who knew the definition scored between 6 and 8, while the majority of those who did not know the definition scored between 3 and 5. All correctly identified the common examples given.

Statistically significant differences ($p = .0001$) were noted for most of the examples between those who knew the formal definition and those who did not, in both universities and schools. Knowledge of the definition proved no advantage in recognition of the familiar quadratic function, in both its algebraic and graphical forms. Nor did it advantage the university students in recognition of an unfamiliar function (in which $y = 1$ if x is rational, and $y = 0$ if x is irrational). It did, however, prove to be of some help to the school group.

Of the different forms of the function definition, the students who displayed knowledge of *both* algebraic and geometric forms were significantly more successful than others who knew the definition in only one form. Those using the geometric definition alone were found to be significantly more successful on most items than those using the algebraic form ($p < .01$).

Problems Involving Functions

The final section of the questionnaire involved students describing the solutions of problems involving function ideas in unfamiliar situations. Students describing multiple image definitions were significantly more successful than those using single images on all problems. Inspection of the contingency tables for each of the questions revealed that students using two and three images were more successful than those using only one. This finding is consistent with the SOLO Taxonomy analysis, which suggests that students using multiple images when thinking about a concept are operating at a higher cognitive level that those using only a single image. This argument is further supported by the finding that those students who displayed knowledge of the formal definition of function in both algebraic and geometric forms were significantly more successful on the item "RS" than those who used only a single form of the definition [$N = 208$, $df = 2$, $\chi^2 = 11.68$, $p = .0029$].

The perception of "function as process" was associated with significant success on the item involving interpretation of a difference equation. This was true for the entire sample [$N = 383$, $df = 2$, $\chi^2 = 7.04$, $p = .0296$] but especially so for the combined university sample [$N = 160$, $df = 2$, $\chi^2 = 10.82$, $p = .0045$].

IMPLICATIONS FOR TEACHING

This study was conceived as a pilot study for a more intensive investigation of these ideas. Nonetheless, certain factors appear to be sufficiently consistent to form the basis of recommendations for the current teaching of functions in the secondary school.

The Function Definition : Primarily important would appear to be the need to teach our students that not every relation is a function. Particularly in Queensland schools, inclusion of the teaching of at least the "vertical line test" and preferably both forms would appear to be a priority. The concept of function is central to modern mathematics, and it is unsatisfactory that our most capable students should be unaware of its nature, and frequently use the term inappropriately. Such students should be capable, not only of correctly distinguishing functions from non-functions, but of explaining the importance of the property of uniqueness in the mathematics they are studying.

Images of Functions : It is important, as teachers, to have some understanding of our students' thinking. From the results of this study, it is likely that most students will think of functions primarily as algebraic objects which can be graphed, and which describe a rule or relationship between things. While such an understanding is generally adequate for most purposes, it has a number of implications.

Students in this study seemed more likely to think of functions as *objects* rather than *processes.* There is a need to encourage an "active" perception, particularly when working with composition of functions and when solving problems which involve function ideas. The idea of *function as process* implies consideration of the domain and range and is probably closest to the image of "function machine" described above, in which students picture objects "going into" a function, and other objects "coming out".

Following from this is the implication that, in our teaching, we should attempt to provide students with as broad a base as possible when they are thinking about functions. Although the graphical image has some advantages over the algebraic, it is more useful still if students are encouraged to think about functions using multiple images wherever possible. The ability to mentally represent a given function variously as a graph, a formula, a table of values, a mapping or even as a "black box" must provide students with the means to view and attack given problem situations more intelligently and creatively than those who are fixed in a more narrow perception.

Functions and Computer Technology : A final implication follows on from the last. As the use of graph plotting software becomes more common in the teaching of senior mathematics, there is a real danger that it may cause a narrowing of the focus regarding function images, rather than the broadening recommended above. While the power of the computer will make it easy to switch from the algebraic form to the graphical, if not used carefully, two problems may arise.

First, students may come to depend more strongly upon a graphical representation when thinking about functions. This has some advantages, but may well prove to be a problem in tertiary studies of mathematics, when many functions are abstract and unable to be visualised. Students too dependent upon a visual image may well be disadvantaged in such cases. Secondly, the perception of "function as object" may be further reinforced, making students less likely to consider functions as active processes which act upon their domains and produce new objects.

The emergence of *multiple representation software* which allows the user to move between algebraic, graphical and tabular form may help to encourage this broadening process, as may the use of the spreadsheet as a tool for exploring algebraic relationships. Allowing the user to view the same mathematical function from various perspectives should increase understanding and improve the versatility of the thinking involved. Here, too, however, may be found the same dangers as described above. Teachers will need to assist their students to reflect upon the processes involved, rather than allow them to use the technology *automatically*, and without thought. The potential is great for computer tools such as these to improve the teaching and learning of mathematics in many ways, but there are dangers too which will need to be anticipated. Classroom teachers, through careful and reflective use of such tools, hold the answers to their effective use. Knowledge of the ways in which students think about and use functions seems likely to be inseparable from any consideration of the effects of such technology.

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TECHNOLOGY ENRICHED INSTRUCTION IN YEAR 9 ALGEBRA[1]

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This paper describes a pilot study involving the use of a computer treatment and a calculator treatment in a unit on linear equation solving in year 9. The computer treatment made use of spreadsheets, while the calculator treatment parallelled the computer treatment but used either previously prepared tables or ones constructed by the student with the aid of a calculator. Although both groups were able to generate tables to solve an equation using guess and check, computer treatment students did not discover on their own how to make full use of the spreadsheet capabilities to refine their guess and check approach.

INTRODUCTION

This is a report of a teaching experiment involving the use of a computer spreadsheet in a Year 9 unit on linear equation solving. The experiment was conducted in a Melbourne metropolitan secondary college during a four week period in 1992. The instructional material developed for the experiment was an adaptation of the computer graphic and numeric equation solving methods which were used in a year 12 equivalent bridging course in mathematics at the University of Melbourne (Asp, 1991). In this paper only results from the spreadsheet component are reported.

The intention of the study was to identify difficulties and problems associated with the classroom use of the material under conditions reflecting typical school and classroom organizational structures and teaching methodology. The material will be revised prior to engaging in further teaching experiments planned for the second semester of 1992. It was felt that through the use of the computer spreadsheet and graphing applications students might gain an understanding of algebraic concepts and techniques even though their skill in the manipulative aspects of algebra were not highly developed. Therefore, the study also sought to determine the effect of instructional treatment upon student learning of some important algebraic ideas.

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IMPORTANCE

Current mathematics curriculum statements within Australia and overseas are encouraging teachers to incorporate the use of computer and calculator technology into the teaching of mathematics.

> ...students should have sufficient experience of calculators and computers as mathematical tools to be able to make informed decisions about whether or not to use them in particular mathematical situations, and to use them efficiently when they wish to do so (A National Statement on Mathematics for Australian Schools, 1990, p. 14)

> ...we believe that students should learn to use the computer as a tool for processing information and performing calculations to investigate and solve problems ... tools that simplify, but do not accomplish, the work at hand ... (Curriculum and Evaluation Standards for School Mathematics, 1989, p. 8)

In Victoria, year 11 and 12 mathematics students are required to use computer technology in their learning of mathematics (VCE Mathematics Study Design, 1989, p. 6). In addition, the nature of recent external assessment tasks at year 12 indicate that the ability to use common computer tools such as spreadsheet and graphing applications would be advantageous. It is too late to leave the development of an appropriate skill and knowledge base in the use of suitable computer tools such as spreadsheet and graphing applications until year 11 or 12. It is better that students become aware of such software in the early years of the secondary school. Hopefully, they will then be in a better position to use the technology as a "cognitive tool" which is not only capable of assimilation into established modes of thinking but also able to provide a medium for thinking and learning (Ruthven, 1992).

Even though the use of computer technology has potential for assisting students to learn mathematics and teachers are being encouraged to use this technology, few have taken up the challenge. As noted by Arnold:

> Mathematics teachers, in particular, seem to have let a golden opportunity slip through their fingers. As the early 'owners' of the computers in schools, they failed to build upon this potential, content to focus upon programming (usually in BASIC) and 'drill and practice' software. The challenge was taken up by others - so much so that it tends to be the Social Science departments, the English departments, and others who dominate the use of 'the computer room' in many high schools today. (1992, p. 32)

Some of the reasons for this situation are lack of teacher knowledge of the technology and its potential use for the learning of mathematics, lack of suitable instructional materials to allow them to incorporate the technology into their teaching, lack of time to undertake explorations into these areas, and lack of facilities.

Research evidence is needed which documents the advantages and possible disadvantages to the learning of mathematics that both computers and calculators may provide. In addition, learning activities that have been proven by research to help students gain fundamental understandings while developing knowledge and skills in the use of this technology need to be developed and communicated to teachers.

THE NATURE OF THE STUDY

The study took place in a northern Melbourne metropolitan secondary college which had recently set up a laboratory of Macintosh computers. All four year 9 teachers agreed to be involved in the pilot study; none had previously used Macintosh computers in their classroom teaching. Two teachers had two year 9 classes, giving six classes in total. These two teachers each taught both a computer and a calculator class, enabling some control of the "teacher variable".

Two parallel teaching treatments for a unit on linear equation solving were developed, one involving the use of the computer and the other the use of a calculator. On the basis of access to the computer facilities, teacher desire to use computers, and the need for those teachers having two classes to have one assigned to each treatment, three classes were assigned to the computer treatment and three classes to the calculator treatment.

A student booklet of written instructions with exercises was produced for each treatment. Each treatment consisted of 15 class lessons with 7 lessons common to both treatments. Six of the common lessons involved the typical formal work with algebraic representation and equation solving found in year 9 algebra. The other common lesson was used to introduce a problem context for the equation solving lessons for each treatment. For the eight remaining lessons in each treatment, five involved the use of tables (spreadsheet) and three the use of linear graphs (graphing application).

The computer treatment made use of the spreadsheet application MS Works and the graphing application ANUGraph to assist students in solving linear equations and hopefully to help them draw links between symbolic, numeric and graphical equation solving. The calculator treatment students also solved equations from tables and graphs which were either constructed by the student with the aid of their calculator as a computational tool or given directly to the student.

An example of numeric equation solving using a spreadsheet is shown in Table 1. The student uses the spreadsheet to generate tables involving F(x) and G(x) by changing the step number or the initial value for x (or both) to solve the equation F(x) = G(x) by a guess and check approach. Each time an initial or step value is entered the spreadsheet automatically recalculates to give a new table. Each spreadsheet was set up in the computer and the student simply accessed it.

Table 1: Spreadsheet for Numeric Equation Solving

Step	x	F(x) = 4 + 0.25x	G(x) = 8 + 0.15x	F(x) - G(x)
10	0	4	8	-4
	10	6.5	9.5	-3
	20	9	11	-2
	30	11.5	12.5	-1
	40	14	14	0
	50	16.5	15.5	1
	60	19	17	2
	70	21.5	18.5	3
	80	24	20	4
	90	26.5	21.5	5
	100	29	23	6

An example of graphic equation solving using a graphing application is shown in Figure 1. Here the student uses the application to draw the graphs for y = F(x) and y = G(x) and solves the equation F(x) = G(x) by reading and interpreting the coordinates of the point of intersection.

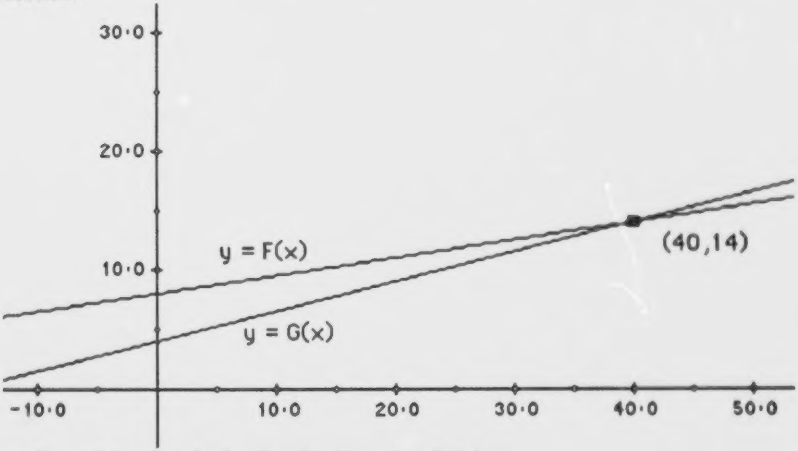

Figure 1: Line Graphs for Graphic Equation Solving

Figure 2 summarises the approach to equation solving used in the two treatments. As well as solving equations, students in each treatment were also asked to find rules which were used to generate a table of values. In the computer treatment students were encouraged to create different tables with the spreadsheet to assist them in finding the rule, and for the calculator treatment, students simply worked from given tables to find the rule. In addition, computer treatment students were asked to use a spreadsheet to create tables in order to compare two expressions and determine if they were sometimes, always or never equal. Calculator treatment students were asked to do the same but they used a table of values which they created by direct substitution into the expressions. The idea of sometimes, always or never was also approached through examination of graphs generated from two given expressions with the computer drawing the graphs (computer treatment) or the students producing one of the two graphs to be compared (calculator treatment).

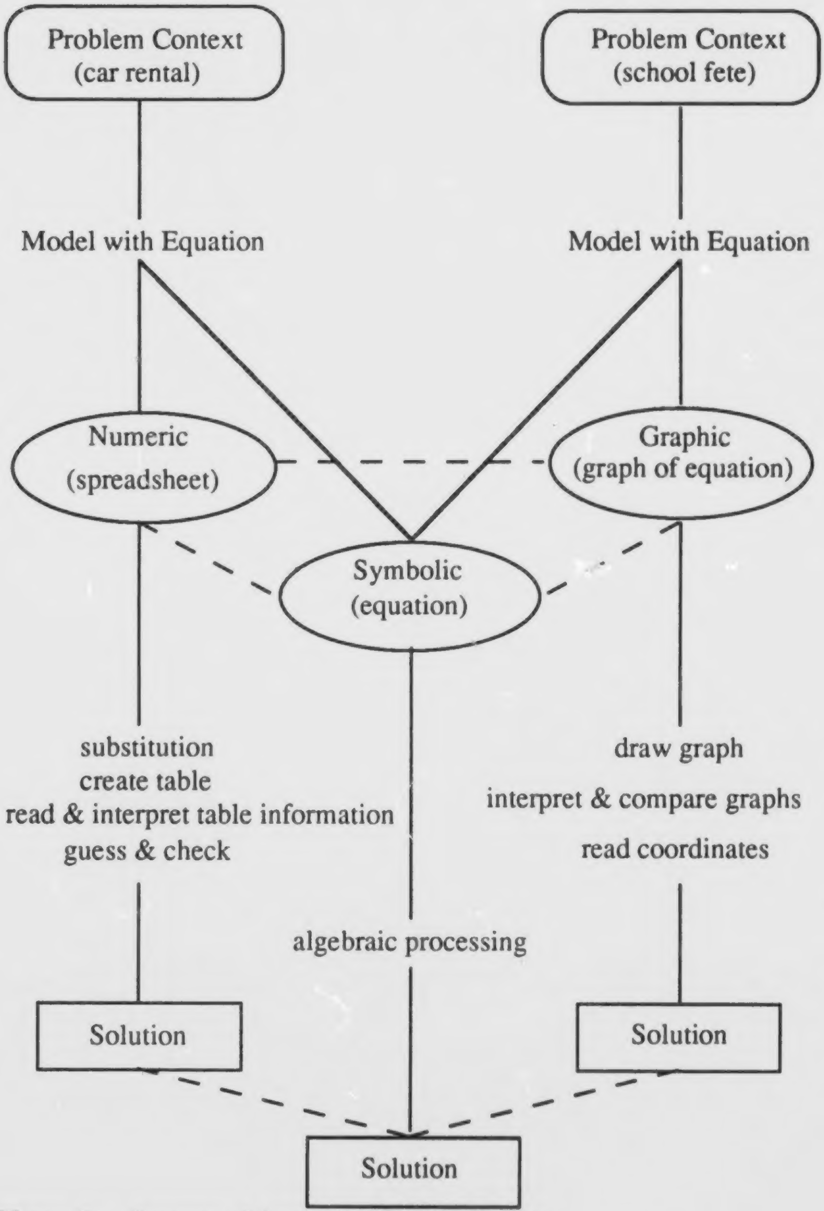
EQUATION SOLVING
Problem Context (car rental)
Problem Context (school fete)
Model with Equation
Model with Equation
Numeric (spreadsheet)
Graphic (graph of equation)
Symbolic (equation)
substitution
create table
read & interpret table information
guess & check
draw graph
interpret & compare graphs
read coordinates
algebraic processing
Solution
Solution
Solution

Figure 2: Summary of Approach to Equation Solving

ANTICIPATED OUTCOMES

It was hoped that computer treatment students could learn how to use the two computer applications to find the solution of equations and explore relationships between algebraic expressions in the time available and, in doing so, they would develop their understanding related to each of the following:

1. A letter is used to represent a variable number.
2. An expression is a variable number, dependent upon the value of the variable in which it is defined.
3. An equation is a statement that the numbers named by two expressions are sometimes (but not always) equal.
4. An identity is a statement that the numbers named by two expressions are always equal.
5. To solve an equation means to find the number(s) which make the two expressions which define the equation give the same number.
6. The solution(s) to an equation can be found by a guess and check approach and this process can be aided and refined by using relevant information in tables or spreadsheets.
7. The solution(s) to an equation can be found by a graphic approach by graphing the functions corresponding to each side of the equation and reading the coordinates of points of intersection of the graphs.

As the material for the calculator treatment parallelled that of the computer treatment it was felt that the above outcomes might apply equally to classes in the calculator treatment.

In order to gain some measure of the effectiveness of the treatments with respect to the anticipated outcomes each class was given a pre-test and post-test. Following the post-test, interviews were conducted with 12 students, two from each class with one at the 80th and one at the 40th percentile on the post-test. In addition, student worksheets were examined, observations were made of classes and, at the conclusion of the program, a meeting with the teachers was held to gain their views.

SOME RESULTS

The results of the written pre-tests and post-tests given to all students will be reported elsewhere. Each of the 12 students who were interviewed completed four tasks. A summary of the results for the first three of these interview tasks follows.

Task 1

The questions for Interview Task 1 are given in Figure 3.

(a) x and y are numbers. $y = 8 + x$ and $x = 10$.
What is y?
(b) x and y are numbers. $y = 19 + (3x + 1)$ and $3x + 1 = 7$.
What is y?
(c) x and y are numbers. $y = 10 + (2x + 15)$ and $2x + 15 = 4$.
What is y?

Figure 3: Task 1 Questions

The computer students did much better on this task than the calculator group with 5 of the 6 computer students answering all parts correctly. Only 3 of the 6 calculator group students answered part (a) correctly. One answered part (b) correctly and did so by finding $x = 2$ and then substituting, a method not used in the computer group. No calculator student answered part (c) correctly. A variety of syntax errors were made by the calculator students (e.g. "$19 + 4x = 23$ " for (b) and $y = 27$ for (c) because "$2x + 15 = 17x$ then add 10"). This question had been designed to probe whether students viewed an expression such as $2x + 15$ as a unit. Such stark differences between the two groups had not been expected. The corresponding item in the written test had been somewhat harder than the interview item (in effect only part (c) was given with no "lead-in") and had not shown a significant difference. This version of the item will be used in the next trial.

Task 2

The questions for Interview Task 2 are given in Figure 4.

In the algebra expression $5c + 1$, the letter c could stand for:

(a)	100	YES	NO
(b)	the number of kms travelled	YES	NO
(c)	cars	YES	NO
(d)	the number of cups on a table	YES	NO
(e)	bottles	YES	NO

Figure 4: Task 2 Questions

Students were asked to explain their answers and for each YES response were asked: "If c stood for that, what would $5c + 1$ stand for?". Three students (two computer and one calculator) believed that c could only be used if the initial letter of the relevant quantity or object was c. Four students (one computer and three calculator) avoided any choices which were objects. Two students (one from each group) said yes to all items but used numbers to explain the meaning of $5c + 1$. There seems to be a mismatch between the answers students give to the written item and their explanations of the meaning of $5c + 1$. Out of the 12, only two students read $5c + 1$ as other than a number ("5 cars + 1" and "5 cups + 1 cup"). Even though students seem to allow "letter as object" most were not confused by the meaning of $5c + 1$. Perhaps they would be more easily misled by the expression $5c$. This item needs work before it could be reliably used to diagnose student misconceptions in a written form.

Task 3

The context for Interview Task 3 is given in Figure 5 and the spreadsheet in Table 2.

The car rental context used in this item was similar to the problem context that the teaching materials had explored. Computer students used the spreadsheet to answer the questions, calculator students were provided with a table which could be used as a basis for interpolation.

All 12 students were able to correctly read F(70) from the table or spreadsheet and 7 gave a satisfactory interpretation of its meaning in terms of renting a car from Mary or Joe. All 12 students were able to find x when G(x) = 299.42 (again a value from the table or spreadsheet) and 10 gave a satisfactory interpretation of its meaning. All 12 students were able to correctly find x from the table or spreadsheet when F(x) - G(x) = 7.38, but only 4 gave a satisfactory interpretation of its meaning.

Students were then asked to interpolate between the initially given values to find x when F(x) = 323.50 and to find the value of x which made the cost of renting equal, i.e. F(x) = G(x). When solving the equation F(x) = 323.50, all of the calculator students were able to use the table to locate initial starting values of x correctly but only 2 were able to use their calculators to narrow down the answer to 87. Both groups of students preferred to compare F(x) and G(x) directly for the second part of the question, and only 2 students (both from the calculator group) referred directly to the data on F(x) - G(x). All computer students confidently created new tables in an attempt to get 323.50, but there was no efficient use of start and step numbers. It seems that direct teaching will be required for students to appreciate strategies such as adjusting start numbers and step numbers alternatively to zoom in on an answer. Three of the 6 computer students found the correct solution (87) to F(x) = 323.50. Only 1 student (computer) found the correct solution (112) to F(x) = G(x), although 1 calculator student used guess and check with calculator and gave 115 as his best answer.

Two questions required algebraic rearrangement before the spreadsheet or tables could be used. Three of the computer students and one calculator student correctly solved F(x) = G(x) + 1.98 (the solution could be read directly from the tables of F(x) - G(x)). Two computer students and one calculator student correctly solved G(x) + 20 = 372.52.

A person can rent a car from either Mary's Car Rentals or Joe's Car Rentals.

x is the number of kilometres the car is driven.

F(x) is the cost of renting the car from Mary's Car Rentals and driving it x kilometres.

$$F(x) = 280 + 0.5x$$

G(x) is the cost of renting the car from Joe's Car Rentals and driving it x kilometres.

$$G(x) = 269.92 + 0.59x$$

Figure 5: Context for Task 3

Table 2: Spreadsheet for Task 3

	Distance	Cost from Mary	Cost from Joe	
	(kms)	(dollars)	(dollars)	
Step	x	$F(x)=280+0.5x$	$G(x)=269.92+0.59x$	$F(x)-G(x)$
10	10	285.00	275.82	9.18
	20	290.00	281.72	8.28
	30	295.00	287.62	7.38
	40	300.00	293.52	6.48
	50	305.00	299.42	5.58
	60	310.00	305.32	4.68
	70	315.00	311.22	3.78
	80	320.00	317.12	2.88
	90	325.00	323.02	1.98
	100	330.00	328.92	1.08
	110	335.00	334.82	0.18
	120	340.00	340.72	-0.72
	130	345.00	346.62	-1.62
	140	350.00	352.52	-2.52
	150	355.00	358.42	-3.42

COMMENTS

There is some evidence to suggest that students can learn to use a spreadsheet with guess and check to solve an equation. However, in their use of the computer students tended to concentrate on a single line of the spreadsheet, ignoring and failing to use other relevant information provided in the full spreadsheet display. Most students were unable to discover, on their own, the joint use of the step number and the starting number.

The instructional material required students to do too much in a short period of time. There was little time for them to discuss and reflect, something which would also have required more stimulus and direction from the teachers. Once students began to use the computers the teachers were reluctant to ask them to reflect on the process they were engaged in or on ideas related to this process. As a result, students tended to work only from the written instructional material with the teacher merely assisting this approach. More direct teacher input into the lessons was needed.

Despite these reservations, the teachers were enthusiastic about the use of computer spreadsheet and graphing applications in their classrooms, and they are planning to incorporate them into their future teaching of year 9 algebra. Student response was also generally quite positive.

The interview results have provided some measures of the differences between the computer and calculator groups. In particular, the computer groups seemed more able to see an expression as a single entity. This and other observations will form the basis for enhanced written pre- and post-tests which will be used to evaluate the success of a revised version of the student instructional materials and teaching approach.

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THE STRUCTURED CLASSROOM INTERACTIONS SCHEDULE (SCIS): A VALIDATION STUDY

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Classroom observation research had its beginnings in the second decade of this century (Hammersley, 1986, p. xi). Rather than representing a single type of research methodology, classroom observation research covers a wide variety of investigations and techniques. By the seventies, the decade when such studies grew exponentially, hundreds of classroom observation instruments had been developed (Borich and Madden, 1977). These have been classified by Delamont and Hamilton (1984) into two categories: systematic observations (usually based on quantitative techniques); and ethnographic studies (usually based on qualitative techniques). Hargreaves (1980) added a third category based on socio-linguistic research (see also Florio-Ruane, 1987).

Quantitative studies usually identify target behaviour(s) prior to observation, develop checklists or other schedules, and apply these instruments on classroom settings to record the frequency of occurrence of the identified behaviour. Qualitative studies usually take the form of an observer immersing himself/herself in the culture of the classroom for relatively extended periods of time and recording as much information as possible in a variety of forms (e.g. field notes and interviews). What is observed in a situation may be contingent on what was observed previously and on the theory behind the research itself.

Both systematic and ethnographic observations have their supporters and have been used extensively in the literature. In spite of the difference and controversy between the two techniques, it can be argued that they are not incompatible. They both fall in the category of research called field research (Popkewitz & Tabachnick, 1981), in contrast to laboratory or controlled experimental research. As such they share an interest in the context of learning.

Spindler (1982) claimed that ethnographic methods can produce valid conclusions when there is a long term intimacy established with the site and with the people being studied. He argued that ethnographic methods can lead to quantitative research designs. He called for the use of both techniques cooperatively. Delamont and Hamilton (1984) noted the different epistemological bases underlying the two techniques, yet acknowledged an overlap between the two traditions.

This paper is concerned with the validation of a Structured Classroom Interaction Schedule (SCIS) developed to record classroom interactions between teachers and students. The instrument was constructed as part of a long term investigation into the social context of mathematics education (Atweh and Cooper, 1989, 1991, 1992a and 1992b). The investigation employed a variety of methodologies, one of which was classroom observations. This paper describes the instrument and a trial of its validity and reliability.

SCIS had the characteristics of instruments used in systematic observations (i.e. quantitative studies) although this need not be the sole use of SCIS. It was developed to allow summaries and averages to be obtained for interactions in a typical classroom.

Although several instruments which have similar purposes have been developed and validated in the literature, SCIS was created to offer three main advantages over these similar instruments. First, it was constructed with a structure to allow it to be easy to use. Other instruments had long lists of codes that placed a heavy memory load on the novice user. SCIS classifications were logically grouped into six categories. Second SCIS had a written code to be recorded for each observed behavior, rather than columns to tick. This made it possible to record the interactions on a seating plan of the whole class such that the information about individual students with whom interactions occurred was not lost but recorded beside the students' name. Third, SCIS was flexible in that, although six aspects were recorded for each interaction, researchers could easily delete or de-emphasise a number of these categories that were not useful for a particular study, or expand others that were crucial.

THE INSTRUMENT

SCIS was a classroom observation checklist that was capable of recording up to six aspects for each observed classroom interaction between teachers and students (a seventh aspect was added after this trial). With respect to each interaction, SCIS identified a **source**, either the **teacher** or a **student** and a **target**, again either the **teacher** or a **student**. SCIS also provided three categories to indicate the specificity involved in identifying the target. The teacher may ask a question naming a specific student. In this case, SCIS categorised the target as being **chosen**. On the other hand, the teacher may choose a student to answer a question from amongst those who have raised their hands when the teacher asked the question. In this case, SCIS categorised the student as a **volunteer** for the interaction. Lastly, a student may call out the answer without waiting to be recognised in any manner. Here, SCIS classified the teacher-student interaction as a **yell**.

In addition to source and target, SCIS categorised teacher-student interactions according to their **type**, **object**, **outcome** and **feedback** given. For **type**, SCIS classified each interaction as a **question**, a **statement**, **exchange** or **non-verbal**. The classifications of **question** and **statement** were straightforward. They were common forms of communication in classrooms. An **exchange** covered extended situations where a question or statement led to a reply which gave rise to at least one further question or statement. A **non-verbal** was any interaction which did not involve talking, e.g. a gesture with the hand, a shake of the head, a look of disgust.

The **object** of the interaction was the purpose of the interaction, what it was about. SCIS classified purpose as **process**, **knowledge**, **management**, **discipline** or **other**. **Knowledge** interactions related to recall of facts or direct applications or processes mastered previously. **Process** interactions related to integration of facts studied previously. **Management** interactions related to the organisation and management of learning, e.g.directions as to the setting out of work, requests to perform certain tasks. **Discipline** interactions were interactions that intend to direct the attention of the subject to the task or to discourage disruptive behaviour. Discipline and management interactions were able to be differentiated by the focus of the interaction - discipline interactions were generally aimed at the student's behaviour while management interactions were generally directed to the task. Interactions classified by SCIS as **other** referred to the many classroom interactions that did not fit under process, knowledge, management or discipline, e.g. social interactions, interactions regarding outside activities.

Outcome is included in SCIS to allow classification of the results of the interactions. This was done by categorising responses in terms of **no response, positive, negative, partly, hints** and **continued**. The classification of **no response** was straightforward. It usually occurred in situations where no response was expected/intended, as was the case for many statements or discipline interactions, or where the target ignored the interaction. SCIS deemed an outcome to be **positive** if it satisfied the source of the interaction. For example, a student answers a question from the teacher correctly, or a teacher grants the student's request for clarification. A **negative** response was deemed by SCIS to be one that was not satisfactory to the source of the interaction. A classification of **partly** was given in the case in which the outcome of an interaction was only partially satisfactory to the source. For example, a student's answer was close but not exact. SCIS provided the classification **hints** for the situation where the teachers give hints to students that help them to give a satisfactory outcome for the interaction. Such hints did not constitute the outcome of the interaction by themselves, yet they were useful information in understanding interactions. Finally, the classification **continued** was used where the classroom interaction did not occur in isolation. At times, one interaction led into another (with different source and target). In this case, the outcome of an interaction was said to have continued in another interaction. For example, a teacher asks a student a question and receives an insufficient response and then turns to another student for further clarification.

The last aspect of the interaction that may be recorded in SCIS was **feedback**. This category was for interactions that did not finish with outcome. It classified feedback in terms of **no feedback, positive feedback** or **negative feedback**, which acknowledge the degree of acceptance of the outcome, or, finally, **praise** or **rebuke**. The classification of **no feedback** was straightforward - it signified the situation where the interaction did not go past outcome. For example, a teacher asks a question of one student, the student gives the correct answer and the teacher moves on to other tasks. A **positive feedback** was a verbal or non-verbal indication by the source of the interaction that the outcome from the target is satisfactory to them. Similarly a **negative feedback** was a verbal or non-verbal indication that the outcome was not satisfactory. Some interactions concluded with **praise**, or affirmation to the target that their answer was correct, or **rebuke**, anger or disappointment about the negative outcome.

TRIAL OF THE INSTRUMENT

SCIS was trialled and validated by using it in a research project in 1991. Six researchers were involved in the trial in classroom settings. Two of the researchers were recent graduates from Diplomas in Primary and Early Childhood education and current students in Bachelor of Education degrees. Both had limited teaching and research experience. Another two researchers were experienced secondary school teachers, one of whom was a current Masters student and the other a PhD student in education. Both of these two researchers had studied subjects on research design and techniques. The fifth researcher was a research assistant at the Centre for Mathematics and Science Education who was currently enrolled in a Master's degree in Chemistry, but who, at the time of the observations, had limited experience in educational settings. The sixth researcher was one of the designers of the instrument. The project consisted of four stages.

1. *Initial Training*

The researchers attended an hour and a half training session on the purposes of the study and the use of instrument. A sample of classroom interactions was analysed by each researcher independently and the codings were compared and discrepancies discussed.

2. *Independent Trial*

Each researcher had the opportunity to try the use of the instrument in two thirty minutes classroom sessions. Year 9 mathematics classes in two city schools were used for these observations. These sessions were conducted as soon after the training sessions as possible. Each researcher discussed with the project manager the outcome of these trials and clarifications were given to some remaining ambiguities about the instrument and the required procedures.

3. *Data Collection*

For the actual collection of data in this study, researchers were first divided into two groups of three and each group assigned to a different school. Each researcher observed two thirty minute sessions of mathematics classes, one with each of the other two researchers in the school. This meant that a total of three sessions were observed in each school, a total of six thirty minute sessions in the project, each observed by two people. Also, each researcher collected data from two sessions.

During each session, the researchers sat at the back of the classroom with a stop watch, a summary of the instrument and blank pages for recording the observations. Researchers were encouraged to record the time at which interactions occurred as well as the code for the interaction. In addition, the researchers were encouraged to record as many comments about the interactions as possible in order to match the interactions for later comparisons. The recording sheets had three columns: one for recording the time of interactions; one for recording the codes for interactions; and the last for comments. Not all three columns need be filled in for each interaction.

After the class, the two researchers met and discussed their codings for the lesson. They were instructed to point out any consistencies/inconsistencies in their recordings as well as discuss any obvious problems encountered. At the end of the project, each researcher wrote a short report of his other experience and observations.

4. *Final Meeting*

A final meeting for about one hour was then conducted to finalise the administration of the project and to discuss the instrument and the observations.

RESULTS AND DISCUSSION

In general the researchers' responses to the instrument and its use were favourable. The structured nature of the instrument meant that a simple card with a listing of the letters of the different categories was sufficient for its use. All researchers reported increased confidence in the use of the instrument within the short time span of the project. The project showed that relatively short training was sufficient for the use of this instrument. However, these trials resulted in the identification of some concerns with respect to the design and use of the instrument.

First, researchers were unable, in the classroom setting, to record some interactions that occurred in private between students and students and between students and the teacher. In most classes, teachers and students were seen to engage in one-to-one interactions that could not be heard by a third party. Arguably these interactions may be of greater value to understanding the classroom than the public interactions. It was felt that these interactions needed to be classifieded in some manner so that they could be followed up and their contents investigated in interview situations after the observation. Hence, a new classification **private** was added to the aspect or category **type** to record such interactions. If the type of the interaction was recorded as private, the other aspects of the interaction were not necessarily recorded.

A second concern was expressed with respect to the difficulty found by some researchers in differentiating between the **type** classifications. One of the difficulties was with types **knowledge** and **process**. One possible understanding of process used in SCIS covered procedures to perform tasks or solve exercises. The definition went beyond the information being learnt in the class observed, the concepts, facts and algorithms; it related to applications of such knowledge. The ability to distinguish between **knowledge** and **process** types of interaction may therefore be problematic if the observer is not aware of what the students have learnt/been taught in the past, as was the situation in this project. However, this may become less of a problem in situations that involve observations of classrooms for extended periods of time, and more experienced teachers or researchers may be able to make this distinction easier than novices.

Another difficulty was encountered in distinguishing between the types **discipline** and **management**. To overcome this difficulty, the definitions of these categories have been expanded to take into account the observations in this study. However, there remain problems with classifying some interactions. Interactions often have a *face* meaning and a *deep* meaning, or what Puro and Bloome (1987) call *implicit* and *explicit* meaning. For example, the instruction "read the problem carefully" may be used as a management instruction in a sense to encourage the student to identify the essential information in the problem. However, the same instruction could be used to rebuke a student who have guessed a solution and gave a wrong answer. These types of interactions are of great interest to socio-linguistic type of research. It is possible to incorporate them in the schedule by using comments next to the interaction or by **underlining** the letter used to code this interaction. Such interactions could then be the source of explorations with the student and/or the teacher at the conclusion of the class.

A third concern arose with the ability of some of the researchers to include all aspects in recordings of all interactions. There was a large amount of information to be recorded on each interaction. This meant that some interactions were missed and some interactions were only partially recorded. As a result, the **outcome** and **feedback** aspects of the interaction were not recorded in many of the interactions recorded.

This last concern is typical of classroom observation research. Observation is always a selective activity and observers are rarely able to observe everything in a classroom. The purposes of the study and the theory behind the research usually dictate what is of value to concentrate upon. Novice researchers, or research assistants who are not very knowledgeable with the aims of the study and its background, may attempt to observe every single interaction as well as to record all aspects for each interaction. Experienced researchers, on the other hand, tend to concentrate only on what they feel to be significant interactions for their study. As argued above, SCIS is flexible enough so that aspects that are not of concern to the individual study may be omitted while others that are central my be elaborated. For example, in investigations that assess the effect of type and degree of feedback on student self concept, it may not be as important to record the object of interaction. For these studies additional categories may be needed to record non-verbal vs verbal feedback (Bull, 1983). On the other hand research that aims at comparing the use of open ended questions in mathematics and social sciences may not be interested in recording the feedback part of the schedule. It is also possible to be selective in the choice of interactions. Time sampling is a technique that has been used in time-on-task investigations in the past twenty years.

CONCLUSIONS

Several instruments similar to the Structured Classroom Interactions Schedule have been developed and used in classroom research. Many of these instruments have been developed within a logical positivist/empiricist tradition. According to this view, the phenomenon observed has its own objective reality. Hence, for these instruments, the establishment of validity and reliability was a major criterion for establishing their usefulness. Invariably, such validity and reliability were established using sound statistical tools on data from well designed and controlled experiments (Medley and Mitzel, 1963).

In this study, no attempts to statistically test validity and reliability have been made. The concept of validity followed in this project has been based on the definition presented by Cronbach (1971) who stated that "to validate is to investigate" (p. 443). Interpretation of results of measurement has been interpreted as the product of negotiation within a community of experts rather than the statistical proofs of experiments. Evaluation has followed the approach of Cherryholmes (1988) in seeing the process of validation as a socially critical activity where "the tacit knowledge of subjects and social scientific knowledge of researchers are resources for discourse on construct validation" (p. 450).

This investigation has attempted to point out any problems with the use of SCIS. These problems have been identified and incorporated into the current form of the instrument. This investigation has shown that the instrument can easily be used for classroom research. The required time for training of research assistants is not excessive. However, more experienced researchers, and assistants who are very familiar with the theoretical foundations and aims of the research project, would make more valid use of the instrument. Further, this instrument's structure implies that it is easy to use and at the same time has flexibility that enables it to be modified to suit a variety of studies.

SCIS has a variety of uses. It may be used as a sole instrument in systematic observation types of research. In the Social Context project, for which this instrument was initially developed, this instrument was used within a study that was mainly based on ethnographic methodology. In that context the instrument was used to identify students within the observed class that had interesting patterns of communication with the teacher, e.g. above or below average number of interactions, or interactions restricted in some manner. It was also used as a basis for in-depth interviews with the teacher about their perceptions of students' abilities and attitudes.

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A PHILOSOPHICAL JUSTIFICATION FOR ETHNOMATHEMATICS AND SOME IMPLICATIONS FOR EDUCATION

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This paper attempts a philosophical basis for ethnomathematics by explaining how it develops, how it is legitimised, and how it is integrated within other mathematical cultures. Ethnomathematics requires a relativistic philosophy. Questioning universal notions in mathematics can be traced back to the French philosopher Gaston Bachelard. He describes an historically relative notion of objectivity which gives rise to changing conceptions of mathematical objects and of rationality. This analysis provides a parallel which is used to explain cultural relativity, not just in mathematical practices, but also for rational thought. The history of navigation is used to illustrate such relativity. The possibility that ethnomathematics exists at the level of rational thought raises important questions in education. Questions about the practice of mathematics education, and about its socio-political function, are discussed.

INTRODUCTION

Ethnomathematics, the study of cultural relativity in mathematics, has been the subject of considerable literature in the last decade. Culturally distinct mathematical practices have been documented (e.g. Cooke, 1990, Gerdes, 1988, Zaslavsky, 1973); the politics of mathematical colonisation have been analysed (Bishop, 1990, Barton, 1991); and the structure of ethnomathematics and its place in mathematics education is the subject of a book (Bishop, 1988).

This paper attempts to establish a philosophical basis for ethnomathematics by explaining how ethnomathematics develops, how it is legitimised, and how it is integrated within other mathematical cultures. In particular we want to know under what conditions it is possible to accept the simultaneous existence of different conceptions of mathematics, particularly if they contradict one another. In addition, we must explain how one mathematics culture has come to be predominant, and, apparently, so highly developed compared with other mathematics cultures.

Ethnomathematics requires a relativistic philosophy. Writers such as Wittgenstein (1956) and Lakatos (1976) have made it possible to question universal notions in mathematics. However, the origins of such discourse can be traced back to the French philosopher Gaston Bachelard writing in the 1930's (Tiles, 1972). He describes an historically relative notion of objectivity which gives rise to changing conceptions of mathematical objects and of rationality. This analysis provides a parallel which can help explain cultural relativity, not just in mathematical practices, but also for rational thought.

The history of navigation, and the mathematics used in navigational systems, are used to illustrate such relativity. If the history of navigation can be rewritten from the perspective of mathematical relativity, and if this history can account for the way we view navigation systems of other cultures as well as the nature of contemporary navigation, then a possible alternative has been successfully argued.

The possibility that ethnomathematics exists at the level of rational thought raises important questions in education. In the final section the practice of mathematics education, particularly its assumptions of 'one mathematics', is questioned. Not only is such a curriculum questioned as to its honesty, but also its social and political effect on the people of various cultures is examined.

For the purposes of this paper, "ethnomathematics" will be assumed to have a broad meaning. As well as describing mathematical symbolisations, applications and practices which are culturally distinct, it will be assumed to include mathematical concepts, systems, modes of thinking, and meta-mathematics, e.g. what counts as a proof, beliefs about how mathematics relates to the world, and values implicit in mathematics. If the philosophical questions can be answered for this large definition, then they will also apply to any smaller definition.

PHILOSOPHICAL BASIS FOR ETHNOMATHEMATICS

Bachelard's Philosophy

Bachelard's key idea is that objectivity is an ideal rather than a reality. At any time we may think that we see clearly how things are, or that we know clearly how to make judgements, or that we know how to discover the truth, or that we undestand what makes a proof. However these ideas change over time, i.e. objectivity is illusory. With respect to mathematics, Bachelard believed that conceptions of mathematics at different times depend on changing notions of rationality, each successive change being regarded as being more objective than the last. Thus there is a progression towards a better, and then a still better, understanding of all the things which must be taken into account to get an objective view.

Thus there are many different historical standpoints from which to view mathematics, each of which are correct *at that time* and each of which explains previous views. Each such view gains its (apparent) objectivity because of its intersubjectivity (the wide agreement amongst mathematicians about the view), its rationality (the structure of causal relations within this view), and because it is seen to arise from previous views and encompass them.

This is an historical explanation. It accounts for the development of mathematics over time, for the changing, creative nature of mathematical notions while retaining the objectivity required of the discipline. We can also see this explanation as sequential: it describes mathematical development as proceeding from one step to the next, each step being an outcome of the previous one and each step being more objective than the last.

Another important idea in Bachelard's philosophy is that of the knowing subject. In order to gain objectivity, or to claim rationality, the person must reason in accordance with logical laws, but also they must be aware of the demand for rational thought. They must realise that they are thinking rationally. Furthermore, Bachelard realised that this created an inevitable link with the culture of the subject: "The detailed features of a subjects' rational cognitive capacities cannot be disengaged from their conception of those capacities, and thus conception is shaped by and revealed in the rational discourse of their culture."(Tiles, 1972, p. 41) Bachelard does not, however, propose a relative philosophy. Although his mathematicians are bound by conceptions of rationality which are revealed in their culture, they can overcome them because they are aware of the 'objective' requirements for rational thought.

Bachelard's view of the vital place of history in the present conception of science can also be echoed for mathematics. He describes a 'recurrent history' of mathematics which is continually retold, continually evaluated and thereby contributes to our understanding of present thought.

Application to Ethnomathematics

Ethnomathematics requires a non-temporal and non-sequential explanation. Culturally different conceptions of mathematics exist at the same time and do not necessarily follow on from one another. (There is a sense of ethnomathematics which describes historical developments, tracing the origins of mathematical ideas from the cultural communities in which they arose. This should perhaps be called the cultural history of mathematics).

It is important to explain how they can co-exist in cognisance of each other. That is, it is not enough that there is a basis on which differing views may exist in far-flung corners of the earth without knowledge of each other. It is also necessary to establish a basis on which holders of these views may know about the other views and accept them as right in some rational sense.

Mathematics carries only internal universality: it is thought by its practitioners to be universal, but this does not prove that it is so. Having arrived at a conception of mathematics where universality is debunked temporally, universality in any sense is brought into doubt. It is possible, and seems likely, that different groups will arrive at different conceptions of number, of shape, of relationships, of proof, of rationality, of mathematics. Bishop's description of 'environmental activities which lead to mathematics' explains this process(..). What must be explained is the relationship between these conceptions: what happens when someone holding one conception becomes aware of another ?

The echo of Bachelard's historical vitality now becomes important. Cultural vitality performs a similar, and parallel, function. As well as a recurrent history, there is a recurrent cultural dynamic which acts in the present in a critical fashion. In this dynamic mathematicians will evaluate their own practices and conceptions in the light of other practices and conceptions, will modify, reinterpret, discard or adopt particular practices. More importantly, they will retain the knowledge of how and why this was done as part of their mathematical understanding.

There may be a consciousness of change, of what motivated particular thoughts, new ideas and so on, but not necessarily a consciousness that this is a culturally relative process. This is possible because of the ethnocentricity of any particular view. It is still the case that most mathematicians regard their subject as universal. From their point of view this is true, for, if a discontinuity arises (be it temporal or cultural) then there needs to be a cognitive shift to accommodate the clash of domains. When this is achieved the sense of universality returns. It is only when this process is reflected upon that we see the relativity of the past situation.

Culturally Different Mathematics

What, then, is the status of a culturally different mathematics. For example, is there such a thing as Maori mathematics? Again a parallel with the historical case is helpful. Particular conceptions of mathematics are temporary: they begin and end although they live on in the critical role played by historical definition of present conceptions. The end occurs when a new conception encompasses the past ones and resolves the conflict of domains which has arisen. Ethnomathematical conceptions are similar. They are temporary in that they end when a new conception arises out of two conflicting ones, but they live on in the critical role now played by that conflict on the present conception. In the case of ethnomathematics, however, there are two simultaneous conceptions held by distinct groups. Thus the possibility exists for one group to recognise and accommodate the conflict while the other does not. It is this possibility which gives rise to judgements about another group's mathematics being inferior (see below) and also explains the widespread belief in *one* mathematics.

Thus Maori mathematics either did exist (and is now part of a wider conception of mathematics adopted by one or both of Maori and European mathematicians) or currently exists but is recognisable only to Maori mathematicians. The term 'Maori mathematician' does not imply that there is, in Maori culture, a body of knowledge recognisably equivalent to European mathematics and conducted by people who hold a parallel relation to their cultural community as European mathematicians do in theirs. It refers to those people in the Maori culture who would identify themselves as having some equivalent types of understanding, or those who perform functions recognisably similar to the functions of mathematicians in European cultures. It is likely that non-European cultures will have different categories of discipline which cannot be mapped onto European categories.

It is also important to draw a distinction between mathematical conceptions and mathematical practices. The interplay between practices, either historically or culturally, is easily seen and many examples can be quoted. Newly discovered practices are very quickly accommodated within existing mathematical conceptions and can usually be transported across historical or cultural boundaries without much difficulty. This is because mathematical conceptions have very broad areas of applicability, are very generalised, and can therefore explain a wide range of activities. However the interplay between conceptions is not so visible, nor is it so easy. This is because the resolution of conflicting conceptions gets played out through the mutual accommodation of many practices. Mathematical practices embody implicitly the mathematical conceptions. It takes the interplay of many practices to carry the full impact of the conceptions behind them. This explains why mathematical conceptions of minor cultures become subsumed or colonised. The mathematical conception with the wider range of applicability will accommodate different practices more readily than the minority conception will accommodate its opposing practices. What is more, in accommodating the practices of a minor culture, the dominant culture will 'rub out' the conceptions behind those practices. The imported practices will become examples of the conceptions of the dominant culture and are often therefore 'devalued' in that their original richness and potential is eliminated.

The sense of one mathematics being 'better', 'more true' than another rests on the status of the understanding. If ethnomathematical community A can see and understand the views of ethnomathematical community B without the reverse occurring, then community A will feel that their mathematics is more advanced. This will be true in the sense that they have achieved a meta-level of development whereby both ethnomathematics' can be understood. It does not affect the quality of the mathematics, it concerns only the level of understanding, and may be equalised at any time if/when community B reach a similar understanding.

However, if there is a time difference in this awareness by the two communities, the enduring feeling is that the second has been subsumed/colonised by the first. What is more, if one community is much larger than the other it is therefore more likely to quickly accommodate many smaller communities. The result is an intellectual 'black hole' into which all other practices fall and are accommodated and from within which it is impossible to see the existence of independent conceptions. This is exactly what has happened in mathematics.

THE HISTORY OF NAVIGATION

The Received History

The development of mathematical models of navigation has been one of increasing differentiation and decreasing integration. Where once the weather, directions, sea conditions and coast were an integrated system of knowledge, there is now a collection of highly developed sciences: meteorology, positional navigation, oceanography and cartography. This is reflected in the etymology of the word 'navigation' (through Latin, past participle of navigare: to drive a ship). Compare this with the current meaning which refers to positional navigation alone.

The accepted history of navigation acknowledges the many skills of ships' captains and their "men of knowledge" sailing in most of the world's oceans up to 4000 years ago. The history records the first use of maps (probably the Greeks in the first century BC although sailing directions (periplus) were common well before then); the development of instruments for observing the elevation of sun and stars (and hence determine latitude); tide charts; the development of lode-stones and compasses (although lode-stones were known up to two centuries BC the compass is not recorded as a sailing instrument before the Crusades); the search for accurate marine clocks (including a large sum of money for the first one of sufficient accuracy and the establishment of the Board of Longitiude in Britain); through to the use of radar and satellite systems. It is now possible for blue-water yachts to use a hand-held, battery powered receiver which gives them an immediate digital readout of their latitude and longitude to an accuracy of a few metres.

At the conclusion of Per Collinder's history (1954), there is a flight of imagination in which a modern ocean-going yacht takes on board an old Arab sea-captain who navigated during the Great Flood. Modern navigational aids are explained to the old man in terms he understands, i.e. by relating them to techniques and equipment which he would have used. The old man's amazement is recounted, but, he concludes "My boat is smaller than yours and does not need all these wonders." The implication is that old sailors did not need the sophistication of modern navigational aids, that their methods were sufficient unto their needs.

But perhaps the old man would have been amazed for other reasons? Perhaps he would wonder at the effect of these new technologies on the art of sea-faring. Perhaps he would have seen those things which had been lost from that art because particular instruments were developed? Perhaps he would have had a wish-list of aids which is quite different from the aids which have, in the event, developed?

These different views of the parable illustrate the different perspective of the relativist philosophy outlined above. The ethnocentric view of the wonders of modern navigation systems is countered by a recognition that the development may have been different, indeed may yet be different in the future.

The sea-farers of old (and of different cultures) navigated using different assumptions. However, as equipment developed to better locate oneself, the early navigation methods became judged using the criteria of accuracy of position, were found wanting and were consequently discarded. It is instructive to ask whether modern techniques (e.g. satellite systems) would be useful if judged on some of these different assumptions? For example, Micronesian sea-paths do not always cover the same ground: they depend on weather, seasons and sea conditions. Thus knowing exact position on a imaginary grid is not useful information. In other words, the development of mathematical navigation proceeded along one particular line out of many possible lines. Some of those discarded methods may well have developed into modern mathematical systems which would have been just as effective in terms of "ship-driving".

The Changing Concept of Navigation

In the history of navigation there have been changes in the idea of what it is to navigate. Three key changes are outlined.

The change mentioned above, that from an integrated body of knowledge and skills to more specialised collections of knowledge and technology (but concentrating on the positional knowledge afforded by astronomical observations), has been a gradual development. The old art of haven-finding has transformed into the technology of nautical astronomy. For early navigators the well-known winds and sailing destinations were used to name directions, stars were used as path-finders rather than position fixers, the plumb-line was used to determine depth but also had tallow affixed to show what the sea floor was like.

The criteria of a good navigation system have changed. Proximity (nearness to land), familiarity (nearness to known conditions), or reckoning (nearness to known paths) were all valid measures of navigation. They have all been replaced by location (nearness to known position). A modern ship's captain reported to me that the mathematics of positional navigation (i.e. the calculations involved) have never been very difficult. The main problem has been gathering accurate data in difficult sea conditions. But this only applies if you are interested in position fixes. If, on the other hand, you are interested in reading the sea to determine your path, or determining proximity to land, then sight is not always needed. Kyselka (1987) and Thomas (1987) report

Micronesian navigators using the feel of swells to maintain a correct path. Collinder (1954) notes the use by Spanish fishermen of the taste of mud attached to plumb lines to determine proximity to land. What would have happened if the technological development had concentrated on these aspects of navigation?

It is true that position lines have always played a role in navigation. Early sailors in most regions are recorded as using sightings on promontories to determine their position, the direction to begin voyages or the entrance to a channel. Such place-lines became augmented by bearings (where the compass rose replaced promontories) and later altitude lines (latitudes) where the sun, moon or stars became the reference points. Clearly these habits led to the development of an intersecting position-line for latitude (i.e. longitude). What limiting effect did this have on the use of position lines? Promontories (or lights) are still best for entering harbours.

This change in the meaning of navigation can be expressed as a change from other-centredness (where is the destination) to self-centredness (where am I).

A second change is that from an oral culture, where navigational skills are memorised and passed from person to person, to a written culture where knowledge is written in books, maps and tables, and is passed on through the media independently of those who wrote it.

Collinder (1954, p. 25) refers to the secrecy of navigational knowledge in early times, and attributes it to the need to have an advantage over enemies or competitors. But in an oral culture the greatest danger is not from enemies, but from poor memory or inaccurate transmission of knowledge. If everyone has access to knowledge then the chances of inaccuracy are vastly increased. If it is retained by a trusted few, and only passed on under stringent conditions of testing and confirmation, then accuracy is preserved.

The result is a close association between the knower and the knowledge. Such an association gets progressively distanced as the ability to draw and write improves. The need of a ship's master changes from "I need a navigator" to "I need navigation". In those sea-faring cultures where oral systems remain, the navigator is the centre of social organisation. Kyselka's stay in Micronesia records such a society in detail (Kyselka, 1987).

A third change is in the 'nature' of ships and shipping. Collinder claims that navigation comes into being with the ship. Because kayaks, canoes or coracles can go as close inshore as they like and can be steered off, and because they are not used to make journeys of more than a day's length, Collinder infers that they do not need navigational knowledge. Larger ships require such knowledge because they are often out of sight of land and cannot always be recovered if they get too close inshore.

In fact kayaks and canoes often make ocean voyages and sail out of sight of land. Just because large ship navigation has dominated development, it does not mean that small ship navigation is valueless. This is an example of the colonisation of an ethnomathematical system as described above.

Another difference is that trade shipping requires particular harbours as destinations and exact berthing facilities. Polynesian navigators seek a whole island and any beach provides berthing.

Navigation Systems

David Turnbull (1991, p. 23) asks the question "What is a navigation system"? when considering Micronesian navigation. Some of the characteristics are: it should be symbolic (and therefore transmittable); it should be manipulable (and therefore adaptable), it should be generalised (and therefore non-localised), and it should be open (and therefore innovating). Gladwin(1970) described the system of navigation on Puluwat atoll. His (and others') descriptions were further analysed by Hutchins (1983) in a way which made it clear that these characteristics are met. To quote Hutchins "The Micronesian technique is elegant and effective. It is organised in a way that allows the navigator to solve in his head, problems that a Western navigator would not attempt without substantial technological support."(ibid, p. 223)

Another question is: on what basis can we judge or compare navigation systems? By considering historical methods and navigation systems from other cultures (e.g. Micronesia) it is possible to come up with several criteria: accuracy, consistency and explainability come to mind. However it is important to see that even these criteria are relative. For example accuracy depends on what you are trying to find. Positional accuracy is different from path accuracy or proximal accuracy.

It is possible to step outside the ethnocentricity of modern navigation both at the level of practice and on a meta-level where different systems can be judged according to their own criteria and can be seen to be valid simultaneously. This raises the question of what other mathematical systems can also be simultaneously acknowledged? Can rationality itself depend on culture?

EDUCATIONAL IMPLICATIONS

Curricula

A relativistic basis for mathematics raises several questions for the way in which mathematics is presented and taught. The 'one mathematics' curricula common in our schools must be seriously questioned because it limits the possibilities of mathematics, it does not reflect our present understanding of mathematical development, and it curtails the debate of the nature of mathematics.

Bachelard's notion of history being critically active in the present, and the parallel notion of a cultural component to our present understanding of our subject, provide important tools for the classroom. By bringing out the interaction between culturally different systems (and historically different ones) our understanding of present knowledge becomes richer.

There is a pedagogical question which also requires consideration. It is necessary for good education to acknowledge the prior learning and cognitive framework of students. If mathematical frameworks are culturally dependent, then mathematics education must start with the framework of the student and use it to reach a higher level framework which incorporates both that of the student and that of the teacher.

Social-Political Considerations

If there is a dominant mathematical culture which colonises (and devalues) alternative mathematical cultures, then good education requires that such notions are at least made explicit, and, preferably, are countered. The difficulty is that this must be done while accepting that the dominant mathematics is that which opens doors in the education system and the labour force. Furthermore, it is not just the content of the system which is relative. Modern mathematics carries a set of techno-scientific values which must also be questioned.

If the colonisation process is as described above then part of an ethnomathematical education must be to convince ethnomathematical communities that their contribution to mathematics is on equal footing. This is done by gaining a meta-awareness of the process, not by evaluating the nature of their input. It is the potential content which is of value.

The identification of this potential content is the biggest challenge to mathematics educators concerned with ethnomathematics. It will not be easy to step outside the mathematical mainstream and 'do' mathematics using other rational systems, other cultural values and other symbol systems. Perhaps the best we can do is to open up resource opportunities for people from other cultures to do this work, educate ourselves so that we recognise it when we see it, and appropriately value the work when it appears?

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GAINING INSIGHTS BY COMPARING PROCESSES

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Starting with apparently unrelated ideas and making connections between them by looking for similarities and differences is one way of gaining insights. In this paper some ideas that are compared are:

mathematical problem solving,
professional development,
research, and
assessment.

The cycles that can be used to summarise these processes are compared, some of the steps of the cycles that are neglected are discussed, and examples of insights that affect assessment are considered as examples of those that might be gained by comparing the processes.

INTRODUCTION

I subscribe to a constructivist view of learning and realize from this theory and from my own experience that topics are often compartmentalized in our minds rather than integrated. As a general aim I would want teachers and students to have a more integrated view of related topics and this paper is one attempt to try to integrate topics. From a mathematics education viewpoint I would want both teachers and learners to make sense of their world and enrich their view of it by seeing it from a mathematical perspective.

At present I am involved in a range of activities in mathematics education including research, professional development and assessment. In this situation I was reading a book by Eisner (1985) on evaluation and this triggered some connections between assessment and research. As I played with these ideas more links quickly emerged both between these two processes and problem solving and professional development. At about the same time I had read an article about the power of metaphor (Knight, in press) and was thinking that perhaps these four processes had a metaphorical relationship. Whether the four were related as metaphors or whether they were aspects of a general cycle was not critical and although I finally did conclude they were examples of the same general process, the insights that emerged were the important facts.

Another reason that emerged for comparing these processes is to attempt to break down barriers for teachers involved in research by making connections between their research role and aspects of their teaching. My hope here is to make the research work seem less intimidating to them.

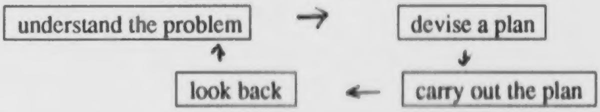
A further intention is to show traditional teachers that when they asked students to "problem-solve" they were asking them to be involved in the sort of task that they as teachers were very familiar with, rather than something that seemed very new. The obvious difference in context remains but I think that looking at similarities would reduce teacher anxieties.

FOUR COMMON PROCESS CYCLES

The four processes I compared were problem solving, professional development, research and assessment. At the start I looked at each separately and jotted down the steps, these are briefly outlined here.

1. Problem-Solving Cycle

This summary of mathematical problem solving is linked to Polya's work (1957) but other problem solving cycles are similar.



- Understand the problem: What is the unknown? What are the data? What is the condition?

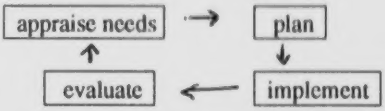
- Devise a plan: Have you seen it before? Do you know a related problem? Look at the unknown.

- Carry out the plan: Carry out the plan and check each step.

- Look back: Can you check the result? Can you check the argument? Can you derive the result differently? If no solution is obtained then start the cycle again and consider alternative procedures.

2. Professional Development Cycle

Professional development (inservice education, staff development, school development) is an ongoing part of school life and has often been regarded as having four parts that can be summarised by this cycle:



- Appraise needs: What are the professional development needs of the school, the department, and the individual staff members? What is the purpose for such development? Do the purposes fit within the school's policies and guidelines?

- Plan: How might these needs best be met?, What models are available? What opportunities exist? What needs to be organised?

- Implement: Implement the professional development activities.

- Evaluate: Were the activities successful in terms of the desired changes? What more needs to be done?

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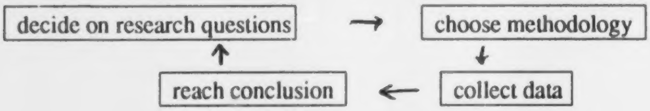
3. Research Cycle

Research can take many forms. The stages of traditional positivist research can be represented on a similar cycle although deciding on the research questions often involves a considerable exploratory stage, these stages are given as:



- Decide on research questions: What are the research questions or the research area to be explored? What are the limitations and context of the proposed programme? What does the literature say?

- Choose methodology: What research methodology would be appropriate? What is the time scale for the project?

- Collect data: Gather the information.

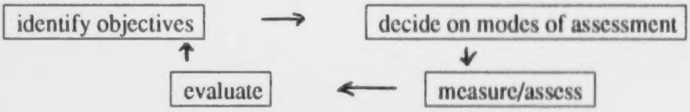
- Reach conclusion: Summarise the results, draw conclusions from them, how can the findings be disseminated, what further research needs to be done in this area?

Often the research starts in different ways. For example, experiences might lead to conclusions which then might lead to a decision to begin some more formal research. Action research is one particular form of research where there is more stress on the cyclic nature of the process but the stages are similar for most research programmes. In the case of critical inquiry the cycle — reflect, plan, act, observe — is virtually the same (Kemmis and McTaggart, 1988). Interestingly action research is viewed by some (Cohen & Manion, 1989) as a form of policy implementation and teacher development as much as it is research.

Another example related to research is the research-development process where the research cycle is followed by a similar development cycle.

4. Assessment Cycle

The process of assessment, in a similar way can be summarised with the following cycle:



- Identify objectives: What is to be assessed? What is the purpose of the assessment? Is it for diagnostic, formative or summative purposes? How will the information be used?

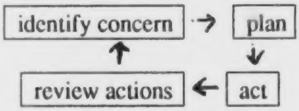
- Decide on modes of assessment: What assessment modes and tasks might be used to get the information? Design appropriate activities.

- Measure/assess: Administer the assessment activities.

- Evaluate: Make evaluative judgements from the assessment data. Make decisions on the action to be taken with respect to the unit of work, the programme as a whole, the learner, or the reporting procedure.

A GENERAL CYCLE

The four cycles described above have many similarities and some differences, they can be summarised with one cycle:



Being cyclic, any of the stages might be the starting point although traditionally in mathematics we start with a problem.

- Identify concern : Both the content and context of the concern are considered with one important aspects of context being reason or purpose.

- Plan: A range of models and strategies need to be considered in terms of whether they are likely to be useful in the particular situation.

- Act: The action stage is often the comparatively easy stage if good planning has preceded it.

- Review actions: This stage is concerned with evaluation, possibly reporting, and making decisions about the next steps.

A general cycle like this fits a number of different situations. For example, if one is trying to negotiate a curriculum (EQUALS, 1989) then one works through four steps – decide on aims, agree on tasks, produce, assess – in a similar cyclic process. Other tasks it can be used for are teaching at the macro level (looking at a year's work), teaching at the micro level (looking at an individual lesson), resource development, and curriculum development.

The cycle could be regarded as the basis of a life-skills programme if we intend to encourage learners to approach challenges systematically.

SOME INSIGHTS AFFECTING ASSESSMENT

By comparing any two of the processes and looking for similarities and differences one might expect to see things in different ways and gain new understandings. I have selected some examples of insights that might be gained when one looks at each process alongside assessment. I chose this one as I was currently working in this area but I am sure that similar insights would be gained by comparing the processes in other combinations.

In problem solving we know that numerous strategies at different levels of sophistication can be used to solve a problem. Calvin Long (1991) presented a good example of this: A farmer knows he has chickens and pigs in the farmyard. He knows that there are 100 animals and 224 legs. How many chickens and how many pigs does he have? This problem can be solved by guess and check, by looking for a pattern, by drawing a diagram (imagine 224 legs pictured as 112 pairs, and then link pairs until one gets only 100 animals), by using a simple equation, by using simultaneous equations, or by more logical and creative methods (such as asking all the animals to stand on 2 legs and raise their other two, 100 animals implies 200 legs on the ground, 24 in the air, therefore 12 pigs). With assessment many teachers often use a limited repertoire of traditional methods rather than considering a whole range of interesting and creative possibilities.

Professional development does not assume that all people have the same needs and starts with an appraisal of needs to decide whether the focus should be beginning teachers, heads of department, school improvement, or the implementation of a new curriculum or whatever. Usually the decision is made that a number of alternatives need to be offered to cater for the range of needs. I see little evidence of the acceptance that students have different assessment needs. We remember Cockcroft's seven-year ability range (DES, 1982) when it suits us, we know (Department of Education, 1988) that essay questions probably favour girls, multichoice seems to favour boys, some groups (such as Polynesians) prefer cooperative activities rather than individualistic tasks, and some students are not good readers or writers, yet the choice of assessment activities rarely takes this range of needs into consideration.

Research methods can be qualitative or quantitative and within each of these paradigms a range of methods exist but in particular in the reporting phase with quantitative results we usually use numbers (statistics) while with qualitative results we are more likely to use the results to tell a story. Even with quantitative work we usually write a report to interpret the statistical data into the context of the original problem and we see our numbers as a tool for the work rather than as the product. In assessment in mathematics we seem to get bogged down with marks - as if 47% meant anything more accurate than somewhere between 40% and 55% or a C, when in fact a sentence or two of constructive remarks might be a more suitable way of providing feedback.

NEGLECTED AREAS

Other insights this comparison gives us concerns the identification of areas of neglect.

With the problem-solving cycle we know that traditionally the emphasis has been given to the "act" step - implementation of a given strategy to a particular problem (drill & practice) while other steps have been neglected. Curriculum developers stressed the need to build up the "planning" stage in terms of having a range of strategies and systematically considering which are appropriate, as well as broadening the range of concerns by presenting challenges that relate to the real world rather than merely textbook type exercises. The review stage becomes more important when the question is broadened and when multiple solutions begin to emerge.

With professional development (Begg, 1992) often little time is given to identifying the concern (needs appraisal? the content and the context of the development? who is it for? the expected outcomes?) Similarly planning often makes assumptions about a model for development instead of realizing that different ones are useful for different reasons and that by considering the factors that make professional development more effective, models may be modified to better suit a situation. Instead, a concern is voiced and a one-day course is provided in spite of our knowing that it is unlikely to cause any change in teacher behaviour and very little evaluation occurs, and when it does little use is made of the information that is collected.

Some years ago most academic research, in particular for higher degrees, was done in the quantitative paradigm and it has taken a considerable time for the consideration of the models to be expanded to include the qualitative methods where the review stage is told as a story rather than as the summary of statistics. One neglected aspect of the review stage with educational research is the reporting/dissemination to the important teacher and development groups rather than limiting this to research reports and journals that are likely only to be read by other researchers. Another area of neglect is when decisions are made about methodology and techniques of EDA (exploratory data analysis) are not considered when statistics are being used. EDA methods seem to fit both with exploratory research and with research that combines aspects of qualitative and quantitative approaches.

In my experience with assessment, stage three (implementation) has been the main focus. Little consideration is given to the purpose for the assessment and whether the results are going to be used in a useful way. Most mathematics teachers use a limited number of assessment activities, and little is done in reporting conclusions in a constructive way to students. Perhaps if teachers look at the whole cycle a more balanced assessment might result.

WHO IS INVOLVED

In professional development the following questions are sometimes asked:

Who is deciding what professional development is needed?
Who is going to provide the professional development?
Who is expected to change?
Who will provide on-going support?

In research similar questions are asked:

Who is doing the research?
Who is involved in the researched?
Who is it for?
Who owns the data?

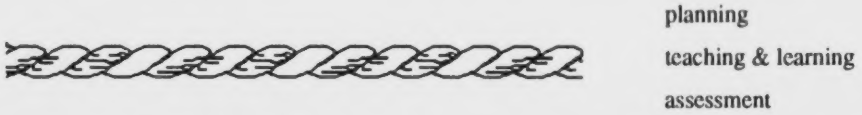
The answers to these "who" questions are partly answered when the need is stressed for teacher involvement in and ownership of all aspects of professional development (Begg, 1992) and similarly when collaboration is discussed in relation to research, particularly action research (Cohen & Manion, 1989). Certainly in both teacher development and research the questions are usually considered.

What about in problem solving? We often talk about the real world (real to whom?) where much problem solving is a group activity. Many problem-solving activities from groups such as EQUALS stress this need for cooperation. Sometimes we give the learners some say regarding the problems they choose, the people they work with, and the methods they use, but as teachers we usually put limits for learner autonomy. It seems that the "who" questions are not given enough consideration.

From a constructivist viewpoint considerable work needs to be done regarding assessment and the roles of assessment may well be different with the different teaching strategies that might evolve. However we would expect assessment to still have a part in our programmes. Educationalists like Carl Rogers (1969) would suggest with experiential learning that the most important assessment is the informal self assessment that the learner does and with some teaching models based on constructivist theory I think we need to consider this idea. However I am not aware of much learner involvement regarding assessment. I would hope that learners are involved in both asking and answering these questions: whether assessment should occur? whether it should be formal or informal? whether it should be teacher, self, or group assessment? whether the results should be reported? and whether the results reflect the quality of the learning or the teaching? In the same way we have very little learner input when we evaluate our programmes? Perhaps we need to answer the "who" questions in assessment with more collaborative approaches too.

ALTERNATIVE MODELS

An alternative model for assessment was offered by Bill Barton at a workshop I recently attended (Barton, 1992) when the discussion on assessment was trying to break the emphasis on summative assessment and focus more on diagnostic and formative. This model seems to fit better with a constructivist view, it puts assessment (objectives, modes, assessment, evaluation) in a closer relationship to planning (curriculum, school scheme and resources), teaching and learning and these can be represented by the strands of a rope.



Applying this model to problem solving we have the strands of problem, strategies, and content links and the more able student often jumps steps because when these related aspects are viewed holistically, a way through the situation often seems obvious.

In professional development too the components of the cyclic process can be merged by the teacher who is constantly considering new ideas, integrating them into a personal framework and trialling them in class. This is more obvious with changes in content rather than changes in teaching style and is probably to do with the greater confidence of teachers in relation to the content of their teaching.

In research, in particular with the exploratory and qualitative research paradigm, a similar merging and interaction exists between the area of study, the methods and the data (and the resulting action in the case of action research or emancipatory research).

Using this rope model the insight for me is the "connectedness" which is something that with a reductionist view from our behavioural upbringing we seem to have lost .

CONCLUSIONS

The similarities between these activities are not surprising as all of them have aspects of a "problem to be solved", consequently the problem-solving cyclic model based (not necessarily mathematical problem solving) for all these processes seems appropriate for the novice and the connected model (the rope) for the expert.

I know that I have regarded these activities as being different and have obtained a richer view by looking at them together both in terms of the components with the cyclic model and holistically with the rope model. Perhaps students need help in making these connections and we need to stress the similarities rather than the differences. The benefits gained by reflection on them should be challenging but not threatening as each person will be integrating their own existing ideas with some less familiar ones.

I am not implying that a general theory exists for these activities as many differences exist between them and a study of the contrasts will similarly provide insights to our students. Perhaps the message is summed up in the familiar assessment task: "Compare and contrast ..."

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STUDENTS' MANIPULATION OF ALGEBRA SYMBOLS AND THEIR AWARENESS OF THE CORRESPONDING CONCEPTUAL RELATIONSHIPS

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INTRODUCTION

> ... substantial evidence exists to indicate that the learning of algebra is addressed by many students as a problem of learning to manipulate symbols in accordance with certain transformation rules (i.e. syntactically) without reference to the meaning of the expressions or transformations (i.e. the semantics). This is of course not surprising, since most algebra syllabuses in the past have paid considerable attention to the syntactic aspects of algebra, precisely because of the central role that symbolic representation plays in algebra work, because power over such representation is crucial to successful performance in algebra, and because the symbolism is both new to the students and an obvious feature of this area of study. (Booth, 1989, p. 58)

Large scale studies in the early 1980s by the CSMS (Concepts in Secondary Maths and Science) and SESM (Strategies and Errors in Secondary Mathematics) project teams in the UK were carried out on thirteen to fifteen year olds in order to ascertain student understanding of the basic concepts and procedures of elementary algebra (Hart, 1981; Booth, 1982). The results confirmed the existence of many difficulties experienced by students in this age range. As a result of these studies and others conducted in the Algebra Learning Project at the University of Georgia (Wagner et al., 1984), teachers now have access to a variety of findings on the diagnosis of learners' errors in elementary algebra. Despite this accessibility, however, many teachers remain as unaware of the difficulties faced by their students as they did almost twenty years ago:

> Too often, when teachers find errors in a child's work, they mark the example wrong, assume that the child did not master the basic facts, and prescribe further drill. Careful analysis of errors through observation and interviews with individual children is essential. (Pincus, 1975, p. 581)

A survey of the teaching strategies employed by over 800 teachers of elementary algebra revealed teaching practices almost totally geared towards rote learning of symbol manipulation (Oliver, 1984). At least two models have been designed in an effort to improve on this situation by avoiding errors pupils make and misconceptions they develop before or during the traditional teaching of elementary algebra (Booth, 1984, Oliver, 1984) and both models have been relatively successful in the short term.

Research has also shown that when commencing algebra instruction, students often possess well established beliefs, some of which are invariably incorrect (Perso, 1992; Claxton, 1987; Osborne, 1984). These conceptual errors are more often than not responsible for students' mistakes (Balacheff, 1984; Bell, 1984) and it is clear that teachers need to be aware of these errors if they are to be effective in teaching. Onslow's view was that:

> Unless teachers are alert to children's errors and misconceptions, situations for overcoming them are easily overlooked. (1986, p. 218)

While Driver and Oldham's constructivist model of teaching incorporates a stage devoted to determining students' errors (1986, p.113).

The purpose of this ongoing study was to examine just a small element of the complex area involving students' errors and misunderstandings in elementary algebra, namely the relationship between students' thinking, based on their visual perception of the physical movement of algebraic symbols and their awareness of the conceptual link thus denoted.

Other matters of interest to the researchers included: students' approaches to reading and transforming relationships expressed by formulae; students' ability to see and express generalisations; the behaviour of capable students when faced with problems beyond their manipulations; the performance of students on a number of paradigmatic problems to identify points of breakdown, and the capability of average students to acquire and retain effective strategies for checking their work on algebra problems.

METHOD

One hundred and thirty-seven pupils of five classes drawn from years 8(1), 9(2) and 10(2) in an Australian senior high school took a one-hour written test (in two parts, A and B). Interviews were conducted with two samples of 10 pupils, one beforehand to guide the test design, and one afterwards, to clarify the methods being used.

The main target tasks may be illustrated by the following item:

Which of the following formulae do you think is most likely to be correct? Underline your answer. Give reasons:

> Pull of the earth on a satellite at height h.
>
> $$P = kh \qquad P = kh^2 \qquad P = \frac{k}{h} \qquad P = \frac{k}{h^2}$$

This involves *reading* the formulae and *recognising* the functional relationships expressed, (a mathematical task), so that it is then possible to consider which of the relations is most likely to fit the physical quantities (this involves some awareness of physical principles).

A different type of task was to read the formula $V = \pi r^2 h$ and to extract from it the various functional relationships - such as the relation between V and r, if h is fixed.

The target *transformation* task was to transform a formula such as $V = \pi r^2 h$ or $S = \pi r^2 h(r-h)$ to give h. Such manipulations are typically performed by the (implicit) application of the transformations of $A + B = C$ (to $A = C - B$ or $B = C - A$), or of $PQ = R$ (to $P = R/Q$ or $Q = R/P$); where the components may be numbers, single letters, or 'chunks' such as πr^2 or $(r - h)$. Other algebraic laws such as the distributive, associative and commutative laws may be required; and much may be subsumed under the principle of 'doing the same to both sides'. Particular known hazards relate to false commuting or reversing of subtraction or division, and the reversal of order exposed by the students and professors problem, in which six students to every professor is symbolised as $6S = P$. This last has been shown to arise in some cases from over-literal translation of the verbal statement, and in others from a perception of S being the big number, by association with the 6, with a degenerate reading of the = sign as indicating a correspondence rather than an equality of numbers (Clement, 1982).

The test questions include some designed to expose these known errors. Others are focused on the basic multiplicative transformations of $PQ = R$, looking both for methods used and for the possible effect of contexts. We shall discuss first the questions demanding reading and writing of algebra, then those concerning transformations.

RESULTS AND DISCUSSION

(a) Reading and writing algebra

Questions covered (1) the translation of a simple verbally described situation involving adding and subtracting, into an algebraic formula; (2) testing for possible false commuting of subtraction, for the 'equals is makes' error and for the 'students and professors' reversal in additive cases; (3) forming an expression for perimeter containing both letters and numbers from an annotated diagram; (4) reading functional relationships from $V = \pi r^2 h$; and (5) choosing plausible formulae for physical situations, as illustrated above.

The first formula construction question was:

1a Clare has earned D dollars. She spent C dollars for clothes and F dollars for food, and has L dollars left. Complete this expression for L, using C, D and F:

L = ____________

(Tick one box)

Are you sure ☐ ☐ ☐

No Fairly Yes

1b Also write a formula to give F in terms of C, D and L.

F = ____________

(Tick one box)

Are you sure ☐ ☐ ☐

No Fairly Yes

(Pupils' degrees of sureness were elicited as shown; these will be analysed later.)

Success rates for the five classes (Classes 8, 9L, 9T, 10L, 10U) were 30, 57, 50, 69 and 70% respectively – showing an increase over the three years. A further 10% of responses were correct except for the omission of brackets, and a steady 5 - 8% each year reversed subtraction, for example writing $L = (C + F) - D$ instead of $L = D - (C + F)$. In classes 8 and 9L, 6 - 7% gave numerical responses, assuming some actual possible amounts of dollars. The 20% or so each year of unclassified responses included D - C, F or D - C and F, D + C, F, and there were omissions declining from about 20% in Class 8 to zero in class 10U.

The second part of the question required a more difficult transformation of the data. In the first part, the formula could be constructed in the order of events; we start with dollars earned, subtract what was spent and finish with the amount left over. The second part requires extracting what was spent on one of the two items, and counting what was left over in the same way as the other spending item. In the year 8 class, the success rate is still 30%, but the omission rate doubles, to 40%; in the other classes, success falls sharply, to 43, 23, 43, 40% respectively.

Question B2 asks for the perimeters of two annotated figures (a) a triangle with sides marked x cm, y cm, 8cm, and (b) a parallelogram with sides x cm, 5 cm, x cm, 5 cm. The error of juxtaposition (xy8) was seen in a few cases (less than 10%). More noticeable was the large minority (20-25%) in certain classes (8, 9L, 10L) who gave numerical answers, by assuming values for the lengths x and y. These pupils appeared not to accept the possibility of an expression as an answer in this case. These were somewhat larger percentages than those failing to give a formula in the previously discussed question – it appears that the mixed expression $x + y + 8$ is less acceptable than the wholly literal one D - C- F.

(b) *Commutativity problems*

A question asking for the value of x in $8 - 5 = x - 8$ showed the false commutative response (5) in 10-15% of cases, with no decrease with increasing age. This misconception clearly does not get treatment adequately in the curriculum. In a question asking for an expression for the time taken by a bus travelling 28 km at 65 km per hour, the correct answer 28 + 65, was given by substantially fewer pupils than gave 65 + 28; in interviews, where pupils were directed to estimate the expected size of the answer, correction often took place, though a few pupils continued to assert the equivalence of these expressions.

(c) *The students-and-professors reversal in additive problems*

Two questions had the potential to expose this error. The first was:

'I have m dollars and you have k dollars. I have $6 more than you.'

Which equation must be true? (Underline one of the answers below.)

$6k = m$ $6m = k$ $k + 6 = m$ $m = 6 = k$ $6 - \mathrm{m} = k$

Correct responses averaged about 50%, and the reversal error about 25%. The second was:

We are told that *a* and *b* are numbers and $a = 28 + b$; which of the following must be true? (Underline one of the answers below.)

a *a* is larger than *b*
b *b* is larger than *a*
c You can't tell which number is larger
d *a* is equal to 28

Success rates here were about 60% (*a* is larger than *b*) with very few reversals, but there were some 15% each of '$a = 28$' (equals is makes) and of 'you cannot tell'.

(d) Reading a Complex Formula

Extracting the functional relations from $V = \pi r^2 h$ produced very low levels of success, except for the first question (r constant, what happens to V when h is doubled); the correct response here (doubled) was given by 50-60% – but it could be argued that this is the obvious response. There were hardly any correct responses to the remaining questions (h fixed, r doubled, V?; V fixed, r doubled, h?; transform the formula to give h.) One of the obstacles here is the kind of acceptance of lack of closure required to envisage a doubling of a general unknown value, and its effect.

(e) Reading and Judging Functional Relations

These were questions asking which was the most likely correct function for each of four physical situations.

They were:

The pull of the earth of a satellite at height h

$$kh \qquad kh^2 \qquad \frac{k}{h} \qquad \frac{k}{h^2}$$

Force needed on pedals to ride bike at speed v

$$kv \qquad kv^2 \qquad \frac{k}{v} \qquad \frac{k}{v^2}$$

Number of marbles of diameter d in a kilogram

$$dk \qquad \frac{k}{d} \qquad \frac{k}{d^2} \qquad \frac{k}{d^3}$$

Stopping distance of a car with speed v mph

$$kv + \frac{k^1}{v^2} \qquad kv + k^1v^2$$

These questions were found very hard; almost no acceptable reasons were given, and there was a majority of omissions. Ignoring the subtleties of linear or square, thus considering only the direction of the relationships, the satellite questions collected, in all, 43 (incorrect) choices of kh or kh^2, and 20 of $\frac{k}{h}$ or $\frac{k}{h^2}$. The bike question showed 45 choices of the correct direction of relationship (kv, kv^2) and 32 incorrect ones. In the absence of valid reasons it is not possible to distinguish errors in the algebraic reading from incorrect choices of relationships. These questions must be redesigned.

(f) *Transformations*

Attention was focused mainly on the very commonly occurring three-term multiplicative formulae. The Resistance, Current and Voltage relations, Speed, Distance and Time, and the similar relation between three numbers A, B and C were used. The last appeared twice, once using division signs, (as the contextual questions), and once using the fraction bar to denote division. One example is given.

The current, voltage and resistance in an electric circuit are connected by the formula: $R = V \div C$

Complete the formula for finding the voltage, given the current and the resistance.
$V =$ ....

Which of these are correct formulae? Put √ or x.
a. $R = C \div V$ ____ b. $V = R \div C$ ____ c. $C = V \div R$ ____

The age and class trend in the results was of a modest steady increase through classes 8, 9L, 9T, 10L, and a drop in class 10U to a level close to that of class 8. Combining the four question items and the five classes gives overall percentage success rates as follows: for D S T 51%; C V R 51%; A B C with ÷ 49%; A B C with fractional bar, 45%. (The drop in the last figure comes mainly from classes 9T, 10L.) It was hypothesised that the DST context would be more familiar and hence more supportive, than CVR; indeed, it was expected that the latter result would be depressed compared with the numerical case, ABC. However, the results refute these hypotheses, showing that the fraction bar notation is *less* easy to handle than the division sign, and that both contexts are equally supportive.

(g) *Mode of Thinking*

The interviews suggested that four modes of thinking were used to effect these transformations; and in the written test pupils were asked about these.

Look back at the questions on C, V, R (no.2). Tick the one or two of the following statements which is closest to the way you were thinking.

a I thought which would be the biggest number, so the others would be divided into it ☐
b I tried some actual numbers in my head ☐
c I just remembered the formulae ☐
d I thought the one on top on the right hand side would go underneath on the other side ☐

The last mode was shown in interviews only in the use of cross multiplication (with insertion of 1 where necessary, e.g., to make $A = \frac{B}{C}$ into $\frac{A}{1} = \frac{B}{C}$. However, it was conjectured that this visualisation of physical movement of the symbols was the dominant mode for experts, and we were particularly interested in its appearance in our sample.

The relative frequency of responses of the four modes (with no great differences between the classes) was 27%, 41%, 21%, 11%. Interviews also confirmed the tendency to consider sizes and actual numbers; and this result relates to the relatively less successful transformation of the fraction-bar version of the ABC question discussed above.

It is also important to consider whether there is any relation between the mode of thinking and success rates. Tentative partial figures, (subject to further analysis) are, for modes ABCD on the CVR question, 2.2, 2.0, 1.8, 1.6 (mean items correct out of 3); and for the ABC fraction bar question 0.9, 1.3, 1.6, 1.7 (these are pupils who said they used this mode for the CVR question; they were not asked about the ABC question). This matter is worthy of fuller investigation.

We may compare these responses with those of a sample of ten 'experts' – teachers of science and mathematics enrolled in an in-service Masters' degree course. These responded to the ABC fraction bar question, (all correctly, except for one error by one person), and to the 'mode of thinking' question. Of the ten, seven reported using mode D, two mode C and one 'none of these'. To check how far this difference in response was likely to be due to the difference in presentation of the two questions, a further small sample of nine novice mathematics teachers was given both the CVR and ABC fraction bar questions, and after doing each, was asked the 'mode of thinking' question. In all cases, they reported the same mode of thinking for the two questions; four mode D, four C and one both C and D. Some of these subjects also reported imagining the CVR relation written with a fraction bar (so that they could use mode D).

Questions which arise from this study are the following. How far is the eventual adoption of vision symbol-moving methods associated with a loss of the ability to recover the awareness of the underlying mathematical operations? Conversely, should the move towards visual methods be encouraged in school, or guided in such a way as to preserve the links?

We hope to explore these in a more extensive project.

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CHALLENGING PRIMARY STUDENT TEACHERS' VIEWS ABOUT MATHEMATICS EDUCATION

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This year a large group of first-year student teachers at the University of Waikato began their introductory mathematics education course by each interviewing a five-year-old child in a school. During the interview they explored the child's understanding of number and the intellectual strategies used by the child to generate such understanding. Data from an investigation into the effects of this approach show that it took most student teachers by surprise and changed the views of many about how children learn in mathematics and about the teaching role. The more important of these changes are outlined in the paper.

INTRODUCTION

As in a number of other countries (see, for example, Nisbet, 1991) a proportion of students entering the primary teacher education programme at the University of Waikato lack confidence in their own understanding of mathematics, mostly because of negative experiences in the subject while at school. Some are understandably anxious about what the first mathematics education course holds for them.

Other students have left school with their confidence reasonably intact, but from their own classroom experiences as learners of mathematics many appear to have gained a narrow reception/transmission view of learning and teaching in mathematics, a view which is inappropriate to mathematics education today.

For these reasons we have, like some other mathematics teacher educators (for example, Crawford, 1991; Klein, 1991) taken innovative measures to try to address these issues. In particular we have developed an introductory course which is non-threatening but designed to quietly challenge traditional beliefs about learning, teaching and mathematics itself. The initial focus is on young children, in particular their number ideas and associated intellectual strategies. After six hours of introductory work, students visit a local school where they each interview a five-year-old child twice, one week apart. In between visits, and following the second, they share and discuss their interviewing experiences in tutorial-type settings.

To facilitate the interviewing process, the author this year devised a new interview schedule, a copy of which is issued to each student. They spend most of the first six hours of the course in workshops becoming familiar with the schedule, making resources to use with it, and developing interviewing skills. An initial session is also spent exploring the various intellectual strategies the student teachers themselves use in performing common operations such as multiplication.

Central questions for the author were whether the early use of the new interview schedule would influence the student teachers' (a) attitudes toward and perspective about mathematics, and (b) their views about learning and teaching in mathematics. This paper reports the results of an investigation into these questions with the first cohort of student teachers to use the new interview schedule.

THE INTERVIEW SCHEDULE

The interview schedule was constructed with four principles in mind:

1. It should involve the children in number tasks set in contexts that are likely to be meaningful to young learners;

2. It should allow a range of understandings to be revealed;

3. It should be structured in a way that is non-threatening to children - hence the inclusion of interviewing techniques;

4. It should, as far as is possible with young children, allow their intellectual strategies in the area of number to be identified.

The number tasks probed children's familiarity with number ("I was wondering if you had a favourite number?") and a calculator, their ability to rote count and count meaningfully, their understanding of the cardinal and ordinal value of number ("Could you hide this 'treasure' under the 4th 'cushion'?"), numerals ("Could you put something on the envelopes to show how many zoo tickets are inside them? The numbers have fallen off my ruler/clock; could you help me put them back on in the right places?"), and the four operations ("Could you share out these 'chips/French fries' so that I get the same as you?"), measurement, and graphing ("using a pictograph of 'pets'").

DATA COLLECTION

Data were collected, by way of an open-ended questionnaire, from 102 student teachers undertaking the course in the first semester of their first year of study. They were asked (a) whether they expected to begin the course by interviewing a child, (b) what aspects of working with the child proved a challenge, and (c) whether the interviewing experience influenced their ideas about both how children learn and teaching mathematics. This questionnaire was administered within one week following the second interview. Further data were obtained from examining the interview-related log-book entries of a sample of 41 student teachers from the above group. These entries included the student teachers' reflections about the impact of the interviewing experience on their thinking about learning and teaching.

RESULTS

For nearly three-quarters of the student teachers (74), interviewing a child early in the course was quite unexpected. Typical responses were: "It took me completely by surprise." "It was quite a shock." "I didn't think we would be let loose [on children] so early." Fourteen thought the course would begin with 'chalk and talk' about the syllabus or principles of mathematics, and nine expected they would be told how to teach mathematics before working with children. "I expected to be trained in a lot of theory before we got near any children."

The interviewing task seemed to have a considerable impact on many of the student teachers.

Initial impact on 'teaching'

1. *Apprehension*

For some, the prospect of interviewing a young child, even with the guidance of an interview schedule, was scary; nine reported that initially they felt apprehensive and nervous, but for most there was a positive outcome. "At first I was rather anxious about teaching mathematics to children as I felt totally inexperienced, but I felt comfortable in the one-to-one [interview] situation." "It sort of felt like being thrown in the deep end at first but when we actually came down to doing it, it was a very enjoyable activity." For at least one student teacher, her apprehension about mathematics itself was dispelled by the interviewing experience.

> I felt apprehensive as maths is not one of my stronger subjects, but after my time with Joshua I realise that maths can be fun, varied, and can relate to almost anything. This also helped me to change my opinion that maths is hard and boring.

For one mature student teacher (with a degree in mathematics), however, the first interview session was not enjoyable. "I felt nervous and unsure of myself. I need to relax, gain confidence, and not get so uptight." Another found it something of an ordeal too, "I was genuinely afraid of making a mistake and scaring the children."

2. *The challenge of listening to children*

Adopting a relaxed, listening approach, rather than dominating the conversation by questioning proved a challenge for five student teachers, "I found it quite hard to just listen to the children's ideas." A further five reported difficulty in thinking at their child's level to understand their responses. "It was challenging trying to think on the child's level, and trying to appreciate and remember that they have not had as much experience."

Nine frankly admitted that they found it a real challenge to remain neutral instead of telling their child s/he was right or wrong. "I found it frustrating trying not to say 'Yes, correct' or 'No, wrong'." A further 15 commented that they found it difficult to resist helping or correcting their child. For example, one wrote, "I learnt to listen to the children's ideas while fighting the urge to put my ideas and answers into their mouths", another reported, "Trying not to put words into the child's mouth was probably the hardest thing", and a third remarked, "I wanted to tell the child when he had made a mistake. I **wanted** to teach, not just interview." Perhaps, as one student teacher commented, this stemmed from, "Wanting to always have something learnt by the end of the lesson."

3. *Probing and rephrasing proved sophisticated skills*

Thirteen mentioned that they realised it was important to probe further to get their child to explain answers, but this was much more difficult than they thought. "I was surprised at how difficult it was to discover the reasoning behind a child's answer" "[It is difficult] knowing how to develop on from their answers. Hindsight revealed a lot of missed opportunities. [The challenge is] to develop further questions that would have revealed how the children reached their conclusions."

Twenty-one student teachers recognised that they needed to rephrase some of the interview items so that their children understood better what was being asked, but to do this in a non-threatening manner proved quite a challenge. "[It was hard] trying to reword questions so the child could understand them." "Questioning has been a challenge to me, [especially] constructing easily understood questions which a young child can relate to."

Impact on views of how children learn

Although three student teachers felt that they didn't know what to expect about children's learning, five claimed that their views remained uninfluenced by the experience, and eight reported that their views had changed in ways which they could not specify, many were quite clear about how their ideas of children's learning had been influenced.

1. *Enjoyment promotes learning*

Fourteen student teachers commented that they now realised that when tasks are interesting and fun for the children then the children are likely to learn more. "I think that children learn better when it's fun; they are more involved." "Children take in more when the lesson is fun, the resources are interesting and they enjoy it." "It seems that learning is more rapid for a child who is well motivated and enthusiastic about what is being taught. All their interest is targeted at what is being learned."

2. *Experiential activities and discussion enhance learning*

Resources for the children to engage in 'practical' mathematics, and discussion about these mathematical experiences were recognised by 22 as assisting children's learning. As one student teacher wrote, "I feel that in the junior school mathematics is learnt through experiences, doing activities and talking about them." Another commented, "I certainly had my eyes opened. Concrete forms are very important; being able to 'see' the object is a major factor." A third reported, "They [children] love to be involved in practical activities rather than a teacher-dominated 'listen and learn' situation." The value of resources other than texts had not occurred to some student teachers.

> Up until now the thought of maths resources hadn't really crossed my mind but since I have done maths using my [interview] resource kit I have decided that from now on I will be making the best use of it, and frequent use of any resources I possibly can.

3. *Linking to children's prior understandings assists learning*

Twenty-five of the student teachers noted that when the tasks were related to things which the children had experienced then they seemed to have greater understanding of them. For instance, "[The children are] more interested in questions when they can relate to something that has happened to them, to experiences they have had." "My child enjoyed the pets' graph because she had many pets at home and could draw on her own experiences."

For some student teachers the idea that mathematics is part of life was quite a revelation: "The children made me realise that maths is everywhere, not something that is solely learnt in class." "I didn't realise that so much learning comes from life experiences; well, at that age it appears as if it does."

4. *Individual children's learning is unique*

Eighteen student teachers reported that it appeared that children can develop their own individual ideas and ways of thinking in mathematics, even at five years of age.

> They learn it their own way, using skills they remember or have invented.
>
> Young children have their own ideas and concepts which need to be expressed and developed. Children are capable of developing their own mathematical strategies.
>
> Children don't systematically learn mathematics ideas. The child I was working with had difficulty with some things I considered basic, but could easily do other things I thought more advanced.
>
> Sometimes children may take you by surprise by the way they think; they may not be right , or wrong for that matter, but have just thought about something in a totally different context to what you, as an adult, would think.

Impact on views about teaching in mathematics

Again a number of student teachers (8) considered that the interviewing experience had not affected their ideas about teaching. Nine felt that it had, but were not specific about how. For example, one wrote that she now realised there are "totally different ways to teach than what I imagined." The majority, however, were clear about the impact it had had on their thinking about teaching. One recognised that teaching is not a simple process, "I've come to realise that teaching is a very complex occupation." In another case it apparently had a negative impact, "It's incredibly difficult. I don't want to do it!" The rest were more positive, if perhaps a little anxious in some cases, "I've got a long way to go [in developing my teaching]."

1. Make learning enjoyable

Twenty student teachers reported that they now believed that a teacher should ensure that the learning sessions for children are interesting and fun. They also realised that they would need to think a bit creatively to achieve this. "You need to be creative and think of interesting ways for the child to learn." "It has made me look for innovative ways to teach maths, especially as I disliked maths at school." "When a new topic is introduced a child should find it enjoyable and rewarding in order to want to achieve more. The teacher must make maths learning fun." "Maths should always be presented as fun and unthreatening so that they [children] can establish the confidence required to attempt work at higher levels."

2. Make learning meaningful

It was recognised by 15 that children should see purpose in the mathematical learning tasks and that these should be related to the real world of the children. "Children need to be able to relate maths to real life; there has to be a point to it." "You have to make a lesson personal by including children's own interests and relating it to their own everyday experiences." Six student teachers reported that using simple language at first would help the children make better sense of tasks and experiences. "I realised that simple language must be used in order for the child to understand."

3. Adapt teaching to the needs of the children

Almost one-third (33) of the student teachers commented that the interview had enabled them to assess their child's understanding of number. Fourteen were now convinced that it is important to identify the children's present understanding and build on that in teaching sessions. "Find out what they know and build on that." "It taught me to adapt my lesson preparation and materials to my student. Each student needs to be taken at his own pace." One way of building on such understanding, 15 student teachers now think, is to provide children with a variety of resources and hands-on experiences, together with discussion about these. "You need to use practical applications that are fun." "I always thought maths was pretty boring, but it can be quite exciting and stimulating if the correct resources are used."

4. Provide the children with opportunities to explore and explain

According to 13 student teachers, children should be provided with space to explore and think things out for themselves, and with challenges and guidance as needed. "I have now realised that children need to be listened to a great deal, rather than told." "Children need to be given a chance to explore, and given challenges to extend their knowledge."

"I had always been just 'taught' maths - or so I thought - but this idea of children doing most of the work, thinking things out for themselves, worked well."

> Young children view things along their own lines and in their own terms (unless someone tells them otherwise) and being able to let them express what they think is a healthy practice as they learn not to be afraid to put their ideas forward.
>
> I didn't realise the importance of not telling the child what the answer is straight away. I thought if a child didn't know an answer, you should tell them what it is straight away. There is much more emphasis on child enquiry and exploration, and working problems out amongst themselves. I think this is an excellent way for children to really understand concepts and ideas.

Five student teachers feel now that, as a teacher, it is important to try to appreciate from the children's perspective what it is like to learn. "To help a child learn, sometimes you have to think differently from the way you normally do. Try to see things in their perspective." "Think of the maths as you would be learning it for the first time; that is, put yourself in their shoes."

> It has shown how you need to be able to adapt to the thinking and reasoning of young children. Expect the unexpected, and don't put words into the children's mouths. Let them explain what they think about it to you.

A further 6 commented that there is a need to provide a supportive classroom climate to encourage children in their learning. This requires teachers to adopt a 'relaxed' approach rather than playing a dominant part during mathematics sessions. As one student teacher noted, "Children seem to respond more if they are relaxed, and feel that they are able to answer questions, whether they be right or wrong." Another commented, "I found that they [children] don't reveal what they know without feeling comfortable in the situation."

5. Exercise patience

Nine student teachers commented that considerable patience is needed to be an effective teacher with young children in mathematics. "It requires a lot of patience on your side as you don't want to put words into the child's mouth but encourage them to try the problem."

The challenge to one teacher's views: A small case study

The results above are reported from a total group perspective. The impact of the interviewing experience on individual student teachers may be better appreciated by reference to the log-book notes of Lynda (not her real name). Lynda worked with a five-year-old boy called Felix and was surprised at both how much he knew and his willingness to make sense of the tasks.

> If he didn't quite understand what we were doing he'd still do it and follow my instruction but be much happier once he'd figured out what it was all about.
>
> If he couldn't do something, for example, count in 2's, he'd try and work it out, and most of the time did. He would then try to explain to me what he'd done, although he didn't have the right language to do this.

Lynda came to realise that simply-worded instructions are needed for young children, and that encouragement to try seemingly difficult tasks is required.

> If you can get a child to try something, instead of just saying they can't, then they will usually benefit from it, and learn more than if you just told them. They will also be on the way to taking responsibility for their own learning. A child should always feel confident to try something and not be afraid that they're going to be wrong or laughed at.

Lynda decided that enjoyment and life experiences play an important role in learning mathematics.

> To learn something children definitely have to enjoy it. They seem to learn an amazing amount by themselves through their everyday activities.

Finally, Lynda decided that there is more to mathematics and mathematics teaching than she had imagined.

> I have come to understand that maths is a much broader subject than I had thought it was. Real maths is everyday stuff. I think it has to be, otherwise there is no point to it.
>
> Beforehand I thought you would just have everyone in the class group doing the same, but now I see that just about every child needs to be working on their own activity, and it must be hard work for the teacher.

DISCUSSION AND CONCLUSION

The data reveal that as a group the student teachers changed many of their views about learning, teaching and mathematics as a result of the two-part interviewing experience. Their comments suggest that many found it quite a struggle to 'shake off' traditional ideas which had become firmly embedded in their belief systems. In some cases, perhaps many, these included the notion that there is a right way of working with children in mathematics - as indicated by concerns about making mistakes with the children.

Several factors appear to have assisted this change process:

1. The surprise element of interviewing a child early in the course seemed to disorient many of the student teachers with respect to their preconceived image of working with children.

2. The constraint of **not** teaching, but instead observing and talking with, the children forced a radical reconsideration of their role - something which teachers often do not come to until they have established management procedures in their classes and feel secure about the mathematics content.

3. The interview schedule items, together with the children's responses to them, required the student teachers to reconsider their ideas about how young children develop number concepts, and more generally how they learn in mathematics. That some came to think that mathematics learning could be fun and could be related to life experiences are telling points.

The data provide further insight into the process of professional change. Three factors that have been found to facilitate change are: (a) a changed philosophy and desire to change practice, (b) an experience of alternative ways of learning (with an alternative teaching approach being modelled) which opens up new teaching possibilities and a rethinking of philosophy, and (c) voluntary involvement in collaborative action research to investigate the effects of different teaching strategies, which can also lead to a rebuilding of philosophy. All take time.

In the present study, imposed constraints, together with a teacher-guide (the interview schedule) required a change of practice. This in turn led to philosophical change. Given the limited time allocated to us for preservice mathematics education (currently one course of 24 contact hours, and another of 50 contact hours), the student teacher professional development which resulted from the interview task seemed good value for time.

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DEMANDS IMPOSED ON CHILDREN BY MATHEMATICS INSTRUCTIONAL MATERIAL

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This research is concerned with the excessive processing demands imposed on children by the presentation format of mathematics instructional material. The effect of presentation format on learning was examined in the context of a paper-folding task. Grade 4 students were interviewed individually and asked to use instructional material as an aid to learning the task. The only difference between groups of students was the format of the instructional material used. The results indicate that some materials, presented with the intention of facilitating learning, may actually interfere with the procedural learning of some tasks.

The inability of students to efficiently comprehend instructions or to successfully complete tasks has always been a source of frustration for teachers (Mason, 1989). Studies conducted by Bennett, Desforges, Cockburn and Wilkinson (1984) and Bennett, Roth and Dunne (1987) revealed that students took enormous amounts of time to complete even simple tasks. The researchers noted that often the demands imposed by the instructions appeared to be a great deal more difficult than the demands of the actual task (Bennett, 1989).

How can an activity presented in a textbook or a teacher-produced worksheet exceed the information-processing capacity of young children? Hegland (1991) describes five separate elements that can require a student's attention while completing a mathematics task:

1. The directions to be followed (e.g. "fold along the line", "circle the one with...");
2. The type of representation (iconic or symbolic; familiar or unfamiliar);
3. The muscular response required (e.g. cutting, circling, colouring);
4. The perceptual arrangement or presentation format; and
5. The inherent difficulty of the concept or skill to be communicated.

The first four components are almost entirely at the discretion of the individual preparing the instructional material. Often factors influencing the preparation of such materials are decided by convention, convenience or aesthetics rather than a consideration for the processing capacity of young children. Hegland (1991) claims that a student's true performance on a mathematics activity may be disguised by errors resulting when any number of these components do not elicit automatic processing.

Recent investigations in the area of cognition and instruction have resulted in theories and findings that have yielded new design principles for the presentation of initial instructional material. Systematic research in this area has seen the development of cognitive load theory (Sweller, 1988). This theory is concerned with the manner in which instructional material determines the deployment of cognitive resources and is based on the fact that some instructional material requires considerable information-processing, not because of the intrinsic complexity of the information, but solely because of the way in which the material is presented. The value of cognitive load theory lies in its power to generate predictions of what presentation format characteristics will impose an excessive processing load. Consequently, alternative design formats can be suggested that may facilitate learning by reducing the cognitive load associated with particular format characteristics. This study is primarily concerned with the excessive processing load imposed on children by the presentation format of mathematics instructional material and with the subsequent design of alternative formats that may actually facilitate learning.

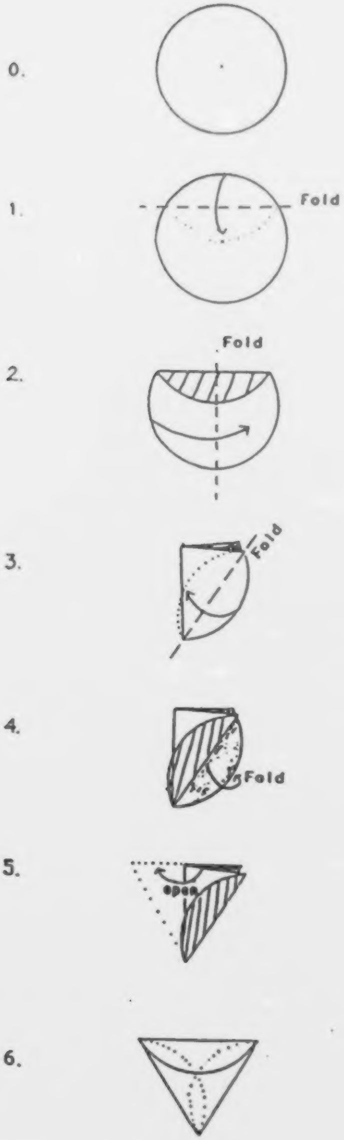
BACKGROUND

The traditional presentation format of instructional material usually comprises some initial explanatory text accompanied by a worked example, diagram or graph presented in a separated format, usually side-by-side or one followed by the other. Such a format can impose excessive processing demands on students in two ways. First, by splitting their attention and secondly, by forcing them to process redundant information. A more detailed discussion of these two characteristics follows.

The split-attention effect

The split-attention effect occurs when students are forced to split their attention between two or more mutually referring sources of information that are physically separated. For instance, a procedure for constructing a geometric shape usually consists of a sequence of geometric diagrams illustrating consecutive folds or cuts arranged vertically on the page (see Figure 1). Accompanying text is normally placed alongside the corresponding illustration. It is unlikely that either the text or the diagrams are completely intelligible when studied in isolation. Sense can only be derived by studying the text in conjunction with the appropriate diagram. It is crucial that an individual simultaneously attend to both the written instructions and the illustrations to gain a complete understanding of the task. This means that a student's attention must be split between two essential sources of mutually referring information. For example, if a statement requires a student to "fold along the dotted line", understanding will only be achieved by referring to the appropriate diagram. The student must read the text, hold it in short-term memory, find the appropriate diagrammatic location and then process the relevant statement in relation to the diagram. The effort involved in searching, identifying and processing the disparate sources of information imposes an extraneous work load on the cognitive resources of the student. This cognitive load is imposed solely because of the manner in which the information is presented and is not a result of the inherent difficulty of the task.

If however, the text is placed directly on the diagram at the appropriate location, rather than appearing adjacent to the diagram, the extraneous processing demands can be greatly reduced. It has been shown in a number of investigations (Chandler and Sweller, 1991; Sweller, Chandler, Tierney and Cooper, 1990) that integration of text and diagrams can enhance learning in two ways: by appropriately directing a student's attention and by imposing a relatively light cognitive load.


0.
1.
Fold
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Fold
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Fold
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open
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Figure 1: Instructional material presented to Diagrams - only format group in Stage 1, 2 and 3

The redundancy effect

The split-attention effect only occurs when the text and the illustration are presented in a separate format and must be mentally integrated to be intelligible. Sometimes, however, a procedure can be learned from diagrams or text alone, any accompanying material being irrelevant, or "redundant". For instance, the construction of a triangular shape from a circular piece of paper may be conveyed by a sequence of diagrams with all necessary folds indicated by arrows placed at appropriate locations on the diagrams. Under these circumstances, any textual information associated with the diagrams (such as describing the task in words) is redundant and students tend to ignore the redundant text, and pay attention solely to the diagrams. When the text and the diagrams are integrated, however, students are forced to pay attention to redundant text. It has been shown in several studies involving high school science students and apprentice electricians (Chandler and Sweller, 1991) that in such cases learning is less efficient than when students are able to ignore the text. This is referred to as the **redundancy effect**.

Mentally integrating mutually referring information that is physically separated and processing redundant information imposes an unnecessary and extraneous work load on cognitive processing resources. If learning can occur with one source of intelligible information, then additional sources may either detract attention from essential information or expend valuable limited cognitive resources processing unnecessary material. Hence, in some instances it may be of more benefit if non-essential information is eliminated, rather than integrating text and diagrams.

In respect to the present study, it might be expected to be beneficial if redundant text were eliminated and essential information incorporated into a diagram.

METHOD

Overview

The study reported in this paper was designed to investigate the redundancy and split-attention effects in the context of a geometric paper-folding task. This task is found in the space strand of many primary school curricula and requires students to gain experience in paper-folding activities to investigate the properties of shapes. The various procedures for achieving particular shapes from other shapes are often presented in mathematics textbooks. In the current investigation, the task required students to fold a paper disc in a prescribed manner. A correct procedure produced a triangle shape with at least two sides equal in length.

Subjects

Throughout the 2 stages of the study, a total of 50 Year 4 students were randomly chosen from four unstreamed Sydney primary schools. Students used in the study had no prior experience of the task they were asked to perform.

Procedure

Two phases were involved at each stage in the study: an *acquisition phase* in which students performed the task by means of the instructional material provided, and a *testing* *phase* in which they carried out the task without aids. The only difference between the groups of students was the format of the instructional material used in the acquisition phase.

At each stage in the study, children were individually interviewed. Each child was given a paper disc and asked to use the instructional material as an aid to completing the task. The children were then given a new paper disc and asked to carry out the task without aids. The time taken to complete the task at each phase and the accuracy with which it was performed was recorded.

Stage 1

Two sets of instructional material were used in the acquisition phase: a "diagrams-only" format that involved a sequence of diagrams intelligible by themselves, and a "redundant format" consisting of the same sequence of diagrams accompanied by written instructions that referred to the diagrams (see Figures 1 and 2 respectively). It was thought that children using integrated instructional material with extraneous information eliminated (diagram group) would learn more effectively than those using a format that includes redundant, but not self-sufficient, written material (redundant group).

Stage 1 results

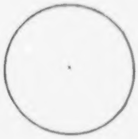
Table 1 shows the mean times in seconds, and the percentages of children correctly completing the task for both the acquisition and the testing phases.

Table 1: Mean times to completion (seconds) and percentages of students correctly completing the task at Stage 1 (n = 15 in each group).

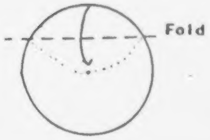
	Acquisition phase		Testing phase	
	diagram	redundant	diagram	redundant
mean time	369.9	463.7	277.8	457.6
% correct	66.7	33.3	60.0	26.7

A t-test indicated no significant differences between the completion times of the two groups at the acquisition phase t(28) = 1.23, SE = 76.42. (The 0.05 level of significance is used throughout this paper unless otherwise stated.) However, there was a significant difference between the mean times for the two groups at the testing phase, t(28) = 1.90, SE = 94.55. Comparison of percentages correct by means of a Fisher Exact test with Overall's (1980) correction yielded a significant difference (p = 0.03) for the acquisition phase, but no significant difference for the testing phase (p = 0.06). These results indicate that the diagrams format was superior to the more conventional, redundant format.

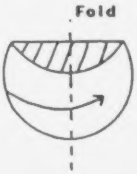
0. Take a circular piece of paper.



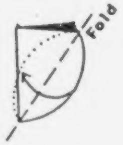
1. Fold along the broken line.



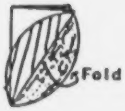
2. Fold this shape in half.



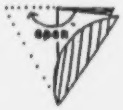
3. Fold back the top edge along the broken line.



4. Fold the other flap back under this line.



5. Open the centre fold.



6. You now have a shape with three sides.

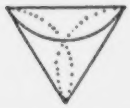


Figure 2: Instructional material presented to redundant format group in Stage 1.

Stage 2

To date, investigations of the redundancy effect have concentrated on text being redundant to a diagram. Stage 2 sought to examine the effect of redundant diagrams. The diagrams-only instructional materials of Stage 1 were compared with a modified version of these materials, in which "back views" were provided for some steps (see Figure 3). Although it might be thought intuitively that this extra information would enhance learning, it may be hypothesised from cognitive load theory that the inclusion of these extraneous redundant diagrams imposes extra cognitive load and has a debilitating effect on learning. It was thought that children using diagrams only would learn more effectively than those using a format in which the same diagrams are accompanied by redundant diagrams.

Stage 2 Results

Table 2 shows mean times in seconds, and percentages of students correctly completing the task for both phases.

Table 2: Mean times to completion (seconds) and percentages of students correctly completing the task at Stage 2 (n = 10 students for each group).

	Acquisition phase		Testing phase	
	Diagram	Redundant diagram	Diagram	Redundant diagram
mean time	244.6	396.0	108.9	314.8
% correct	90.0	60.0	90.0	60.0

A t-test indicated that the diagram format group spent significantly less time to complete the task than the redundant diagram group at both the acquisition phase and the testing phase, t(18) = 1.92, SE = 78.95 and t(18) = 2.12, SE = 97.10, respectively. Thus, the diagram format was superior to a format in which the same diagrams were accompanied by redundant diagrams. An inspection of the percentages of students correctly completing the task indicates that students studying the diagram format performed better than students studying the redundant diagram format. However, a comparison of the percentages correct by means of a Fisher Exact test with Overall's (1980) correction yielded no significant difference for either the acquisition phase or the testing phase (p = 0.07).

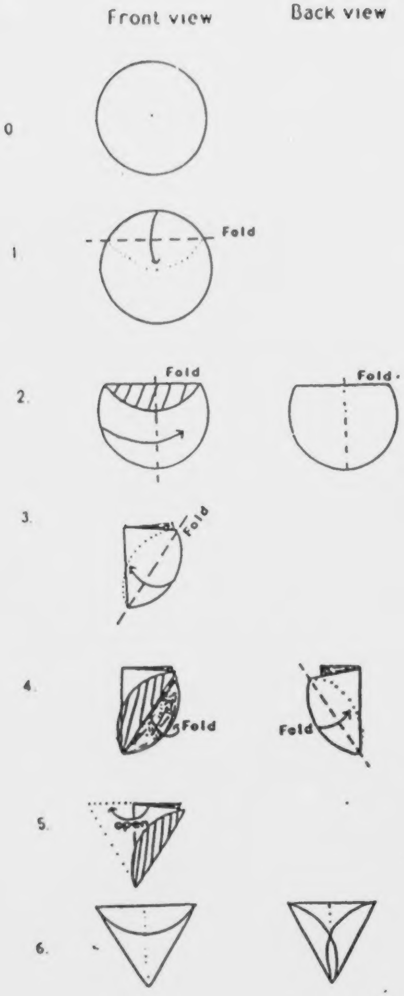

Front view
Back view
0
1
Fold
2.
Fold
Fold
3.
Fold
4.
Fold
Fold
5.
open
6.

Figure 3: Instructional material presented to Redundant - diagrams format group in Stage 2.

DISCUSSION

The results of this study indicate the inadequacy of conventionally formatted instructional material and emphasise the debilitating effect redundant information and attention splitting presentation formats can have during initial instruction. The benefits of presenting information with redundant material removed were demonstrated at Stage 1 and Stage 2. At Stage 1 students studying the diagram format processed the instructions more quickly and completed the task more successfully than students studying formats that contained redundant material. Because the instructions for the diagram group and the redundant format group both contained the same self-explanatory diagrams, it can be inferred that the effect was caused by the additional textual information provided for the redundant group. This information, rather than assisting comprehension as normally is intended, had an inhibitory effect. Processing the text merely required additional cognitive resources but did not confer any detectable explanatory advantages.

Stage 2 demonstrated that the redundancy effect is evident not only when text is redundant to diagrams, but also when information is conveyed solely by means of diagrams; this extends the generality of the redundancy effect to redundant diagrams. Providing additional diagrams detailing a perspective unnecessary for the successful completion of the task, proved to inhibit rather than facilitate learning.

There are three general conclusions. First, the redundancy effect is not a function of a text versus diagrams comparison - there was no text in Stage 2. Secondly, the physical integration of essential information into one source of information will usually facilitate the learning process. Thirdly, extreme care must be exercised when providing students with additional, seemingly useful, information. If the mental integration of additional material with essential information imposes a heavy extraneous cognitive load, it may have adverse rather than beneficial effects. For optimum effect, the usefulness of additional information must outweigh the consequences of having to process it.

IMPLICATIONS FOR TEACHING

For teaching and learning, the findings of this study have implications for the manner in which initial instructional material is presented to students. Teachers need to exercise extreme care when providing students with additional and seemingly worthwhile information. If the processing of additional information (whether it be textual or diagrammatic) with essential information imposes an extraneous cognitive load, it may have detrimental rather than beneficial effects on learning.

It should be noted that material which is redundant for some students may be important for others and any decision to eliminate seemingly redundant information should take into consideration the information to be conveyed in respect to the audience it is intended.

While it is not possible to make predictions concerning the facilitatory or debilitating effects of all possible student/instructional material combinations, the knowledge that redundant and split attention formats can have negative consequences should act as a check to instructional designers.

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ACQUISITION OF CONCEPT OF VARIABLE IN A TRADITIONAL AND COMPUTER-INTENSIVE ALGEBRA CURRICULUM

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INTRODUCTION

With the advent of computer technology and calculators that can perform symbolic algebra, graph functions and relations, and display tables of values the question arises: how should mathematics be taught in the presence of these technologies? In the United States one answer to this question was given by James Fey (1985) in the form of a curriculum, called the Computer-Intensive Algebra curriculum (here after referred to as CIA). In it the idea of algebra was radically changed by focussing on the concept of function and variable instead of equation and allowing an algebra system, DERIVE, to perform all symbol manipulations. For the curriculum a special program was developed that could graph functions, display tables of values, compare several tables of values, and perform certain simulations. The idea of the CIA was to teach students to use the available algebra tools as intelligently as possible and therefore to have more time available to study the concepts of algebra, variables and functions, and the modelling of problem situations.

The research reported here concerned itself with determining, whether the introduction of technologies in the algebra curriculum improved conceptual understanding of the concept of variable. To define what it means to understand variables after a first-year algebra course, algebraic activities and the role variables play in those activities were examined. This process resulted in a division of conceptual knowledge of variables into four aspects:

1. knowing the ways that symbols can be used to represent elements of a numerical domain;
2. interpreting variables in different contexts;
3. using variables in modelling problem situations and in translating one representation of a problem to another;
4. using variables in justification and proof.

These four aspects of understanding of variables were studied among students following a traditional algebra course (TA) and the CIA by interviewing six students from each course twice during the year and by giving a diagnostic exam at the end of the year to three CIA and two TA classes.

METHOD

Study Design

At the individual level six students from a TA (named T1 through T6) and six from a CIA class (named C1 through C6) were interviewed twice during the academic year: once half way and once near the end of the year. The interview students were selected from the average-and higher-ability students in each class. Ability was based on the January grade received in Algebra I during the year the research took place. Interviews were standardized, although routes of questioning could vary depending on the responses of the individual students. Interview questions were designed to measure the four aspects of conceptual knowledge of variables as described above. Interview I consisted of 8 questions and interview II of 6 questions respectively. Interviews were tape-recorded and lasted between 30 and 45 minutes. Examples of questions from interview I and II are given in Appendix I.

At the class level the pretest/posttest design was utilised. The pretest was the county wide Criterion Reference Test (CRT). This test measures competence in pre-algebra skills and was given in the first week of class to all ninth-grade algebra students in the county where the research took place. The post-test was given near the end of the school year to five classes: two TA classes having the same teacher (TA) and three CIA classes having three different teachers (CIA1, CIA2, and CIA3). It was specifically developed for this study. The post-test instead of being an achievement test, was diagnostic in character. It consisted of three parts, each having a different question format and probing one of the first three aspects of conceptual knowledge of variables. In this paper the results of the third part of the post-test will be given and discussed. The third part was mainly concerned with students' understanding of modelling and translating.

Post-test Analysis

Part III of the post-test consisted of eight free response questions. Each question was scored separately and partial credit was possible for some problems. Because different problems measured similar aspects of modelling and translating skills, the scores of some questions were combined into one score. This process resulted in four different scores and reduced the possibility that significant statistical differences would be found by chance. The first score, X1, reflected modelling skills. X2 was a score representing the ability to read and interpret tables of values and graphs. X3 gave insight into the students' ability to relate equations with their graphs, their understanding of the solution set of a system of two linear equations, and into the students' idea of the size of the solution set of a system of equations. The last score, X4, tested the students' ability to translate between a table of values and a graph (problem 17c, see Appendix II).

Because the three CIA classes were taught by different teachers and none of the teachers taught both curricula, possible effects of the teachers needed to be taken into account. Therefore a one-way analysis of variance was done with independent variable 'teacher' and dependent variables the four scores X1, X2, X3, X4, and the pre-test score.

RESULTS

The interviews and post-test revealed two different profiles of use and understanding of variables acquired by students in CIA and TA curricula. To reveal some of those differences the results for two of the four aspects of conceptual knowledge of variables will be discussed based on information gained from the interviews and post-test.

Interpreting Variables in Different Contexts

What is a variable? In both interviews the first question asked students "What is a variable?" from several angles by letting them talk about what algebra is, about the use of variables in algebra and in equations, the importance of variables, and how you would tell a friend what a variable is.

In their explanation of what a variable is all TA students mentioned the idea of an unknown number that you have to find. The main idea of variable for TA students was that of a static concept. What was most noticeable in their ideas of variables was that, for none of the TA students had the meaning of variables changed from the first to the second interview. The only change that had taken place was that T1 thought in the second interview that variables stood for large numbers, a view that T6 had expressed in the both interviews.

Another important idea of variable for TA students was that of a place holder. This was expressed by T3, T4, and T5. In answer to the question why variables are important they said: "if you did not have them in an equation or expression you might make a mistake in simplifying or solving." The idea was that "if you don't have them you'll make a mistake." In their opinion they seemed important in the process of solving.

In both interviews, the idea of variable expressed by CIA students was more varied than that of the TA students and made a distinct development from the first to the second interview. For CIA students variables could be input, output, varying quantities, generalisers, and missing numbers. Several CIA students gave practical examples of variables. By the time of the second interview, four of the six CIA students had a concept of variable resembling a generaliser, but at the same time understood variables could be missing numbers. Included in their concept of variable was the idea that two variables could be dependent on each other. The development of their idea of variable can be seen from the responses of one student C2. In the first interview this student mentioned two aspects of variables. First of all a variable was a substitution for a list of numbers, (a varying quantity) and secondly in an equation a variable was a missing number. The variable as a varying quantity she explained as follows:

> It's like a substitution of a number. Like if you have a list of numbers, like say they are from 1 to 10, and put the n and you can pick the number you want to use, and then you get 5n and you have a list of 1 to 10 and you want to pick 2, it would be 5 times 2. And if you didn't want to use all numbers you didn't have to. Or if you only want to use certain numbers at certain times you can do that.

In the second interview she (C2) responded to the question "Why variables are important" as follows:

> You can try all different numbers for variables, and if you just use numbers you have to do the problem all over again. You know with a variable all you have to do is write out that problem once and you can figure it out.

When this excerpt is compared with the response C2 gave in the first interview it shows that by the time of the second interview C2's idea of variable had sharpened. This change in idea of what a variable is from the first to the second interview could be detected in the responses of four of the six CIA students.

Variables in equations. From the interpretations that students gave for the equation $S = 6T$ (problem I6, Appendix I) several observations can be made. Firstly, a coefficient other than 1 in front of a variable seemed to confuse the students into using the variable as a label. If students interpreted variables as labels it was more often T than S.

Secondly, students had varying ideas about the dependency of S and T. Some thought that the two variables were dependent on one another and therefore saw S and T as varying quantities. Others saw only S as varying and thought that the number of teachers was fixed at 25, independent of the number of students. The last category of students thought of both S and T as labels. They were the students that reversed the interpretation of the equation. In the latter case S and T could be dependent on each other, but not always.

Thirdly, filling in numbers for the variables led four students to a correct interpretation of the equation. One student held on to his reversed interpretation after replacing the variables with numbers. For that student substituting a value for one of the variables changed the problem situation.

With respect to the difference in results between the two curricula the following remarks can be made. There were three CIA students who gave a correct interpretation of the equation, two of which needed numbers in order to do that. Only one TA student came to a correct interpretation after substituting numbers. Two CIA students saw S varying independently of T, which they thought to be fixed at 25. None of the TA students thought this way. One CIA student was unclear about his interpretation. Three TA students reversed the interpretation, none of the CIA students did that. One TA student could not give an interpretation because there was not enough information. And one TA student held two very different interpretations depending on whether you used numbers or not.

Variables in Modelling and Translating

Results from interviews. On questions similar to I3 (see Appendix I) where students were asked to model a problem situation, TA students seemed to remain thinking of variables as unknowns. In problem I3, when asked to represent the relationship between two varying quantities of the table of values in the form of an equation or rule, five of the six TA interview students wrote an equation for one instance of the table of values. For example, they wrote: $4x = 16$. Some commented that there was no need to write an equation, because they knew what the relationship was: "it was four times." When specifically asked to use two variables, one for the first and one for the second column, only one TA student succeeded in writing a correct equation. The other four gave a similar equation as before. TA students might not have thought of the first and second column of the table as standing for two varying quantities. Their choice and use of variable was consistent with the definition of variable in their curriculum. All CIA students were able to give the functional relationship for the table of values in problem I3.

Results from post-test. One-way analysis of variance on the pre-test, CRT, with the teachers as independent variable did not indicate any significant differences ($F(3, 97)=1.07$, $p<.37$) between the teachers, neither was there any differences between the X4 scores (ability to translate from table to graph, $F(3, 105)=.36$, $p<.78$). One-way analysis of variance on the other three scores with teachers as independent variable all reached significance (see Table 1 for the mean scores and standard deviations). For X1 the F-value was 2.62, $p<.055$, for X2 the F-value was 7.50, $p<.0001$, and for X3 the F-value was 6.94, $p<0.0003$. The degrees of freedom for all three analyses were 3 by 105. Because the latter three scores were significant, contrasts between the teachers were checked for their significance.

Table 1: Mean and standard deviation (in brackets) of CRT, X1, X2, X3, and X4 for the students of each teacher and the number of students (N) on which the mean and standard deviation are based.

SCORE	CIA1	CIA2	CIA3	TA	TOTAL
CRT	17.04 (5.0)	16.72 (5.4)	19.18 (4.9)	18.10 (4.5)	17.64 (4.9)
X1	1.89 (1.75)	2.69 (2.07)	2.81 (1.83)	1.70 (1.55)	2.18 (1.82)
X2	1.91 (.84)	2.32 (.53)	2.51 (.51)	1.66 (.84)	2.03 (.79)
X3	.13 (.26)	.35 (.34)	.58 (.67)	.14 (.29)	.26 (.42)
X4	.45 (.51)	.46 (.51)	.42 (.51)	.34 (.48)	.41 (.49)
N	29	26	19	35	109

Table 2: *T*-values and probabilities in brackets for contrasts in teachers on the scores X1, X2, and X3. For all *t*-values the degrees of freedom are 105. * indicates that the *t*-value is significant.

contrasts				*t*-values and probabilities		
CIA1	CIA2	CIA3	TA	X1	X2	X3
-1	-1	-1	3	-2.08* (0.04)	-3.92* (0.000)	-2.62* (0.010)
0	-1	-1	2	2.61* (0.01)	4.60* (0.000)	3.66* (0.000)
-1	-1	2	0	1.11 (0.27)	2.08* (0.04)	3.32* (0.001)
-1	2	-1	0	0.78 (0.44)	0.61 (0.55)	-0.08 (0.93)
2	-1	-1	0	-2.03* (0.05)	-2.93* (0.004)	-3.61* (0.000)

In Table 2 the contrasts, their *t*-values and probabilities, based on the pooled variance estimate, are listed for each of the three scores X1, X2, and X3. Scheffé tests ($p<0.5$) indicated that the TA teacher differed significantly from the CIA2 and CIA3 teachers on the X2 score, and from the CIA3 teacher on the X3 score. CIA1 and CIA3 differed significantly from each other on X3, $p<0.5$. Table 2 further shows that the first two contrasts, were significant for all three scores. These two contrasts measured essentially the differences between the curricula. The last contrast, the difference between the teacher of

CIA1 and the other two CIA teachers, was also significant for all three teachers. Teacher of the CIA3 class taught the curriculum for the third time during the research, CIA1 and CIA2 teachers were teaching the curriculum for the first time. CIA1 teachers was observed to be much less organised than the other two CIA teachers. The TA teacher was considered to be a very good teacher in her school and county.

Conclusion. From the statistical analysis of the third part of the post-test one can draw the conclusion that on problems testing the ability to model problems situations (X1), the ability to read and interpret tables of values and graph (X2), and the ability to relate various aspects of equations with their graphs, CIA students significantly outperformed TA students. Some of the variance was due to teacher influence, but contrasting the curricula proved also significant.

CONCLUSION

The most important findings of this study are the gain four of the six CIA interview students made in their understanding of variables as generalisers and as varying quantities that are dependent on other quantities and the conclusion that CIA students were better able to model problem situations and read tables of values and graphs than TA students. It seems to indicate that technology can provide the opportunity for teachers to teach algebra with more emphasis on conceptual and less on procedural knowledge.

This research did not set out to prove that the TA curriculum was ineffective and it did not show that TA students had gross misunderstandings with respect to the concept of variable. Within the limits of their curriculum, TA students did reasonably well or at least not differently from what has been established in previous research (see, for example Küchemann, 1981). What has to be decided by secondary mathematics teachers and educators is: "What concept of variable do we want algebra students to learn?" the exclusive idea of variable as unknown, or a broader concept of variable including the idea of generaliser, missing number and dependency of varying quantities. The second question would be "What kind of skills do we want algebra students to learn?" the exclusive ability to solve equations or the ability to model problem situations, solve equations related to that model, and to be able to represent and interpret the model in other representations like tables of values and graphs. The CIA curriculum proves to be an option for the latter alternatives.

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Appendix I: Examples of Interview Problems

I3. Suppose a student did an experiment in science to determine the relationship between the number of tomato plants planted and the number of kilograms of tomatoes harvested. The student made the following table of results.

# of Plants	4	5	9	25	11
# of Kilograms of Tomatoes	16	20	36	100	?

(a) How many kilograms of tomatoes do you expect the student will harvest with 11 plants?

(b) What can you say about the relationship between the number of tomato plants and the number of kilograms of tomatoes?

(c) Can you express that relationship using variables?

(d) If the student harvested 24 kilograms of tomatoes, with how many plants did she start out?

I6. The relationship between the number of students and the number of teachers in a particular school is given as:

$$S = 25\,T$$

Where S stands for the number of students and T for the number of teachers. Explain what this relationship means to you?

Questions from interview II.

II1. This year you've taken algebra and you've worked in different ways with variables. Can you give some examples of how you have used variables in algebra this year?

If you had to tell a friend who had never taken algebra what a variable is, what would you tell him/her? How would you explain that?

What can you do in math now that you know about variables? Why do you think variables are important? Why do you think you learned about variables?

Have you any idea how you might use variables outside this class, for example in other school subjects, outside school? How variables might be useful in those situations?

Appendix II: One Question from the Third Part of the Posttest

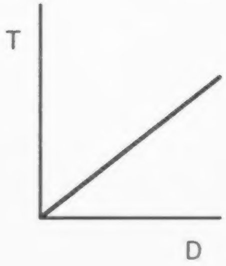
17. A bicyclist is approaching a hill. When he is at the bottom of the hill, the time (T) is set to zero. As he is going up the hill the distance (D) between him and the bottom of the hill is measured every 10 seconds. Below in the table are the distances for the first ten measurements.

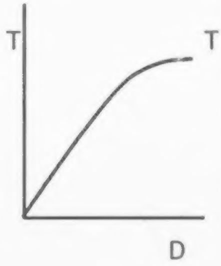
T in seconds	0	10	20	30	40	50	60	70	80	90	100
D in meters	0	100	190	280	360	430	480	520	550	575	590

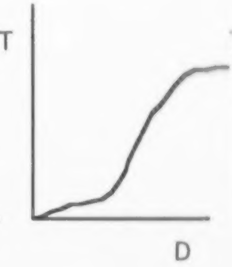
(a) What is D for T = 50? ______________________

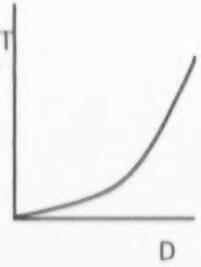
(b) During which 10 second period does the bicyclist cover the largest distance?
____

(c) Which graph below most accurately represents the relationship between time and distance for the
bicyclist? Check one alternative.









THE GRAPHICS CALCULATOR IN TERTIARY MATHEMATICS

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INTRODUCTION

Technological advances are so rapid that hand-held calculators which perform symbolic algebra and graphically display functions will soon be affordable for most students. Already relatively cheap graphics calculators are available. Their use on a large scale by teachers and students at the secondary and tertiary level might therefore not be far off. Of importance before the wide-scale introduction of such technology into the classroom are several questions of which two are "How do students perceive of the calculator as a tool in mathematics?" and "How do students use the graphics calculator?"

In 1991 the TI-81 graphics calculator was introduced at Swinburne University of Technology to all first-year Applied Science students in their calculus courses. At the same time as the calculator was introduced, research was initiated to study the impact of the graphics calculator on the teaching and learning process. Several sources of information were tapped. These were: student surveys, a student 'brainstorm' session about the calculator, and the study of final examination scripts. In this article we will discuss some of the results of this research. Before presenting the research methodologies, results and discussion of the results, the calculator itself and the calculus subject in which it was used will be described.

SUBJECT

The course was a tertiary first year calculus course based on the book "Calculus with Analytic Geometry" by Hunt (1988) . The course content was similar to that of previous years, but wherever the graphics calculator could be used, adaptations in the form of lecture notes were made to the presentation of the content (Barling, 1991). For example, the calculator was used in the topics of graphing functions, equations and inequalities, and limits. In addition, particular emphasis was placed on the capacity of the calculator to act as a checking device for analytically derived knowledge. For example, derivatives of functions could be graphed and graphically compared with analytically determined derivatives.

THE CALCULATOR

The calculator used was the TI-81 graphics/scientific calculator. Graphs are produced by entering a function into the alphanumeric key board and by using the various scaling functions. A 'zooming' facility enables the user to move rapidly from the macroscopic to the microscopic view of the function and vice versa. The calculator can simultaneously display the graphs of up to four functions. Facilities exist to enable the user to read off the co-ordinates of any point on a graph with a high degree of accuracy. This is extremely useful when using the calculator to solve equations. The graphing capacity of the calculator is limited in certain ways. For example, for the function $\frac{\sin x}{x}$ with a point discontinuity at zero, the calculator will not show a gap around the point $x = 0$ unless the range of x is chosen such that $x = 0$ falls exactly on a pixel of the screen. Students not aware of such discontinuities would most likely not recognize the discontinuity from the calculator alone. Another limitation is that only functions can be graphed; relations such as circles and ellipses have to be entered in two parts, one for the bottom half and one for the top half.

There are other peculiarities that are beyond the scope of this article to discuss, but which force the students to use mathematical judgement before they can use the calculator as a mathematical tool, particularly when operating at the calculus level.

STUDENT SURVEYS

Method. In April and September of 1991 a survey was given to all calculus students attending the tutoring sessions during one particular week in each month. Except for two questions, the surveys were identical so developments in students' attitudes and opinions could be studied. The two questions in April that were deleted in September asked the students whether they had bought the calculator or not and if so how long ago. Because 98% of the students had bought the calculator at the time of the first survey, the two questions were in September replaced with items of more interest at that time.

The first four questions of the survey dealt with the students' major, sex, and first language. The remaining twenty-six items were statements concerning attitudes towards mathematics, learning mathematics with the graphics calculator, and the graphics calculator itself. Students responded to the statements by indicating their attitudes or opinions on a five point scale ranging from strongly agree to strongly disagree, with not sure in the middle. The complete questionnaires and results can be found in Boers and Jones (1992b).

Results and discussion. Several issues related to the graphics calculator were addressed in the survey, such as whether students were anxious about mathematics and whether the graphics calculator made them more confident, whether learning mathematics at the same time as learning the technology was difficult to do, whether the calculator was difficult to learn how to use, did they think the calculator was worth the price, and how the students used the calculator in the class and during problem solving: were they following on their own calculator what the teacher did in front of the class, did they use the calculator to explore problems and to check problems solved by algebra, and had making a graph of the function become an integral part of their problem solving routine. A summary of students' responses during April and September is given in Table 1.

From Table 1 we can read that the effects of the calculator were mostly positive for the students. And if there were developments from April to September they were generally in a positive direction with respect to the influence of the graphics calculator on students' attitudes and behaviour. Some statements caused significantly different reactions between males and females. The gender effects in those survey items are summarized in Table 2.

Table 1: Student opinions about mathematics, the TI-81, and learning mathematics with the TI-81 expressed in two surveys during April and September. The September responses are recorded in brackets. (SA-Strongly Agree, A-Agree, NS-Not Sure, D-Disagree, SD-Strongly Disagree) Items indicated with * had a gender effect in April, September or both months.

Issue	% of students giving response				
	SA	A	NS	D	SD
* Calculator is difficult to use	3 (3)	24 (14)	12 (13)	47 (56)	13 (13)
Prefer scientific to graphics calculator for doing numerical calculations	20 (13)	22 (19)	10 (7)	34 (43)	14 (17)
Simultaneous learning of mathematics and how to use the calculator confuses me	1 (1)	16 (7)	11 (9)	59 (67)	13 (14)
Graphical and algebraic analysis helps me to understand mathematical ideas	13 (14)	59 (63)	17 (13)	10 (9)	2 (2)
Time spent in class learning how to use the graphics calculator is worth it	10 (9)	50 (52)	20 (18)	16 (15)	5 (6)
* Due to the TI-81 I find myself exploring a mathematical problem rather than just trying to get that answer	5 (8)	39 (39)	26 (26)	23 (23)	7 (4)
The TI-81 is not worth the price we had to pay for it	18 (19)	40 (36)	23 (28)	16 (16)	3 (2)
In class I follow on my own calculator what the teacher is doing when he is using the TI-81	18 (11)	55 (56)	9 (10)	15 (17)	3 (4)
* I am generally anxious about mathematics	8 (7)	39 (29)	26 (29)	22 (30)	4 (4)
Since using the TI-81 I am more confident in doing mathematics	4 (7)	33 (40)	32 (25)	27 (25)	4 (4)
* Because of the TI-81 I am less afraid of mathematics	2 (6)	22 (28)	29 (29)	38 (30)	8 (7)
My TI-81 is useful for checking solutions to problems I solved using algebra	19 (20)	55 (58)	13 (11)	11 (9)	2 (2)
Making a graph of the problem situation on my TI-81 is becoming an important part of my problem solving	(10)	(47)	(22)	(17)	(5)

Table 2: Gender effects for survey items. Numbers in brackets indicate the responses for September

Issue		% of students giving response				
		SA	A	NS	D	SD
Calculator is difficult to use April $p = 0.0003$, Sep. $p = 0.06$	male	2 (3)	19 (9)	9 (13)	52 (58)	18 (17)
	female	5 (4)	33 (20)	18 (14)	37 (54)	6 (7)
Due to the TI-81 I find myself exploring a mathematical problem rather than just trying to get that answer April $p = 0.02$, Sep. $p = 0.06$	male	7 (11)	43 (43)	26 (24)	17 (19)	7 (3)
	female	3 (2)	30 (33)	27 (29)	34 (30)	6 (5)
I am generally anxious about mathematics April $p = 0.3$, Sep. $p = 0.003$	male	9 (3)	34 (34)	29 (21)	23 (35)	5 (6)
	female	6 (11)	46 (26)	22 (36)	22 (26)	3 (0)
Because of the TI-81 I am less afraid of mathematics April $p = 0.2$, Sep. $p = 0.07$	male	4 (7)	23 (33)	28 (27)	37 (25)	8 (8)
	female	0 (4)	21 (20)	30 (39)	40 (39)	8 (6)

'Brainstorm' Session

Method. One of the main reasons for introducing the graphics calculator was that it would support the learning of mathematics. Feedback from the students on this issue was sought through a 'brainstorming' session in which a number of students (n=18), selected at random from the calculus classes, participated in a structured group discussion. The purpose of the discussion was to identify and rank in order of importance both positive and negative aspects of learning mathematics with the graphics calculator. Input from the researchers was held to a minimum; they only guided the discussion and provided the starting questions. The two questions discussed were: What are the positive aspects about learning mathematics with the TI-81? and What are the negative aspects of learning mathematics with the TI-81? For a more detailed description of the procedure see Boers and Jones (1992b) or Jones (1992).

Results and discussion. The five most important benefits of studying mathematics with the calculator from a list of fifteen this group of students came up with were, in order of importance:

1. the ease of sketching and gaining information for graphs;
2. being able to quickly check the correctness of derivatives, integrals, and answers;
3. the understanding and interpretation of graphs and derivatives is made easier;

4. screen display helps in calculating and checking difficult formulae;
5. increases confidence and enthusiasm.

From three of the five most important positive aspects of studying mathematics with the TI-81 (points 1, 2, and 4) it becomes clear that the students value the calculator as a tool for helping them 'do' mathematics. The students also valued the graphics calculator as an aid to understanding and learning (point 3) but this would appear to be of less importance to them than its technical capabilities. Yet, from the initiators' point of view, the potential of the graphics calculator as an aid to learning was the key reason for its introduction.

The fifth positive feature identified by the students was attitudinal in that they claimed that learning mathematics with a graphics calculator increases confidence and enthusiasm. This point was also addressed in the survey mentioned earlier with around one third (36%) in April and almost one half of the students (46%) in September reporting that with the graphics calculator they were now more confident in doing mathematics and 24% in April and 33% in September saying that, because of the graphics calculator they were less afraid of doing mathematics (see Table 1).

With respect to the problems of studying mathematics with the TI-81 students made five points of which the two most important were:

1. possibility of calculator dependency - removes the need to know why
2. tendency to rely on the calculator

These points related to fear that using the calculator could lead to deskilling. It is of interest that this was less of concern with the teaching staff, possibly because they tended to view the calculator primarily as a teaching and learning aid rather than a mathematical tool.

USE OF CALCULATOR UNDER EXAMINATION CONDITIONS

After requesting students' opinions about the calculator the question emerged: how do students actually use the graphics calculator while solving mathematics problems? An attempt at answering this question was undertaken by studying the examination scripts of a representative sample of 37 students chosen from 304 students who took the 1991 calculus examination and who scored 15 points or more on the exam. A quota sampling technique was used to select the students for this analysis so students from all levels of achievement were represented in the sample. For a detailed description of the selection procedure see Boers and Jones (1992a). In this paper the selected students are indicated with a number (1 through 37) assigned to them during the study.

Results and discussion. Three main results were found from studying the examination scripts of calculus students. For a complete report on this study see Boers and Jones (1992a).

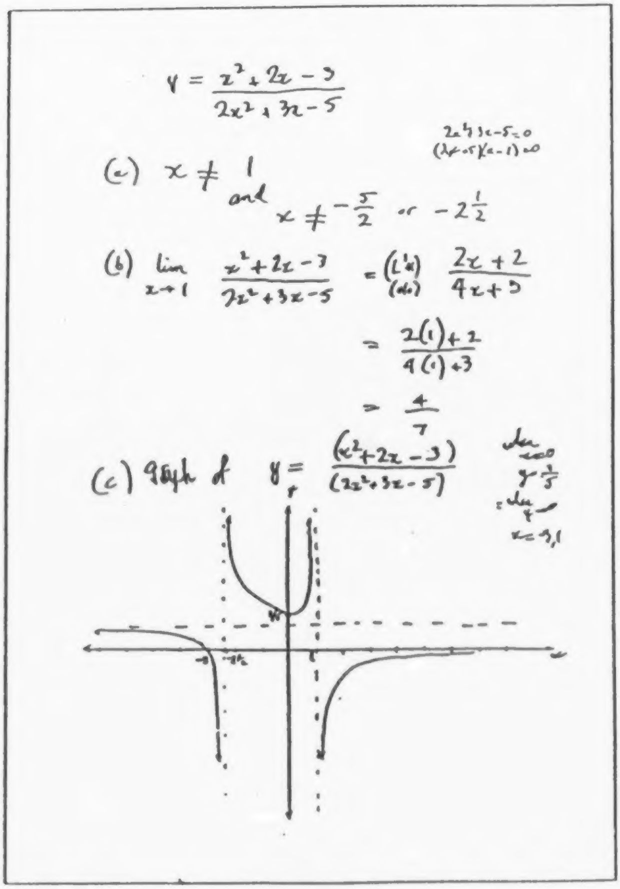
Firstly, the full potential of the graphics calculator was not utilised by the students during the exam. On questions that did not specifically ask for a graphical response, such as, on a question asking for the limit of a function at a certain point, little evidence was found that the graphics calculator had been used to arrive at a solution. The calculator's potential as a graphical checking device for algebraically derived information was not realised either. None of the errors made in derivatives, limits and integrals were corrected.

On a question where a graphical response was required *and* where a purely graphical solution would have sufficed, less than half the students (43%) solved the problem by calculator alone. For this question, which required students to sketch a graph of the function $x^{1.4}e^{-x}$ and find the stationary points, most students (49%) began finding the stationary points algebraically, with about a third changing to a graphical solution when their algebra broke down. However, irrespective of how the students attempted to find the stationary points, students did use the graphics calculator to obtain their sketch graph. Apparently, while students were taught how to find stationary points using a graphics calculator, many preferred an algebraic strategy. Under examination conditions, for many, the calculator was seen as a last resort.

Under-utilisation of the calculator was therefore a much greater problem than over-utilisation. Banning the graphics calculator on the grounds of over-use seems therefore unfounded.

Secondly, the presence of the graphics calculator could make solving the mathematical problems on an exam harder instead of easier. The presence of an independent source of information could possibly bring students into conflict, which they needed to resolve. On a question that asked students to give the points where the rational function $\frac{x^2 + 2x - 3}{2x^2 + 3x - 5}$ was not defined (part (a)), the limit of the function at one of those points with a bounded limit (part (b)), and the graph of the function (part (c)), the majority of students (73%) ignored the information gained from the first and second part of the problem, or were unable to reconcile the apparent differences between their algebra and the graphical information from the calculator. The graphics calculator does not show a gap in the graph at a point where a function has a bounded limit, unless that point falls exactly on the centre of a pixel of the screen. The students who saw the conflict (27%) between algebraic and graphical information, either ignored the graphical information from the calculator or ignored the information gained from answering parts (a) and (b) algebraically (see for example the work of students #21 and #30 on this problem in Figures 1 and 2). Only 14% of the students were able to integrate algebraic and graphical information.

A third finding about the use of a graphics calculator under examination conditions is that it can give the examiner a deeper insight into a student's understanding. To illustrate this point we can look at the work of student #4 on the question that asked students to graph the function $x^{1.4}e^{-x}$ and indicate the two stationary points in the sketch graph. Student #4 determined the stationary points algebraically. She then most likely entered the function $x^{1.4}e^{x}$ into her calculator (which is an exponentially growing function), but still marked the algebraically found stationary point on her sketch-graph where there obviously was none (see Figure 3). If the student did not see the conflict, she must have lacked understanding of what a stationary point means graphically. Another possibility is that she saw the conflict, did not know how to resolve it, ignored the graphical information, and relied on her algebraic work instead. From her algebraic work one might have drawn the conclusion that she basically knew what she was doing. The combination of the graphics calculator and algebraic work told a different story.



$$y = \frac{x^2 + 2x - 3}{2x^2 + 3x - 5}$$

$2x^2 + 3x - 5 = 0$
$(2x+5)(x-1) = 0$

(a) $x \neq 1$ and $x \neq -\frac{5}{2}$ or $-2\frac{1}{2}$

(b) $\lim_{x \to 1} \frac{x^2 + 2x - 3}{2x^2 + 3x - 5} \overset{(L'H)}{=} \frac{2x + 2}{4x + 3}$

$= \frac{2(1) + 2}{4(1) + 3}$

$= \frac{4}{7}$

(c) graph of $y = \frac{(x^2 + 2x - 3)}{(2x^2 + 3x - 5)}$

$\lim_{x \to \infty} y = \frac{1}{2}$
$= \lim y$
$x = -3, 1$

Figure 1: The work of student #21 on question 1 showing Strategy IIA, failed integration: algebraic preference.

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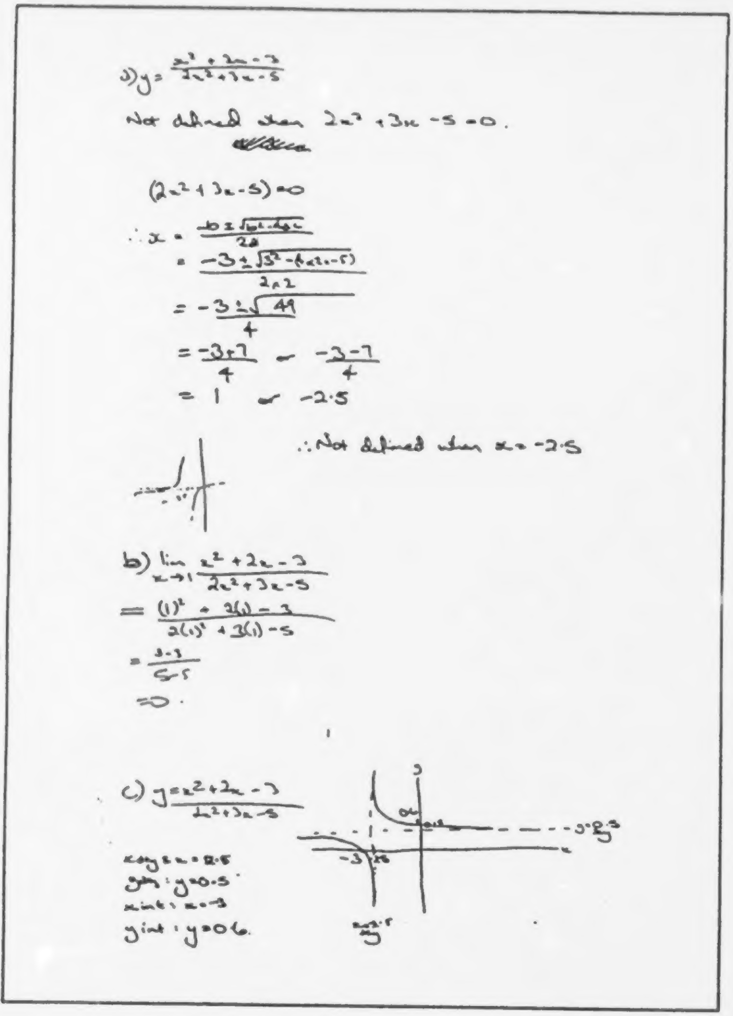


a) $y = \frac{x^2+2x-3}{2x^2+3x-5}$

Not defined when $2x^2+3x-5=0$.

$(2x^2+3x-5)=0$

$\therefore x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-3\pm\sqrt{3^2-(4\times2\times-5)}}{2\times2}$

$= \frac{-3\pm\sqrt{49}}{4}$

$= \frac{-3+7}{4}$ or $\frac{-3-7}{4}$

$= 1$ or -2.5

∴ Not defined when $x = -2.5$

b) $\lim_{x\to1} \frac{x^2+2x-3}{2x^2+3x-5}$

$= \frac{(1)^2+2(1)-3}{2(1)^2+3(1)-5}$

$= \frac{3-3}{5-5}$

$= 0$.

c) $y = \frac{x^2+2x-3}{2x^2+3x-5}$

x asy: $x = 2.5$

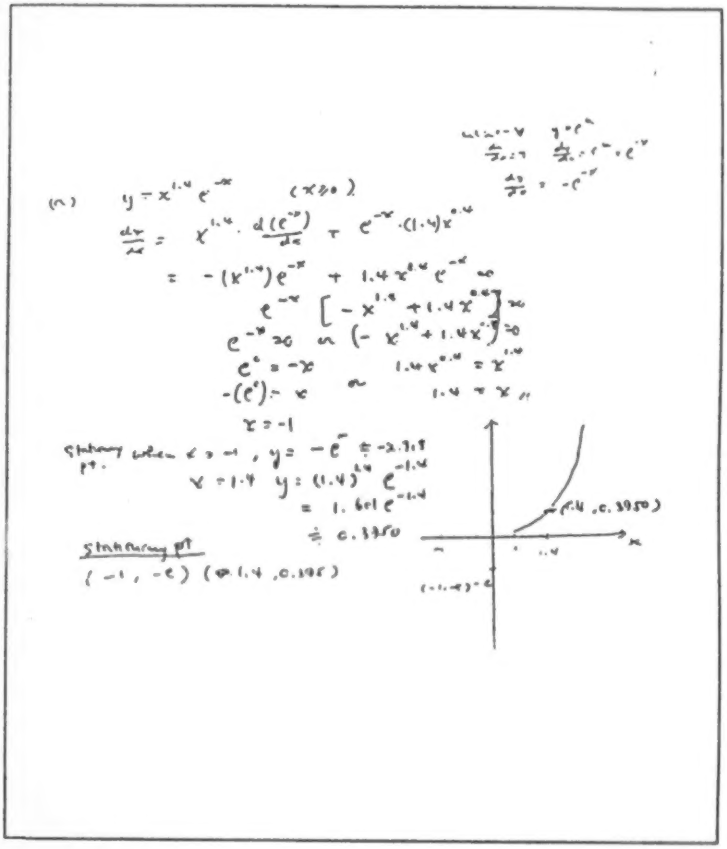
y asy: $y = 0.5$

x int: $x = -3$

y int: $y = 0.6$.

y
0.6
0.5
0.5y
-3
-2.5
x
x=-2.5

Figure 2: The work of student #30 on question 1 showing an example of Strategy IIB, failed integration: graphical preference.



let $u = -x$ $y = e^u$

$\frac{du}{dx} = -1$ $\frac{dy}{du} = e^u = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

(a) $y = x^{1.4} e^{-x}$ $(x \geq 0)$

$\frac{dy}{dx} = x^{1.4} \cdot \frac{d(e^{-x})}{dx} + e^{-x} \cdot (1.4) x^{0.4}$

$= -(x^{1.4}) e^{-x} + 1.4 x^{0.4} e^{-x} = 0$

$e^{-x} \left[-x^{1.4} + 1.4 x^{0.4} \right] = 0$

$e^{-x} = 0$ or $(-x^{1.4} + 1.4 x^{0.4}) = 0$

$e^{0} = -x$ or $1.4 x^{0.4} = x^{1.4}$

$-(e^{0}) = x$ or $1.4 = x$ //

$x = -1$

Stationary pt. when $x = -1$, $y = -e \doteq -2.718$

$x = 1.4$ $y = (1.4)^{1.4} e^{-1.4}$

$= 1.601 e^{-1.4}$

$\doteq 0.3950$

Stationary pt

$(-1, -e)$ $(1.4, 0.395)$

(1.4, 0.3950)
x
1.4
(-1, -e)

Figure 3: Work of student #4 on question 2(a) showing reasonable analytical skills but lack of integration of the analytical and graphical meaning of stationary point.

CONCLUSION

From the survey, one might draw the conclusion that the graphics calculator has had positive influences on the students' mathematical behaviour and attitudes towards mathematics. The 'brainstorm' session, although reinforcing those positive aspects found in the survey, pointed also to some of the dangers of the graphics calculators, such as, calculator dependency, which is often cited as a reason for banning its use. The evidence from studying the use of the graphics calculator under examination conditions seems to suggest that over-use and dependency were not a problem, at least during the exam. Students who did rely on the calculator as their main source of information, did not score highly in this traditional calculus subject; their strategy did not pay off. Of more interest to mathematics educators should be that the presence of the graphics calculator might make the mathematics more difficult because students might need to integrate two independent pieces of information, and that it can help examiners to gain insight into the students' levels of understanding.

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CONSTRUCTIVISM, LEARNING AND COGNITIVE REPRESENTATION; THE CASE OF FRACTION IDEAS[1]

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The assumption that mathematics is learnt by the individual construction of ideas, processes and understanding rather than through the transmission of pre-formed knowledge from teacher to student is now a commonly held belief among mathematics educators. An essential feature of this view is that existing conceptions, whether gained from everyday experiences or previous learning, guide the understanding and interpretation of any new information or situation that is met. As a result, there is often a resistance to adopt new forms of knowledge or to give up or adapt previously successful thinking, and the intuitive conceptions of children may appear very different to accepted mathematical practice. While much of the early support for constructivism has come from observations of situations where new knowledge has arisen from concrete situations, constructivism also needs to account for the more complex mathematics which has been formed by the processes of abstraction and generalisation of earlier ideas. The conventions that have emerged cannot simply be replaced by the idiosyncratic building of a host of individual learners; they need to be acquired in the same social context from which the mathematical concepts are to be drawn. This paper reports research which set out to establish a constructivist approach to the learning of initial fraction ideas, focussing on the social setting and activities which could lead to the negotiation and reconciliation of mathematics formed from historically and culturally determined generalisations.

CONSTRUCTIVISM AND MATHEMATICS EDUCATION

By viewing teachers and learners as active meaning-makers, continually giving contextually based meanings to each others' words and actions as they interact, constructivism challenges the assumption that meanings reside in words, actions and objects independently of an interpreter (Cobb, 1988). It suggests that the sharing or exchanging of mathematical thoughts and ideas is dynamic, reflecting a continually changing fit between the meaning-making of active interpreters of language and action. It also highlights the fact that old ways of thinking are not usually given up without resistance and emphasises that their replacement by or extension to new ways of thinking is guided by already existing conceptions (Duit, 1992). In contrast, traditional approaches to mathematics teaching and learning assume that students' everyday and prior conceptions have to be replaced by more mathematical ones, and there often appears to be no assumption of any prior knowledge or experiences at all. Seemingly, any existing ideas should simply be erased in the process of inculcating 'proper' procedures.

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[1] The data reported here is drawn from a BT(Hons) study conducted by Barbara Messinbird during 1991.

Much of the support for the belief in constructivism has come from observations of the learning of elementary mathematics (Steffe, von Glasersfeld, Richards & Cobb, 1983; Kamii, 1990; Cobb, Yackel & Wood, 1992). Further evidence has been drawn from the common difficulties that occur as complex procedures are built with inappropriate or inadequate generalisations of the concepts and processes that gave initial success (Confrey, 1990; Graeber, Tirosh & Wilson, 1990). It seems a short step from seeing how ill-formed constructions are made to assuming that even 'correct' mathematical forms can also be constructed individually. But seeing how appropriate constructions can be formed is not the same as knowing that they necessarily occur in this way. There is a compelling need to show and explain instances of children constructing for themselves these advanced ideas from among the various possibilities that could arise.

In particular, this requires viable models of cognitive development in specific areas of mathematics so that a learner's conceptions can be inferred and activities that might lead to a restructuring of conceptual understanding towards a particular mathematical goal can be selected or formulated. An awareness of ways in which learners might discuss, negotiate and resolve their possible constructions is essential for this, so that much current thinking in mathematics education is directed at the social conditions of learning and the specific activities in which the learning is situated. At the same time, constructivism reminds us that mathematical concepts are human inventions rather than objective truth; that there are different possible conceptual frameworks for explaining and acting on phenomena in a consistent manner. There is a need to account for the actual possibilities that arose in the development of mathematics as well as to be aware of the various possibilities that children might feel constrained to develop. In this way, the building of mathematical cognition can be viewed as a process of social (re-)construction with the conventions which rule mathematics acquired in the same social context from which the mathematical concepts are drawn. Consequently, the notion of situated, social knowing can replace the concept of knowledge as the development of routines and strategies which are only viable for coping with reduced complexities, are non-adaptive, and which have a damaging influence on self-concept and self-confidence (Bauersfeld, 1991).

LEARNING FROM A CONSTRUCTIVIST PERSPECTIVE

Constructivist approaches to learning seem to have taken on one of two forms; either conceptual change is avoided to some extent by enlarging or partially restructuring students' thinking, or some form of cognitive conflict is involved in the teaching sequence to create a situation where students exchange their initial constructions for something more akin to the mathematical viewpoint (Duit, 1992). From the perspective of constructivism as enlargement, a teacher starts with a learner's existing knowledge and, by means of a continuous chain of development helps him or her "climb up the different steps of the intended construction ... [so that] each step is an extension of accrued knowledge and this endows the learning process with continuity" (Herscovics & Bergeron, 1984 p. 195). This form of learning sees the teacher's role as a guide and raises the issue as to how much freedom the individual learner will have for her or his own constructions. But, so long as the formal end-points that the teacher conceived are seen just as that, end-points to which the learners might be oriented when they have constructed the essential mathematical meanings themselves, the danger of the teacher becoming simply a transmitter of formal knowledge can be avoided. Although, there is also the possibility that the most influential prior learning a student brings is a belief that there are exact answers, correct and efficient methods, along with the view that the major, perhaps only, task of the teacher is to hand these methods over to the learner.

The cognitive conflict form of constructivist teaching focuses on situations in which discussion, pertinent examples and examination of the limitations and advantages of particular points of view lead to the displacement of inadequate or inappropriate constructions by mathematically superior viewpoints. Situations are arranged in which contradictions in the students' own constructions can emerge in contrast to the alternative mathematical view raised by the teacher, text or some other arbiter of mathematical method. However, this is not always easily achieved, for, as Confrey (1990, p.37) says, "students cannot see their solution as erroneous until they have constructed a new problem and new solution ... this usually means revising their previous belief" and not just their method. Perhaps the notion of cognitive conflict is more appropriate when an inadequate or inappropriate construction has become ingrained, for, at an earlier stage, what is often called cognitive conflict is really just the process of argumentation and justification that is required to discuss and negotiate the mathematical meanings of a given situation. As Duit (1992) reminds us, the conflict arises first in the various possible constructions that might be formed:

> It is a central idea in constructivism that there is a dialectic relation between conceptions and perceptions. Conceptions guide perceptions and perceptions develop conceptions. Where the guidance of perceptions is concerned, one could say that humans tend to see only what their current conceptions allow them to see (p. 11).

COGNITIVE REPRESENTATION OF MATHEMATICAL IDEAS

The replacement of the notion of forming a conduit between the teacher's mathematics and the students' mental conceptions as a perspective for the learning of mathematics by a model of individual construction has had profound implications, not only for practice, but also for the relationship between mathematics learning and cognition. No longer is the object to be one of implanting an ideal structure in each student's mind, but to understand the student's conception for what it is. Cognition is necessarily concerned with individual conceptions of mathematical ideas, personally constructed from the experiences to which each individual has been subject but reflected in the particular way of viewing or interpreting these experiences based on prior knowledge and ways of looking at the world. New cognitive structures result from both the social and physical environment, and cognitive development needs to be explained by reference to principles of self-regulated change and interaction, in particular, how the artifacts and forms of social organisation, conceptual, symbolic or material products which have emerged over the course of social history, come to be interwoven with and are intrinsically related to the nature of children's intellectual constructions (Saxe, 1991 p. 4).

This is very different from the view of assimilating and accommodating new ideas to a knowledge structure by simply looking at the structure of the particular aspects to be taught to see how to best re-present them so as to replicate corresponding mental structures in learners. However, as Matthews (1992) highlights, too often "constructivism has failed to appreciate the reality or objectivity of human intellectual activity" and mathematics education also needs to be "conceived in terms of the appropriate introduction of individuals into this world of concepts, understandings, techniques and community standards". Saxe (1991) reminds us of Vygotsky's observations that scientific concepts are interconnected, comprehensive systems of understandings which have been elaborated and refined over the course of social history. They cannot simply be internalised but must undergo a complex transformation in their inward movement from artifacts external to the child's activity to the mental processes which make up an individual's cognition. Knowing is an evolving social practice, continually constituted by constructive activities of individuals, mediated by social interactions which bring about the meaningful adoption of mathematical conventions, and which allows some common perceptions to be taken-as-shared.

At the same time, by drawing attention to the need to account for cognition in terms of the learner's experiences and interpretations, a constructivist approach reveals the distinctive manner in which ostensibly similar tasks may be carried out in quite different ways or with differing degrees of success. The activity and context in which the learning takes place is more than just a pedagogical vehicle for discussion and meaning-making. Rather than being separable from, or ancillary to learning, the situation is integral to what is learned, co-producing knowledge through activity; learning and cognition are fundamentally situated (Brown, Collins and Duguid, 1989). Knowledge is, in part, a product of the activity, context and culture in which it is developed and used. Hence, to account for the developing cognitive representations an individual is building, it is necessary to consider socio-historical precedents for the ideas, the relativity of the context to the learner and the authenticity of the activity in terms of the experienced culture of the learner.

DEVELOPING FRACTION IDEAS

Fraction ideas are the first abstracted mathematics met by the young learner, and the difficulties experienced by child and teacher alike as the differences and similarities to the earlier whole numbers are met, grasped and reconciled are well known. A full understanding of fraction ideas would seem to require exposure to numerous rational number concepts (Kieren, 1988; Behr, Harel, Post and Lesh, 1992). An analysis by Messinbird (1991) provided a framework for the development of initial fraction ideas through the use of models, manipulative materials and games involving chance designed to give situated meaning as opposed to rule-like procedures that often dominate the learning of fractions. It was postulated that this would allow the children to:

> construct and reconstruct their knowledge, returning to earlier stages of development as reflection and observation identify areas of incompleteness. The complexity of the situation, with its established social norms, actions and interactions, cognitive conflicts and agreement ... is best experienced in a collaborative classroom. (Messinbird, 1991)

Although many children did manage to construct their own understanding of the fraction concept, including meaningful naming and manipulation of the ideas, the provision of rich experiences, both mathematical and social cannot be assumed to readily lead to mathematics governed by socio-historical conventions. Even with the use of carefully planned materials and engaging activities, children can acquire rote routines for forming answers rather than build for themselves the underlying concepts. Further, not all children will engage in the social actions required for negotiation of meaning, let alone participate in the conflicts needed to overcome cognitive obstacles resulting from inadequate earlier concepts. Indeed, children may well alter the conditions of the situation in which they are placed to avoid perceived difficulties altogether or simply to allow them to achieve their own objectives, which may be very different from those mathematical end-points the teacher had in mind.

These points are illustrated in the following excerpt from the video-taped records of the study in which Jo, Lizzie, Rebecca and Lisa played a game which used cards showing fractions more than one in pictorial form. Each player was required to read a card in two ways: in common fraction and mixed number form.

Jo: *Picking up a card*
Um. Three and ... yeh, three and one third

Rebecca: *And what's the other way? You got to say the other way to say it too.*

Jo: *Do you?*

Rebecca: (picking up the card and pointing)
Yeh, and you got to count how many are shaded and then say the fraction.

Jo: *Okay, one third.* (ignoring the fraction that made up the three completely shaded rectangles)

Rebecca: *No, like count all them that are shaded.* **Rebecca tries to impose her knowledge**

Jo: *Oh, four thirds* (counting the completely shaded rectangles each as one third instead of as three thirds) **Rebecca and the others did not appear to notice. Where does argumentation and resolution enter into the constructive activity when there is not a knowledgeable guide present?**

Rebecca: *Yeh, that's better.*

Rebecca: (picking up a card for her turn, counts) *Thirteen eighths or one and ... one and five eighths* (Jo watches carefully) **Rebecca models her knowledge for Jo.**

Lisa: (Picking up a card) *Um ... one and three fourths or um ...*

Jo: *Three fourths* **Despite watching Rebecca, her conception persists.**

Lisa: *Yeh* (seeming to follow Jo's cue and looking only at the partially shaded shape but then reconsidering) **Is it her level too, or does she just go along?**

Rebecca and Lisa together: *No*

Rebecca: *Count up...* **Direct instruction again - if teacher does not transmit, children often will!**
Jo seemed to guess instead of counting as Rebecca had suggested:

Jo: *Thirteen fourths?* (The fraction under consideration was 1 and 3 fourths)
Or had she? She appeared to have generalised from her recent experiences with decimal fractions, taking one whole as 10 equal parts, then added the three fourths to make thirteen fourths.

Rebecca: *No, like um ...*

Lisa: (Looking at Rebecca and asking questioningly) *Seven fourths?*

Jo: *No, I thought ... No, it would be three fourths, one and three fourths is three fourths.* **What is a fraction to Jo? Only part/whole? This belief is rife in out-of-school experiences and is situated in the introductory fraction activities with which she had success.**

Rebecca: *Yeh, but ...*

Lisa: *Yeh, but when ...*

Rebecca: *You gotta count up how many to get a four ...*

Lisa: *So there'd be four, five, six, seven, ...***Lisa has internalised the relationship of wholes to parts in general (in this case, 1 one is 4 fourths) and counts to show why, rather than counting all like Rebecca.**

Jo: *Oh, Yeh.*
Jo gives in to the pressure of the other girls' insistent answers

Rebecca: *Four and three ... seven fourths.*
Whereas Rebecca begins to reconstruct her view from Lisa's explanation and by re-thinking her own procedure.

Lisa: *Seven fourths ... Yes, that's what I said.*

Lizzie: (Picking up her card) *One and three fifths.*

Jo: *Or what else?*

Lizzie: *Um ...*

Rebecca: *Or* ... (waving her hands)

Lisa: *What's five and three?* **Lisa tries to impose her method on Lizzie in order to get to the social purpose of playing the game.**

Jo: (Answering for Lizzie) *Or eight fifths*
Beginning to respond to Lisa's procedure.

Lizzie: *Yeh.* (Putting down the card)

Jo: *No, it wouldn't.* (picking up the card)
No, because ... **This conflicts with her earlier view**
But Rebecca overrules the doubt, now using the thinking she has constructed from Lisa.

Rebecca: *Yeh, because it's five in it and three, five and three is eight.*
Jo seldom persisted when she did not understand. Even when she could solve a problem successfully, she was quite prepared to be overruled by the more dominant students. She seemed to regard her social position in the class as very important and rarely created or continued a conflict. To her, capitulation was more acceptable than confrontation, negotiation and resolution. As a result, Jo's construction of fraction knowledge remained incomplete.

Jo: (Picking up the card for her turn) *Okay* (pointing and counting) *Um, one, two, three, four. Four and one third or ...* (pointing and counting again) *one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve and one fifth ... Oh! ... or thirteen and three, thirteen ...* (looking puzzled)

Rebecca: *Thirteen thirds if they're equal ... Yeh, thirteen thirds.* (sounding impatient)

Jo: *Yeh, would it be?* (in a questioning tone to Rebecca)

Although Lisa and Rebecca constructed appropriate, sophisticated strategies for renaming improper fractions and mixed numbers, Jo and Lizzie often avoided situations where other players checked each person's results, ostensibly to speed up the playing, but actually to avoid the conflict with previous ideas. In this way, the social demands of playing were used to avoid constructing and coming to terms with the social conventions of the more advanced mathematics.

TENSIONS BETWEEN THE SOCIAL CONSTRUCTION OF MEANING AND THE SOCIAL CONVENTIONS OF THE CLASSROOM

This episode shows how, even with the use of carefully planned materials and engaging activities, children can acquire rote routines for forming answers rather than build for themselves the underlying concepts. Further, not all children will engage in the social actions required for negotiation of meaning, let alone participate in the conflicts needed to overcome cognitive obstacles resulting from inadequate earlier concepts. Indeed, children may well alter the conditions of the situation in which they are placed to avoid perceived difficulties altogether or simply to allow them to achieve their own objectives, which may be very different from those mathematical end-points the teacher had in mind.

It also raises concerns about the manner in which we might need to structure a constructivist classroom if the type of learning theoretical considerations lead us to anticipate are going to occur. Our object as mathematics educators is now to provide rich situations and experiences out of which particular mathematical concepts might be able to emerge, to guide and facilitate the construction of mathematical ways of looking at these situations, and to come to terms with the ways of thinking that each individual has formed in order to do this. In this way we may be able to make sense of each student's mathematical world-view and endeavour to assist students to build for themselves the socially-accepted concepts, processes and, sometimes, even the procedures of what is felt to constitute mathematical knowledge. In time, and with further experience, we may be able to develop situations in which "the outcomes or products, which from a psychological perspective are described as "mathematical knowledge" appear as social accomplishments of the specific culture" (Bauersfeld, 1991 p. 11).

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MATHEMATICS EDUCATION IN FIRST YEAR PRE-SERVICE PRIMARY TEACHERS

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What are these students' approaches to mathematics education and, in particular, what are the differences in approach, between passing and failing students? The term "approach" is used to suggest a broad underpinning of a student's thinking about knowledge, a relatively persistent characteristic, changing only gradually, and, as proposed here, plays a powerful part in a student's learning behaviour.

THE INVESTIGATIONS

Students' views of mathematical knowledge were investigated to determine to what extent mathematics was integrated into their ways of thinking about the world, to what extent they appeared to value knowledge for its intrinsic interest and its usefulness to them. Did the students failing the First Year Mathematics unit regard mathematics as a series of compartmentalised rules and processes, and separate from their informal ways of quantifying? Were there successful students with these views?

Finding evidence of these views was investigated in three ways; A and B with 92 students, 15 of whom failed, 77 of whom passed, the First Year mathematics competence and methodology unit; and C with these groups and two case studies.

A **Error analysis of working and modelling division problems**
This was done to assess procedural skills and instrumental understanding, and whether students could use models to show how and where an operation might be used.

B **Connection-making**
Which students were making connections in mathematics? Success in a concrete version of the problem and lack of success in an abstract one was taken to mean that connection-making had not happened and that the student's view of mathematical knowledge was likely to be compartmentalised. Relational understanding was less likely (Skemp, 1976). Evidence of successfully completed operations on division and ratio in different contexts was assumed to mean that connection-making could have taken place.

C **Learning behaviours**
In this part, how students said they learned, the language they used to describe their learning and mathematics, and their associations with mathematics were compared for two attributes, their quality of learning and their orientation on mathematics and learning generally.

i) *Quality of learning*
Saljo's concept of Quality of Learning Scale (Figure 1) was used here to gauge learning behaviours that the case studies described. This is a scale

produced for assessment of how students learn; to distinguish between a quantitative change and a qualitative change, the acquiring of principles and control over knowledge, rather than just the amassing of facts.

Saljo describes behaviours, such as remembering a few terms from a lesson, or simply rote learning information, at levels 1 and 2. Here, where there was evidence of faulty, rule-bound behaviour, a lower quality of learning was assumed.

Level	Type of learning
1	A quantitative increase in knowledge
2	Memorising
3	The acquisition of facts, methods etc which can be retained and used when necessary
4	The abstraction of meaning
5	An interpretive process aimed at understanding reality.

Figure 1: Saljo's Quality of Learning Scale (1984)

The quality of learning above Level 2, is of a more interactive nature. At the middle level, one indicator is a student's tendency to select and manage a number of mathematical procedures appropriately. It was assumed that many of the students just passing the First Year Unit would be learning in this way.

At the upper levels, students would understand the effects of operations, make sense of procedures, ask why these work, connect them with earlier experiences, and apply them.

ii) *Orientations on mathematics and learning*

What are barriers to these more interactive learning behaviours? Are they attributable to students' orientations on knowledge? Current research indicates that a student's knowledge definition and the ways he or she acquires knowledge, are interrelated (Wilkinson, 1989; Copes, 1985; and Buerk, 1985).

The schemes of Perry (1968) and Belenky, Clinchy, Goldberger and Tarule, (1986), were used as a guide to views typically held about knowledge. These are developmental schemes, devised as a result of making longitudinal studies of undergraduates of Humanities at Harvard, and women returning to tertiary education, respectively. In this study of pre-service primary teachers, their orientations on mathematics were grouped into three broad brackets, I, II, and III (see Figures 2, 3, 4).

Where students' viewed mathematical knowledge as discrete rules and procedures, necessary to pass examinations, perhaps made more acceptable by being seen as a part of the primary school curriculum, but nevertheless, as separate from themselves, this orientation was placed in Bracket I. The quality of learning associated with this, is likely to be restricted. According to Belenky et al., student with this orientation, were less likely to value their own thinking, their "subjective" knowledge and relate it to the "objective" knowledge being taught.

THE CASE STUDIES

Two students who had failed the mid-year examination in their First Year, were interviewed, to discover to what extent they held such views. Alison had one of the lowest marks in this examination, and Adam had only just failed. Alison was interested in learning what she understood or found relevant:

Alison — I've always liked learning - new things - but then there are just some things that I find irrelevant or boring.

(Maths) just - it's never been my forte.

Even at primary school I never liked it, and like, fractions I hated, and decimals up till now, which .. It's because I understand them now, that I like them - so that if I understand something, I like it in maths."

This sounds as though she is ready to learn, as though the opportunity is there to improve this relationship with mathematics, but her preparation for examinations involves a heavy load of rote learning, and apparently little of the making sense behaviours.

Interviewer — Could you tell me generally about your learning, not necessarily in connection with maths?

Alison — Um - I can learn, like in P.E. for example last year I would learn my notes word for word off by heart, and write word for word in the exam.

That's how I would learn things whereas I was told it would be much easier if I summarized notes and then learnt the basic things about - even English and things like that, if I summarized notes and then just basically learnt what they were saying it would be much easier, but I found it easier to learn the text word for word.

Interviewer — This year...?

Alison — This year I've done it again. This year in P.E. I've learnt word for word definitions and things. And if I don't ... If they're not on the exam I think, Oh, it was a waste of time, learning all that but ... For exams I learn everything I have done

Alison has been given advice to make notes but is not relating to the subjects well, and has not done this, perhaps because of her orientation, rather than lack of motivation.

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The quality of her learning is clearly at Levels 1 and 2 of Saljo's Scale, in Bracket I.

Adam, on the other hand, has procedural skills, characteristic of Level 3, ("...in my Skills Test, I got 100%"). His discussion of several calculations that he made, suggested that he worked well at a rule-driven level, but was not concerned with understanding the effects of division, Bracket II. So that in spite of his competence, his quality of learning was not what one would expect of learning at Levels 4 and 5.

Alison in the classroom:

Her approach to learning for examinations / assessment appears to play a large part in her learning behaviour in class.

Interviewer	Does that mean you work very hard?
Alison	Yes. for exams and assignments I'll do very hard work whereas in the classroom I mightn't feel like learning anything and I'll do it in my own time.
Interviewer	Does that mean you don't have a lot of confidence about what's going on in the classroom?
Alison	Oh, no! No, I just - some things I find irrelevant, like Base Five. I learnt to do it. But then I knew I wouldn't have to do it when I taught. It's not in the school curriculum.

The base five topic was intended to provide an opportunity to look at the four operations, and algorithms, in an unfamiliar base, from the perspective of a beginner, and in order to focus on the structure of our decimal base system. Her response, like many others', to this, is not positive. Her reasons, very much those associated with a "received knowledge", Dualist perspective, for neglecting it are reasonable, considering her view of the primary curriculum and mathematics. They are not helpful in developing her understanding of the number system and how children learn about it. Her view is a limiting factor in the development of her learning.

THE SCHEMES OF INTELLECTUAL AND PERSONAL DEVELOPMENT

The first three major subdivisions of Perry's scheme are set into the three brackets.

BRACKET I: THE DUALIST

1 Basic Dualism: World viewed in polarities of right and wrong, passive learners, dependent on authority to teach right from wrong.

BRACKET II: THE MULTIPLIST

2 Multiplicity: Awareness of diversity of opinion and multiplicity of perspectives. Authorities may not have the right answers at least in humanities; also everyone has a right to his own opinion.

BRACKET III: THE RELATIVIST

3 Relativism Subordinate: Cultivation of an evaluative, analytical approach to knowledge in an academic area.

4 Relativism: Full shift to relativism where student comprehends that truth is relative, that meaning of an event depends on context and on the framework used. Relativism pervades all aspects of life not just the academic. Students understand that knowledge is:

constructed	not absolute,
mutable	not fixed.

5 Affirmation of Identity and Commitment: These follow with the belief that until the framework changes the student will act in the current light.

Figure 2: Perry's Scheme: Intellectual and Ethical Views of Knowledge

Adam shows several characteristics of Perry's Dualist and Multiplist categories. Adam is more assertive than Alison, and the way he makes decisions indicate a muscular view of "Authority" that Perry describes.

Adam's approach to C & M Mathematics half way through First Year

Interviewer ... in the months that you have been here, is there anything that you have found significant to you as a learner?

Adam There are several other ways of doing questions and tackling questions.

For example ... I've always thought right before the course ... I'm going to teach it this way, because it's done this way.

I've always learnt that way and it always will be done that way, but there are other ways - like subtraction - there's decomposition, equal additions.

Adam's new found interest in mathematics education was in the possibility of showing more than one way to do an algorithm. During this interview he mentioned "different ways" sixteen times, an indication of the delight and relief he felt at finding these alternative algorithms. This is the expression a Multiplist, Bracket II, willing to cope with diversity, in certain areas of thinking.

Adam Look, if I'm teaching a grade I'd do it both ways. Now I'd do it this way _ equal additions _ now we can always do it another way, and I've done the decomposition and whatever feels right to them they could do. I mean I'm not going to force them like we were forced to do equal additions.

His approach to mathematics knowledge is still, in some senses, Dualist, concerned with what "Authority" allows in this area, but he is, what Belenky et al. call invested in the knowledge of this operation.

Alison is probably better described by the concepts of Belenky et al.'s scheme, because of its focus upon "subjective" knowledge. She is very much the Recipient of Bracket I, in this area of mathematics.

Belenky, Clinchy, Golberger and Tarule's scheme of the ways women construct their experience, 1986

In this a strong element of personal development, assertiveness, and integration of public and private knowledge influences learning behaviours.

BRACKET I: (RECIPIENT)

1 Silence: Women as mindless, voiceless, subject to the whims of external authority.

2 Received Knowledge: Women capable of reproducing knowledge of their own on their own.

BRACKET II: (COMMUNICATOR)

3 Subjective Knowledge: Concept of truth and knowledge, personally or privately known or intuited.

4 Procedural Knowledge: Women are invested in learning, applying objectives and procedures for obtaining and communicating knowledge.

BRACKET III: (CONTEXTUALISER AND CREATOR OF KNOWLEDGE)

5 Constructed Knowledge: View of knowledge as contextual, women experience themselves as creators of knowledge, and value both subjective and objective strategies of knowing.

Figure 3: Scheme II: Women's Ways of Knowing

SUMMARY OF MAIN SAMPLE QUESTIONNAIRE RESULTS

Connection-Making in Mathematical Workings

The following questions were set to note what connections were being made; and to decide whether connection-makers were successful in passing the unit.

i) An abstract problem in decimal division that students typically get wrong 2 -- 0.4 =

ii) A problem that students can do, related to this, estimating how many items they can get for a sum of money.

"How many 41 cent stamps would you get for $3?"

iii) What images do they make for themselves of a decimal division problem in mathematical code?

"Write a real world problem that accurately reflects the mathematical sentence, 2 -- 0.4 "

(eg. Set in a money context:

$2, how many lots of 40 cents?)

iv Do they manage to relate decimal division to a number line model of the operation? "Show how this division problem would look modelled on a number line."

Figure 4: Assessing Competence and Connection-making

Mathematical Workings and Connection Making

Connection-making would be considered at least in Bracket II, and could indicate Bracket III orientations and quality of learning. The results here are mixed, but point to the likely compartmentalisation of mathematics by the Repeats and a number of the CM200's.

At most, a third of passing students are possible connection-makers.

Most students could do the Stamps problem (ii), with the correct change, CM200's 90% and Repeats, 80%.

40% of CM200's could dot the abstract division problem (i). None of the Repeats could do either, confirming their weakness in procedural skills in abstract contexts.

A third of CM200's could supply a real world model for the abstract problem. 20% of Repeats could do this, and yet did not relate it back to the earlier question, to get the answer to the abstract problem. Connection-making, as an indicator of orientation on mathematics, suggests here that 20% of Repeats do this on occasion. In neither group is it very evident.

Students' Associations with Mathematics

About a third of both groups associated mathematics with "Boredom". More than half of the CM200's associated it with "Rote Learning", Bracket I learning behaviour, whereas, fewer, (40%) of Repeats did this, perhaps because they knew they were not such successful rote-learners at this stage, the beginning of their Second Year in the course.

A third of CM200's associated mathematics with "Imagination" and "Creativity", (Bracket III) possibly the third here are the "Connection-makers" above. Less than 7% of the Repeats did this.

Sources of Help when Stuck

CM200 students mentioned more sources of help than Repeats. Repeats (6%) were less inclined to call upon a lecturer for help than CM200's, (22%), and more inclined to go to their families for help, (12%, as compared with 3%).

Preferred Method of Instruction in Mathematics

40% Repeats wished to be shown how to do an algorithm, and how only, (Bracket I), whereas 14% of CM200's wanted this.

Over half of both groups wished to be explore several ways to solve a problem, suggesting that not everyone who failed had such Dualist orientations.

CONCLUSION

A clear picture of orientations and the relationship between orientations and learning behaviours, came from the interview data. It was apparent that the two students had orientations in Bracket I, particularly in relation to learning for examinations. These could have played a major part in their learning difficulties.

The similarities and differences which appeared, between the passing and failing groups are informative, but suggest that the Repeats cannot be thought of, in every respect, as one Bracket below the passing students, and vice versa. The stages described explain in part the success and lack of success in their formal college assessment. A follow up survey, given nearly two years later, as students were finishing the course, should when analysed, supply information about trends in the areas surveyed.

If students do appear to progress through the stages of the schemes, further study of individual students' views of mathematical knowledge would be worthwhile, as from such a longitudinal study, results could be used to devise a more appropriate curriculum, to promote more effective orientations.

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SENSITISING PRIMARY STUDENT TEACHERS TO YOUNG CHILDREN'S MATHEMATICAL IDEAS

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Research has shown that young children possess a considerable store of mathematical ideas (particularly about number) when they start formal schooling. This paper describes an attempt that was made to use these ideas as the starting point for a course in mathematics education for first year students at university. Students were given an interview schedule and were asked to interview one or two five year olds over two sessions. Between interviews the students met back at university to discuss and reflect upon the children's responses, and to prepare for the second interview. In a questionnaire administered one week after the second interview, the university students were asked to comment on what surprised them about the five year olds' responses. These were, in order of frequency, that the children knew so much "maths", that they were so open and co-operative, and that their ideas and strategies were se different from those of the interviewer. The implications of these are discussed in the paper.

INTRODUCTION

Young children possess both a store of ideas about number and the ability to use these ideas. This has been consistently shown by a number of studies (Carey, 1991; Carpenter and Moser, 1984; Hughes, 1986; Young-Loveridge, 1991) and is now generally accepted by mathematics educators. How might a group of university students embarking on their first mathematics education course perceive the mathematical ability of young children?

We decided to begin a new mathematics education course with first year university student teachers by taking them into schools to interact with five year olds. There they would interview one or two children. The reasons for this approach were that we wanted the university students to become sensitive to children's ideas in mathematics early on in the course, and we felt it might be a suitable introduction for those students who were themselves maths anxious - a problem with many teachers in training (Nisbet, 1991). As well, we wanted to have the course members gain a wider view of mathematics education than the one they might have developed from their own classroom experiences.

These were the major questions for the author:

1. were the students at all surprised by the mathematical knowledge of the five year olds, and

2. what was surprising about what the children did/said (from the student teacher's perspective)?

INTERVIEWING THE CHILDREN

An interview schedule was devised by a member of the teaching team (Biddulph, 1992). The schedule contained not only questions that might be asked, but also described suggestions for interacting with the children. In particular, guidelines were given for how the students might ask questions, put the interviewee at ease, ask gentle, probing questions, and avoid telling.

The interviewing took place on consecutive weeks in March, so many interviews were held in an appropriate outdoor setting. Most were audiotaped and each lasted about 40 minutes.

METHOD

102 first year university students were administered an open-ended questionnaire. The subjects, 75% of whom were female, ranged in age from 18 to 45 years of age, with a mean of 20. The questionnaires were completed in class time at university under the supervision of the course lecturer. The answers to one of the questions is the topic for this paper.

The question asked was, "What things surprised you about the way the child(ren) responded?"

This question was part of a wider questionnaire that tapped into a number of issues to do with the interviews. These included whether they expected to begin a course by interviewing children, what the most difficult aspects of working with a child were, and what influence the experience might had had on their ideas about the learning and teaching of mathematics.

As well, the students' course log books were used to get information about what aspects of the interviews surprised them.

RESULTS

The student teachers were surprised by 18 different aspects of the child(ren)'s responses. They were as follows (in order of frequency):

1. Thirty-four students wrote that the children had a greater-than expected knowledge of mathematics. Typical statements from the questionnaires that reflected this were:

> I did not expect my child to be so knowledgeable in the area of mathematics. She was well informed on numbers and she could count in twos, fives, tens and so on. She knew many types of shapes including hexagons, pentagons.
>
> Children have a great deal more mathematical knowledge and skill (at a junior age) than I previously thought.
>
> Emma had only been at school 5-6 months and was far more advanced and beyond my mere mortal range of questions. Unlike most kids

> Emma not only gave me answers but a lot of the time she explained to me why this was so without any prompting from me.
>
> My child seemed to know an awful lot. One time I was really surprised when she did something which I still don't understand how she did it.

Fifteen students who replied in this way were asked to elaborate on their statements. Their replies indicated three main areas of "number strength" that surprised them. These were the child's ability to count in a rote and/or meaningful fashion, their knowledge of the operations, and their competence at solving simple word problems. Responses from the interviewers were:

> The thing that surprised me was that a five year old could count so high and could establish where a number would be placed - before or after.
>
> My child could rote count for ages. I stopped him at 100 but he could have gone on.
>
> Meaningful counting - it was interesting that she didn't get confused when things meant something to her.
>
> I was surprised at how well the child I was given, when asked the question, "If mum washes 2 pairs of socks in the family for everyone, how many socks will be hanging on the washing line?", could answer. He did struggle to start with, but basically knew what I meant and could make a start.

2. The second most-mentioned aspect referred to the interaction between the adult and child- seventeen university students were surprised at the openness, candour and co-operation of the children.

> Very talkative and wanted to show me everything that she could do.
>
> The child I was working with was very open and co-operative and did not hold back in the least.
>
> They were keen to work with me.
>
> Their (the children's) answers were often explained without asking - no inhibitions.
>
> He always tried. Even if he didn't know how or what to do, he'd look away and say, "Let me see ..." He always came up with some way of doing it.

3. Twelve students commented on the "unusual" nature of the childrens' ideas and explanations. Some of them realised this was because they were comparing the child's responses with their own.

> How they use logical, simple methods to solve problems without using the ways I had been taught. My boy didn't know about using a ruler to measure the difference between our strides, but he marked where our strides ended with leaves and stood back and saw the difference and then could accurately conclude whose stride was the longest (sic).
>
> For one of the activities he turned it into a game which I wasn't expecting, and would sometimes do things just a little bit differently than I thought I'd explained them.
>
> Some answers to questions were explained to me in a logical way that I'd never explained to myself. But they were very clear to the child.
>
> Their logic seemed totally up the creek at times, and ridiculously straight forward at others.

4. Eleven of the 102 students were surprised by the children apparently "knowing" something one week, only to find on their return a week later that the child had forgotten. As well comments were made about "knowing then not-knowing" within the interviews.

> Sometimes he knew what a rectangle or circle was called in one session, then 10 minutes later referred to it as an oblong or a round.
>
> She also surprised me in the change in her ability through the amount she knew - one week she counted higher than the week before.

The different answers that some children gave to the same question were a concern for some. One student commented, "If I asked the same question again later in the lesson, the answer given was nearly always different from the original. This made me worry about the **true** understanding of any mathematical concepts" This is a perceptive comment that gets to the core of many of our concerns about assessment.

5. The next most often mentioned aspect was how **little** mathematics these young children knew. It might be interesting to know of the background of the students with this view. Were they school-leavers? Or more "mature" students who had had experience in interacting with children?.

> It surprised me how little my child knew about maths.
>
> I was surprised with what they didn't know. I guess I expected New Entrants to have the knowledge of a more advanced child.

The remaining 13 categories of surprises came from smaller numbers - ranging from six to a single respondent. They included:

- how confident the children were
- their ability to use a calculator
- the strong influence of home
- how easily the child became bored and restless

DISCUSSION AND CONCLUSION

The data show that a group of 102 student teachers gave very diverse views on what surprised them about the mathematical knowledge and understandings of five year old children. However, the four most common responses accounted for over three quarters of the total. By far the most common "surprise" for the group was the extent of the knowledge and ability that the young children possessed, particularly in the general area of number. Looking towards the future, this may help these student teachers become more sensitive to the learning needs of children. It may also help them decide on appropriate programs for five and six year olds, and look critically at proposals that de-emphasise extensive work with number (in a variety of contexts) on the basis of "readiness" arguments.

The data suggest that the student teachers were forced to think about the characteristics of children's thinking and mathematical ideas. Earlier quotations from the questionnaire in this paper support this view. This outcome was perhaps assisted by the group discussions and reflections that were held in the school immediately after each interview, and five days later back at university.

Comments reported here also suggest that the student teachers were having to think about how they might work with children. One wrote, "The children knew some jargon without seeming to really know what it meant (eg "square"), but somehow seemed to understand a concept without the technical words. This surprised me in that what the children knew or understood was quite different to surface appearances of what their knowledge was." This student teacher may now be more wary of a "telling" style of teaching where the learner's existing ideas are often ignored, and where they may be few attempts to probe below the surface. As well, s/he may become more sensitive to the role of language in mathematics (cf Hughes, 1985).

From the data reported here, and a course evaluation that was administered to the 102 university students at the conclusion of the course, we now believe that the students have grown in their ability to:

1. listen carefully to children and not be satisfied with initial responses;
2. appreciate that children have their own views of the world;
3. recognise the advantages of working with children in the school setting;
4. work with children in a relaxed and supportive manner;
5. use the technique of interviewing in order to better understand children's mathematical ideas.

The course, of which the interviews were a significant component, continues to be evaluated by the teaching team.

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GUIDED PROBLEM SOLVING FOR FIRST YEAR TERTIARY STUDENTS

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Although over the last few years there has been much effort devoted to introducing problem solving and modelling into our teaching of mathematics, up to now there has been little progress on this at the tertiary level. At this level there appears to be just too many problems to overcome and little research has been undertaken. In this paper reasonably efficient problem solving guided approach that addresses some of the issues is proposed and some accompanying resource material presented.

INTRODUCTION

Teaching mathematics using a problem solving approach has been with us for a long time and much has been written on the subject but at least at the tertiary level not much has been achieved in the way of implementing a problem-based program. A serious obstacle may be that it is difficult to integrate problem solving in subjects (such as the typical first year tertiary mathematics) that are necessarily content driven. In this paper a solution is presented - a problem guided approach where problems are used to motivate ideas and theory.

THE PROBLEMS

Here are some of the well-known symptoms associated with the teaching of mathematics at the first-year tertiary level:

1. High attrition rates in many mathematics units.
2. Decay in numbers of students actually attending lectures and tutorials over the semester or term.
3. Many students not doing enough work in their own time (homework or assignments).
4. Students' lack of motivation.
5. A low pass rate or "lower standard" in many mathematics units.
6. Mathematics courses typically having small enrolments.

Although there are notable exceptions, these symptoms appear to be somewhat independent of the style of the lecturer (within reason) and even of the country (see, for example, MS2000 Committee, 1991) so it seems that something else is wrong. Often students themselves are blamed and while it cannot be denied that there has been an overall drop in the quality of students attending tertiary institutions - they are coming in record numbers and the overall quality must therefore drop - as their teachers we must work with what we have, not with what we wish we did have. We must make some changes now and there are many possibilities. Here are some problem areas that have been and continue to be addressed by many teachers at many institutions.

1. The lecture/tutorial system. This doesn't seem to work for many (most?) students studying mathematics. Mathematics can[not] be learned by watching someone else do it correctly" (MS2000 Committee,1991).

2. The lack of preparation of incoming students. Many teachers at the tertiary level are concerned that they are continually seeing a drop in the level of preparation of students.

3. The subject matter of mathematics subjects often does not relate to students' other subjects.

These are obvious problems and there are some obvious "cures"; for example, having students in smaller groups and/or forming special learning centres, implementing bridging mathematics subjects, splitting groups of students up into discipline-based groups. But there are other, more serious and fundamental, problems, that are not so often tackled though they are just as well-known.

4. There has been and continues to be an undue emphasis on purely manipulative techniques and associated "standard" problems that can be done by having the students follow a "recipe". And while it is definitely important that students become proficient at manipulative skills it is obvious that simply teaching them in association with the "standard" problems doesn't work. Many (most?) students will be unable to cope if they are given a slightly novel problem that involves just relatively simple techniques.

5. Mathematics is traditionally taught as a series of seemingly unrelated topics, usually in different subjects (Precalculus, Calculus, Vectors and Matrices, Probability and Statistics) yet we wish the students to ultimately be able to draw on all tools as needed.

6. The problem mentioned as problem 2 actually goes much deeper than is often recognized. Many students lack skills as "elementary" as the ability to work with fractions and square root and .... This deficit becomes especially clear when students are required to use elementary techniques in a novel problem (as mentioned as problem 4) but even a purely mechanical question dissociated from any practical problem can cause trouble for many students.

7. The problem mentioned as problem 3 is actually a little too specific - the real problem is that there is often an almost complete lack of any motivational material, course specific or otherwise.

8. Textbooks are mostly all the same (and take an approach that only makes the above problems worse).

PROBLEM SOLVING AS A CURE

At the tertiary level, a problem solving approach to teaching mathematics seems to offer cures for each of the problems listed above. The "cures" are numbered to indicate the respective problems that they address.

1. Problem solving is best done in smaller groups (workshops) and thus "forces" (to some extent) teachers to abandon the lecture/tutorial system.

2. Problem solving forces students to develop and use all their mathematical skills together. Thus it provides a mechanism for students to work on their basic or elementary skills (problems 2 and 6) as well as the new material they are supposed to learn.

3. Problem solving should increase and maintain motivation provided that the students can relate to the problems studied (problems 3 and 7).

4. By using a problem solving approach we obviously help students in learning to use mathematical tools to solve problems (problem 4). And if we are careful we can still cover the range of techniques and mechanical skills that they would be presented with (and fail to learn!) in the more traditional approach.

5. Problem solving forces an integrated approach to the teaching of mathematics (problem 5). It is almost always the case that a problem involves the use of techniques from a range of traditional topics.

6. If problem solving is to be used, new textbooks and other resource material must be developed for students and teachers.

Given the above advantages of the problem solving approach, it may seem somewhat surprising to find that most teachers at the tertiary level are reluctant to try it. Indeed, many teachers are reluctant to try any new ideas at all and instead adopt the traditional approach (lecture/tutorial system with one of the standard textbooks that does not use a problem solving approach). There are many reasons for this:

1. Tertiary mathematics subjects, especially the all-important service subjects in Science, Applied Science and Engineering are necessarily content driven and teachers at this level are under considerable pressure to cover a large amount of required material. Unfortunately, a standard problem solving approach is more demanding in terms of the main resource, time, than is the traditional approach. Students simply do not have the luxury of being able to discover and experiment for themselves. A solution is presented in the following section.

2. There is a serious lack of resource material. Current textbooks nearly all take the traditional approach and are more-or-less all the same. Potter (1990) points out textbook publishers may be partially to blame. A solution is presented in the following section.

3. There is a general lack of knowledge and experience as to what techniques for teaching problem solving actually work at the tertiary level.

4. In spite of much rhetoric, there is still very little incentive for teachers at the tertiary level to experiment and improve their teaching. Promotion is based primarily on research, not teaching. In fact there appears to be so little incentive for experimentation that about the "best" that many teachers at the tertiary level have so far managed to do is introduce some assignments or tasks that involve the use of problem solving techniques to some extent while keeping the rest of the subject more-or-less the same.

THE PROBLEM-GUIDED APPROACH

In this section we present a program and related support material for an approach to teaching first year tertiary mathematics that employs an essentially problem solving approach while taking into account the essentially content driven nature of this subject. The basic idea is to use what may be called a problem guided technique and relies very heavily on a textbook (1992), which was written especially for the subject. The book consists of two parts - a problem solving part and an overlapping theory part.

The problem solving part of the book consists of a set of carefully chosen problems (applications) that are presented one after another in a sequence designed to introduce key ideas in a logical order ("elementary" to "more advanced"). There are many associated problems for the students to do, too. Together these applications are designed to form a more-or-less complete book on their own, but, and this is what is designed to satisfy the content driven nature of the subject, interspersed within them is another set of notes that contains the relevant theory and skills and techniques (and many associated drill exercises) that arise in or are needed for their solution. Ideally this "theory" sub-book is intended to tag along as needed. Sometimes sections appear at the end of an application (if the solution is sufficiently short); other times they may pop-up in the middle of the solution to an application because some new theory or idea needs developing. In reality, however, the sub-books were written together so that ideas are developed in a logical order and, in fact, the "theory" sub-book often was designed before the associated application in the "problem solving" sub-book was found.

So the whole book is therefore formed by interspersing two intimately related sub-books - "problem solving" and "theory". It is sometimes difficult to separate the two and they are necessarily heavily interdependent, but the logical difference is usually clear. They are clearly distinguished from one another in the entire set of notes by having them printed in different colours or on different coloured backgrounds.

An example of the problem - guided approach will help make the idea clear. The topic is L'Hopital's rule and the application used to introduce the idea was actually used earlier in the book to motivate the whole notion of a limit - quite a large section of work followed where some elementary techniques (using a calculator, graphical techniques, use of "algebra tricks", etc) were presented together with the use of a symbolic manipulator and a discussion of the Squeeze Theorem. Now the problem is tackled again. In the book the application and its solution were printed on coloured paper to separate them from the associated theory. Here this material has been placed in a box. Some details have been omitted and additional comments added (in italics).

Example (taken from Carr (1992))

Recall Application 310:

This does not mean that there have been 310 applications - all items (Applications, Definitions, Examples, etc) are numbered sequentially.

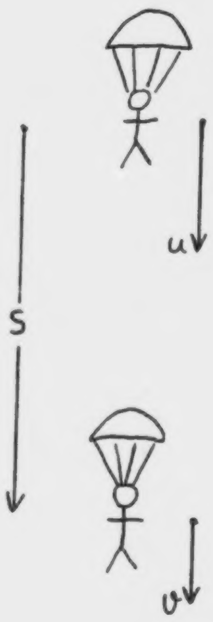
APPLICATION 310. After opening a parachute, the distance s travelled by a parachutist of mass m can be shown (see **449** - this refers to a previous problem) to be related to the velocity v by

$$(*) \qquad s = \frac{m}{2k} \ln \left(\frac{mg - ku^2}{mg - kv^2} \right)$$

where k is the drag constant and u the speed of the parachutist when the parachute is opened (v can be greater than or equal to u or less than u). This should be a generalization of the familiar physics formula relating velocity with displacement without resistance:

$$v^2 = u^2 + 2gs$$

$$(**) \quad = \qquad s = \frac{v^2 - u^2}{2g}$$



Check that (*) simplifies to (**) if k = 0.

SOLUTION 512. Unfortunately, we can't plug k = 0 directly into (*) (why?), so what we really have to show is

$$\lim_{k \to 0^+} \frac{m}{2k} \ln \left(\frac{mg - ku^2}{mg - kv^2} \right) = \frac{v^2 - u^2}{2g}$$

(Continued in Solution 520)

The limit in Solution 512 cannot be done (by hand) using any of the methods covered previously (Sections 12.3.8.1-4), so we need something new -

12.3.8.5. Evaluation of limits using L'Hopital's rule

In earlier parts of Section 12.3.8 we looked at some simple methods for evaluating limits (that involved, for the most part, the use of graphs, etc, on a computer, simple algebra tricks or the squeeze theorem). But there are still many limits (like the one in Solution 512) that we cannot evaluate this way. In this section we will look at a very useful result that will help us further.

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THEOREM 513. L'HOPITAL'S RULE for the indeterminate forms 0/0 and ∞ / ∞ .

Let x_0 be a real number, + ∞ or - ∞ , and let f and g be two differentiable functions so that

$$\lim_{x \to x_0} f(x) = \begin{cases} 0 \\ \pm\infty \end{cases} \quad \text{and} \quad \lim_{x \to x_0} g(x) = \begin{cases} 0 \\ \pm\infty \end{cases}$$

(ie, 0/0 or $\pm \infty / \pm \infty$)

If

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

(where L is a real number, + ∞ or - ∞) then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

PROOF
The "fake proff" and the formal proof using the Mean Value Theorem are presented - details omitted.

Some examples showing how to use the theorem and some "drill-type" exercises are given (3-4 pages) - details omitted.

SOLUTION 512 (continued) 520.
To verify this, note that we can write

$$\lim_{k \to 0^+} \frac{m}{2k} \ln\left(\frac{mg - ku^2}{mg - kv^2}\right) = \frac{m}{2} \lim_{k \to 0^+} \frac{\ln\left(\frac{mg - ku^2}{mg - kv^2}\right)}{k}$$

with

$$\lim_{k \to 0^+} \ln\left(\frac{mg - ku^2}{mg - kv^2}\right) = \ln(1) = 0 \quad \text{and} \quad \lim_{k \to 0^+} k = 0.$$

So L'Hopital's rule applies:

$$\lim_{k \to 0^+} \frac{m}{2k} \ln\left(\frac{mg - ku^2}{mg - kv^2}\right)$$

$$= \frac{m}{2} \lim_{k \to 0^+} \frac{\ln\left(\frac{mg - ku^2}{mg - kv^2}\right)}{k} \qquad (0/0)$$

= . . . . (Details omitted)

$$= \frac{v^2 - u^2}{2g}$$

as expected.

NOTE 521. Derive is able to evaluate most limits that use L'Hopital's rule (it may not acually use the rule, but the effect is the same).

For example, the above limit could have been determined using Derive:

SOLUTION 512 (continued) 522.

To verify this use Derive:

```
* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

Author lim(m/(2k) ln((mg-ku^2)/(mg-kv^2)),k,0,1) ↵

                                            2
                      m            ┌ m g - k u  ┐
        1: lim     ──────── LN     │──────────── │
           k→0+     2 k            │          2  │
                                   └ m g - k v  ┘

Simplify ↵

              2    2
             u  - v
        2: - ────────
               2 g

* * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
```

So $\lim_{k\to 0^+} \frac{m}{2k}\ln\left(\frac{mg - ku^2}{mg - kv^2}\right) = \frac{v^2 - u^2}{2g}$ as expected.

COMMENTS ON THE PROGRAM

There are some interesting points and some difficulties with the program that may be apparent even from the above small example.

The first is that computer programs (including spreadsheets, graphical packages, numerical equation solvers and symbolic manipulators) are used very often as mathematical tools (not just as teaching aids) and students are expected to learn how to use them, too.

Another point is that some topics (the above one is not one) are introduced in a fairly non-traditional manner. Definite integrals, for example, are first evaluated numerically (using a computer, for example) rather than using the Fundamental Theorem of Calculus (and the topic may actually be best done before differentiation, as Courant (1938) does). The exponential function is defined by

$$\exp(x) = \lim_{n\to\infty} (1 + x/n)^{1/n}$$

rather first defining the natural logarithm via a definite integral. Probability is introduced in the context of an hypothesis test (though it's not called that, of course).

One difficulty (and this is a problem with any problem solving approach) is that we must rely on students having a sufficiently good background in other non-mathematical subjects. In the above example the given equation was derived in a separate problem that relied on the use of Newton's $F = ma$ law.

A more serious problem with the approach is that because theory is so dependent on the applications and appears only as it is needed, it is difficult to use the notes as a reference book. It is difficult even to read the notes in small pieces. Often an idea becomes clear only when a whole slab is read. The first impression can be that the work appears disorganized and difficult to follow. A serious flaw for a textbook for first year students ! - and it does require considerable perseverance and careful reading.

The system has been developed and trialled by the author over the past few years and currently the notes contain all the relevant material found in a typical first year "calculus/linear algebra/probability & statistics" sequence. Limited initial trials using the approach seemed to indicate that there was no dramatic, if any, improvement in students' performance over the year. However, there seemed to be no reasons for not pursuing the program further and the current version of the notes is complete enough for a full trial. This is in progress at the moment, but unfortunately any effects will necessarily be difficult to detect because the program is being followed by a whole group of new students with no control group for comparison (the introduction of the Victorian Certificate of Education makes it impossible to sensibly compare this year with previous years). However, it is probably not worthwhile to attempt to quantify the approach yet. There are still some serious design problems in the notes that need to be rectified.

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PROFESSIONAL DEVELOPMENT AND THE SECONDARY MATHEMATICS TEACHER: A CASE STUDY

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INTRODUCTION

This paper reports a case study of the development, implementation and outcomes of the "Active and Reflective Teaching In Secondary Mathematics" (ARTISM) professional development program. This development arose through the collaboration of teachers on three school sites, the school system (Catholic Education Office), and a tertiary institution (Australian Catholic University). An evaluation has been undertaken to document the development process, the presentation of the program, and participants' responses, including the long-term take-up of the program's key features in the participants' classroom practices. Details are presented of development and delivery and the preliminary findings of the evaluation.

The ARTISM professional development program is predicated on the belief that change will arise from the classroom experiences of teachers who have undertaken to field-test some of the new techniques. Adopting this perspective, the teacher's commitment to field-test some of the techniques becomes an essential condition of a teacher's participation in an in-service activity. This model of professional development assumes that the value of the new approach will emerge most clearly and with greatest personal impact when applied in the teacher's own classroom, and that this will fuel a change in teacher's beliefs, generating on-going commitment to further professional development. The professional development cycle so established (Clarke, 1988, p. 188, see Figure 1) will hopefully become an upward spiral.

Two points must be made:

1. As Figure 1 makes clear, each stage is mediated by factors peculiar to the particular teacher's situation. As a result, the nature of each teacher's professional development will be a unique product of their personal history and their present situation.

2. It is possible to join the cycle at any point: a chance question or an improvised activity may demonstrate the inadequacy of previous methods and lead to a reassessment of beliefs about good teaching; or participation in a professional development program may raise the unexpected possibility of attractive new methods. It is also possible to leave the cycle at any point should the mediating factors provide insurmountable obstacles. For instance, a new approach may be successful in achieving goals which are not valued by the school community, and the professional development subside for lack of support, recognition or encouragement.

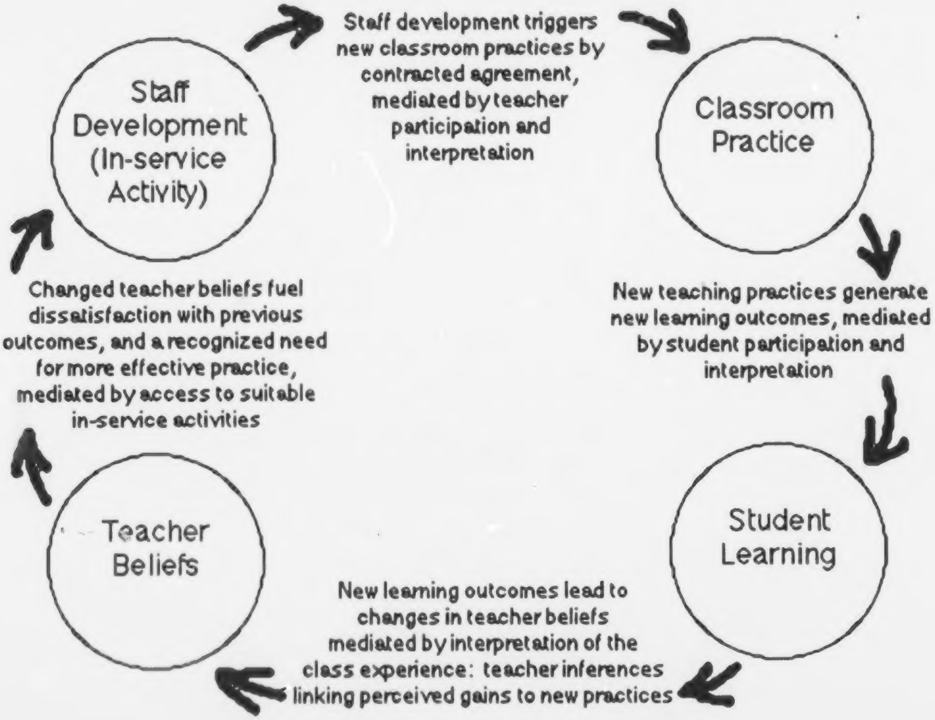
Staff development triggers new classroom practices by contracted agreement, mediated by teacher participation and interpretation
Staff Development (In-service Activity)
Classroom Practice
New teaching practices generate new learning outcomes, mediated by student participation and interpretation
Student Learning
New learning outcomes lead to changes in teacher beliefs mediated by interpretation of the class experience: teacher inferences linking perceived gains to new practices
Teacher Beliefs
Changed teacher beliefs fuel dissatisfaction with previous outcomes, and a recognized need for more effective practice, mediated by access to suitable in-service activities

Figure 1: A Cycle of Professional Development (from Clarke, 1988)

This view of professional development employs the same elements as the linear model developed by Guskey (1985), and embodies a picture of the teacher as learner, constructing personal professional practice from in-service activities and classroom experiences in a way which reflects the teacher's personal history and present situation.

THE PROGRAM

Basis for the program

ARTISM was intended to make the participating teachers aware of the current developments in the learning and teaching of mathematics while acknowledging the factors inhibiting implementation, making realistic demands of teachers, while providing appropriate support. This professional development program was predicated on the belief that change will arise from the classroom experiences of teachers who have undertaken to field-test some of the new techniques. The application of the key content of the sessions in the teachers' own classrooms is an essential element of the course.

Program structure

An initial questioning of the mathematics co-ordinators of the three schools indicated a clear preference for a program structure providing models of teaching and learning as well as certain content issues that are regarded important in secondary mathematics.

This led to the following program structure:

Unit 1: Mathematical Problem Solving and Modelling

Unit 2: Communication and Classroom Structures

Unit 3: Technology in the Mathematics Classroom

Unit 4: Assessment Alternatives and Module Planning

Unit 5: Reflection on Action and Further Planning

Units 1 and 2 were offered at the commencement of the school year on two separate evenings, one week apart. A four week classroom exploration period was provided between units 2 and 3 to enable the teachers to undertake significant exploration of the course themes and techniques.

As ongoing follow-up and reflection on the implementation of the intended change are crucial for successful and effective professional development (Zigarmi, Betz, & Jensen, 1977, p.551), the fifth session about reflection and the school visits of consultants and presenters at the three school sites between the sessions on a regular basis were included as an integral part of the course. These school visits, which occurred during the four week break between the second and the third session and between unit four and five, served three principal functions: to support teachers in their trialling of the strategies offered in the course; to provide a measure of accountability, increasing the teachers' sense of obligation to make use of the course materials; and, to collect evaluation data through informal discussion with teachers regarding their classroom successes and concerns.

Participants' commitments

* Maintain a journal/portfolio

* Complete readings for each week

* Undertake classroom tasks specified for each week

* Develop or rewrite a unit of work

* Complete additional feedback sheets, or provide information as required

Desired outcomes of ARTISM

* An expanded repertoire of teaching strategies, practice and success with new ideas

* Improved student learning

* Willingness of teachers to experiment

* Changes in school classroom practice

* Changes in the school mathematics program

* Resources (and where to get them)

* Teacher affective outcomes

EVALUATION

Evaluation Design

The development of the ARTISM professional development program was characterised by the collaboration of teachers on three school sites, the school system (Catholic Education Office of Victoria, and a tertiary institution (Australian Catholic University). The evaluation of the program was required to address the needs of all three collaborating groups, while simultaneously assessing program outcomes within a structure informed by earlier research into teacher professional development. The evaluation guidelines set out by Guskey and Sparks (1991) have informed the evaluation of the ARTISM course specifically.

The evaluation was conceived and undertaken at the same time as the development of the professional development program. From the outset, the development team, the educational system, and the participating schools were represented in the establishment of the program's goals and the criteria by which the success of the program might be assessed. The program design and the evaluation were integrated to maximize the opportunity for the preliminary findings from the evaluation to inform the implementation of the program. The identification of objectives encompassed system goals, school goals, departmental goals, and the goals of the individual participants. It was seen as essential that each of these goals be documented at the commencement of the program. An obligation to access multiple sources of data was seen as an imperative for research into educational contexts, where control of environment and treatment is limited and where the reality of participation, motivation and consequent action resides with the individual (for instance, Clarke, 1984) In such research situations it is the responsibility of the researcher to document the individual realities of the participating subjects from as many perspectives as possible, since the authority regarding "what really happened" does not reside with any individual data source, but rather the assessment of both experiences and outcomes is most appropriately expressed with independent reference to each of the participating groups: program developers and presenters, the facilitating educational system, and the participating teachers and their respective school communities.

Participants' perceptions will be sought regarding their current practices, their perceived needs, their current concerns, their opinions of the value of program components, perceived program omissions, the practicality of suggested practices, and the classroom outcomes attributable to their participation in the progam.

Data collection categories for the evaluation of the ARTISM program can be characterised as related to:

- **Context**: the existing administrative and environmental constraints affecting participants, and pre-existing beliefs, practices and expectations;

- **Participation**: the nature of teacher participation in the program, consistency in attendance, completion of work requirements, and teacher perceptions of program components;

- **Implementation**: the details of the implementation of the program, the identification of what constitutes the "treatment" or "innovation" to which participants are being asked to respond, fidelity to the program design, and changes to the program arising during implementation;

- **Satisfaction**: the satisfaction of the participants with the program, enjoyment, interest, practicality, and the extent to which the program met their needs and offered something of value;

- **Change in teaching practice**: the changes in the practices of the participating teachers attributable to program participation, changes in teacher beliefs, and consequent organizational changes in participating schools;

- **Student outcomes**: the student outcomes which can be associated with teachers' participation in the program, through changes to the classroom environment or to teachers' work practices.

Results

The results which follow are organized around the above data collection categories. A more extensive discussion of the results of this study can be found in Clarke, Carlin and Peter (1992).

Participants

Thirty-three teachers from three Catholic boys' secondary schools enrolled for the program.

CONTEXT

In completing the questionnaire "My Classroom Structure", teachers were asked to "Think of a junior secondary mathematics class which you have taught over the last 3 - 4 weeks. Circle the words in the table below which best describe that class." The distribution of teacher responses is given below in Table 1.

Table 1: Number of teachers (Percentage in brackets) responding in each category.

Year level of class	7	8	9	10		
No. teachers (percentage)(n=18)	1(6)	4(22)	6(33)	7(39)		
Most common teaching group	Whole class	small group	individual pupil			
No. teachers (percentage)(n=22)	15(68)	3(14)	4(18)			
Most common working group	Whole class	small group	individual pupil			
No. teachers (percentage)(n=24)	7(29)	7(29)	10(42)			
Most common emphasis	Problem solving	Applications	Skills			
No. teachers (percentage)(n=21)	2(10)	6(29)	13(62)			
Most common materials (circle one or more)	Text	W'sheets	Aids	Cal'tors	C'puters	Others
No. teachers (percentage)(n=24)	10(42)	13(54)	5(21)	10(42)	1(4)	1(4)

Some classroom "scenarios" were provided. These scenarios were clustered into groups, which had certain characteristics in common. Teachers indicated which scenario most resembled their classroom, and then which other scenario in another group they would consider using. The results for each of the groups of scenarios are recorded in Tables 2 and 3.

Table 2: Number (and percent) of teachers recording each scenario group as representative of their current practice (n=24).

Group	**Scenarios**	**Characteristics**	**No.**	**%**
A	1 and 2	whole class, skills and bookwork	13	54
B	3, 4, 5 and 6	variations on small group work	2	8
C	7 and 8	problem-based instruction	1	4
D	9 and 10	skills then applications	5	21
E	11	vocational mathematics, few new maths skills	0	0
F	12	individualized maths programs	3	13
G	13	vertical modular grouping and multi-age structures	0	0

Table 3: Number (and percent) of teachers recording each scenario group as one (different from their current practice) which they would consider using (n = 24).

Group	Scenarios	Characteristics	No.	%
A	1 and 2	whole class, skills and bookwork	1	4
B	3, 4, 5 and 6	variations on small group work	6	25
C	7 and 8	problem-based instruction	7	29
D	9 and 10	skills then applications	2	8
E	11	vocational mathematics, few new maths skills	0	0
F	12	individualized maths programs	4	17
G	13	vertical modular grouping and multi-age structures	0	0
None		none of the above	3	13
Any		any of the above	1	4

Pre-existing concerns with the School Context

Teachers nominated a range of concerns regarding their present school mathematics programs. The most commonly cited concerns were grouped into categories sharing common features. These categories were: Staff; Resources; Content; The school mathematics program; Senior mathematics; and, Classroom practicalities.

Perceived strengths of the school program

The most commonly cited strengths were grouped similarly into categories sharing common features. These categories were: The school mathematics program; Staff; Co-ordination; Teaching environment/Resources; Professional support; and Students' motivation.

Expectations of the ARTISM Program

The expectations of the three mathematics co-ordinators could be related to specific issues or educational constructs:

- increase teaching "expertise";
- engender teacher "enthusiasm", "ability" and "confidence";
- to equip teachers to "evaluate effectively";
- to increase the focus of mathematics programs on "real world experiences";
- a clearer picture of what is "problem solving and modelling";
- increase the repertoire of classroom management techniques;

The curriculum co-ordinator at one school expressed the desire that the program: "affirm the things that staff are doing well" and "establish a network of mathematics teachers across the three schools".

The teachers indicated a highly diverse list of desired outcomes for the ARTISM program, which included: Classroom Practice, Problem Solving and Modelling, Assessment, Curriculum Planning, Professional Support, Classroom Environment and Learning.

At this stage in the development and evaluation of the ARTISM program the data collated is primarily descriptive and specific to the existing context. Data relating to the teachers' participation in the formal program is reported in the following sections. Teachers' application of the strategies advocated in the program is the subject of on-going study.

This report presents preliminary data of a case study into the development and evaluation of a collaborative professional program which is still in progress at the time of writing. In summary, four critical stages can be identified:

Stage 1 - the initial input of the first four sessions;

Stage 2 -
  a. applications of new practices in the classroom;
  b. group preparation of mathematics teaching modules to be taught at some stage during terms two or three;

Stage 3 - the collective review and reflection on the first four sessions and the classroom application of new ideas;

Stage 4 -
  a. continued application of new practices in the classroom;
  b. teaching of the designed unit of work (teaching module) and reflection on the outcomes.

It was intended that the teaching module would provide a vehicle for the assimilation of the new practices met in the course.

PARTICIPATION AND IMPLEMENTATION

Together with a growing willingness to try new approaches, the emergence of a new collegiality among mathematics staff on all three school sites was seen as the major outcome of the ARTISM program by those responsible for the administration of mathematics within the three schools.

The level of teacher participation in the ARTISM program can be gauged by the continued attendance of the initial participants, by the classroom trialling by participants of the classroom exploration activities (documented through discussions during school visits), by participant engagement in the activities and discussion of the formal sessions, and by the participants' endorsement of the ideas and strategies discussed and modelled during the workshop sessions as expressed in teachers' journals and completed course review questionnaires. Participants' opinions regarding valued components of the course were sought in very practical terms: "I have tried the following activities ... " and "I have found the following useful ... ".

Most frequently cited course components were: problem solving approaches; 'Good Questions' (open-ended mathematics tasks); the 'Think, Pair, Share' strategy; and specific calculator activities. Other course components identified by participants as having been used in the classroom (the endorsement of use), or as having been found to be useful (the endorsement of value) constituted an effective summary of the program content. It appeared that almost every aspect of the program received the explicit endorsement of at least some of the participants.

School visits made between course sessions provided an opportunity for conversation with teachers about their use of strategies such as those listed above. In addition, mathematics co-ordinators reported that such school visits provided a crucial accountability and a pretext for regular meetings of participating teachers. A record was maintained of these conversations, documenting key features of all such discussions, and illustrating these key features with specific teacher anecdotes. Classroom application of the new ideas and strategies were enthusiastically recounted. It was clear that many teachers felt that they were producing positive teaching and learning outcomes.

A consistent impression was gained of general interest and of specific enthusiasms. Participants engaged in enthusiastic and focussed discussion and analysis of teaching and learning. It became clear that participation did not mean total and unquestioning acceptance of the ideas and strategies being promoted in the sessions. The universal approach was adaptation rather than adoption of the new ideas.

PARTICIPANT SATISFACTION

Participants were asked to express their concerns regarding the ARTISM program. The responses were informative and occasionally contradictory. Only a small number of teachers expressed any concerns.

Frustration was expressed regarding attempts to implement the ideas presented in the program in the face of limited school facilities and lack of student motivation. Participants were asked to identify additional course components, or existing elements requiring greater emphasis or time allocation. Commonly cited as requiring greater attention were:
dealing with diverse student abilities; strategies for motivating students; planning techniques; classroom management techniques; additional examples of assessment alternatives; and, more specific classroom examples of content-specific teaching, applications of technology, and available resources.

CHANGES TO CLASSROOM PRACTICE

> It [the ARTISM program] made me think about what I am doing in my classroom and it is really challenging. The things that have been presented and taught during the sessions are valuable and really working. But we are quite stable and set in our ways and need more time to change, because it challenged the heart of our teaching.

From the outset, the major goal of the ARTISM program was to facilitate change in classroom practice. Participants were asked to indicate ways in which they felt that their classroom practice had changed.

Three key factors were most commonly cited:

greater instructional diversity,

greater diversity in assessment strategies,

and attempts to make mathematics instruction more student-centred.

These three factors represented major themes within the program. The prominence given to changing classroom practice with respect to these three factors is a strong endorsement of the approach taken by the program, and an indication of participants' attempts to put into practice the major themes of the ARTISM program.

PROFESSIONAL DEVELOPMENT: ONE TEACHER'S EXPERIENCE

The progression in professional development anticipated in Figure 1, can be seen in operation in the changing practice of one participant as recorded in weekly journal entries relating to the teaching of one class during the first four months of the ARTISM program. The content of each week's journal entry is summarised under the headings: Mathematical Content, Relation to ARTISM, Teaching Strategies, Assessment, and Teacher's Comments. The evaluation of these journal entries led to the following summative statements:

1. An increasing number of elements and strategies from the formal course were integrated progressively into the planning, implementation and evaluation of the lessons.

2. The school visits provided confirmation of progress through the sharing of experiences, and support for further progress.

3. Success with the new strategies encouraged this teacher to go further and try to explore more ideas from the ARTISM program, but the teacher always tried to adjust the ideas carefully to the content and special needs of the class.

4. The teacher was very concerned about appropriate assessment strategies and procedures. Other important issues which were mentioned several times include: control of work, supervision, student motivation.

5. The developmental progression documented in this case study demonstrated significant congruency with the ARTISM goals in terms of teacher affective outcomes (the teacher is gaining more confidence, enjoyment, enthusiasm, and professional satisfaction). Willingness to experiment increased as the teacher experienced success with the new ideas and improved student learning was observed. There was some evidence of a change in classroom practices.

The teacher also reported that other mathematics staff had begun to prepare more meaningful lessons, on the grounds, "We do not want to be left behind."

CONCLUSIONS AND IMPLICATIONS FOR FURTHER STUDY

The following comments can be made about the ARTISM program to the end of stage 3:

1. The course appeared to have met some important needs and expectations of both participants and school administrators. This observation was based on high levels of course attendance and enthusiastic participation, positive feedback from school administrators, and evidence from school visits by presenters and evaluators that a significant number of participants are trying new ideas and strategies. Teachers' written responses to course review questionnaires, and in their journals, also supported this conclusion. It also appeared from the accumulated body of written, oral and observation data that many of these teachers were producing positive teaching and learning outcomes.

2. The enthusiastic and focussed discussion and analysis of teaching and learning of mathematics which was observed during visits to schools appeared to support early impressions of the success of the program. It should be emphasized that enthusiastic participation does not mean total and unquestioning acceptance of the ideas and strategies being promoted in the sessions. There have been strong challenges to some of the practices being advocated. These discussions and reflections have the appearance of the beginnings of a "mini-college" of teaching within the three schools.

The kinds of long-term outcomes the program developers were aiming for included:

1. A continuing enthusiasm by teachers to continue their learnings and to try to reflect on new ideas and strategies in their teaching;

2. An increased capacity among teachers to write and teach better units of work based on their experiences in the ARTISM program;

3. An increased inclination among the teachers to meet together to review and reflect on their teaching practices in order to confirm and consolidate good practice, and to identify those aspects of their teaching or their schools' mathematics programs which require further development.

On the basis of the data collected so far, it appears likely that these goals will be achieved for many of the program participants.

ACKNOWLEDGEMENTS

A review of the research literature relating to the professional development of teachers (Clarke, 1991) usefully informed the development of this program. The co-operation of the schools and teachers whose concern for professional excellence led to the development of this program is gratefully acknowledged.

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STUDENT RESPONSE CHARACTERISTICS TO OPEN-ENDED TASKS IN MATHEMATICAL AND OTHER ACADEMIC CONTEXTS

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INTRODUCTION

The research reported in this paper represents one component of a larger project addressing the consequences and implications of the use of a new class of open-ended mathematical tasks called "Good Questions", devised by Sullivan and Clarke (1991b). The particular characteristics of this type of task are the content-specific focus, and the opportunity for answers at different levels of sophistication. An extended research program is underway to investigate the use of this form of open-ended mathematics task for instruction and assessment in mathematics. The characteristics of these Good Questions have been discussed elsewhere (Sullivan and Clarke, 1988; Clarke and Sullivan, 1990, Sullivan and Clarke, 1991a and b, Sullivan, Clarke and Wallbridge, 1991). Each of the three postulated characteristics of Good Questions derives from a specific educational stance.

1. The task should require more than the recall or replication of a fact or procedure.
2. The task should be educative.
3. The task should be open-ended.

The rationale is that if mathematical power is to be associated with the ability to solve non-routine problems (NCTM, 1989), then our assessment and instructional tasks should have some of the characteristics of problem solving tasks. The Good Questions approach has been proposed as a step in this direction.

Perhaps the most consistent and significant finding to emerge from previous research (see Clarke & Sullivan (1991) for an overview of these findings) was the reluctance of pupils from grades 6 to 10 to provide more than a single answer to such open-ended questions as:

- The average of five numbers is 17.2, what might the numbers be?
- A number is rounded off to 5.8, what might the number be?
- A rectangle has a perimeter of 30 units, what might be its area?

The term "reluctance" is used here to indicate the disinclination of the pupils tested to give either multiple answers or a general statement. As was demonstrated in the earlier studies, this disinclination should not be confused with an inability to provide multiple answers. If open-ended tasks like those above were reworded so as to require pupils to list all possible multiple answers, then the number of multiple responses increased substantially. This demonstration of a capacity to provide multiple answers was also evident in the responses of pupils when the tasks were administered in interview situations. It is clear from these earlier studies (Clarke and Sullivan, 1991) that, while few students are equipped to produce general statements as solutions to open-ended tasks, many pupils have the capability to produce multiple answers but do not choose to do so. Key results from these previous studies are summarised below.

PREVIOUS STUDIES

Mathematics Only

In a typical administration of the investigation of open-ended mathematics tasks, a set of four questions was given to participant classes of schoolchildren. The criterion for selection of classes was the willingness of their teachers to participate. Even though no teacher declined the invitation no claims are made about representativeness of the results for other schools.

In the first administration of tasks, the questions asked, the procedure for administration and the response coding system were as follows:

Subtraction

Last night I did a subtraction task. I can remember some of the numbers.

```
   1 4 __
 -   7 __
 --------
     __ 4
```

What might the missing numbers have been?

Rounding

A number has been rounded off to 5.8 What might the number have been?

Area

A rectangle has a perimeter of 30 m. What might be the area?

Fraction as operator

$\frac{2}{5}$ of the pupils in a school play basketball. How many pupils might there be in the school and how many might play basketball?

The format for administering the questions was the same in each class:

i) The question: "___ + ___ = 10 What might the missing number be?" was posed, and the responses suggested by the pupils were written on the chalkboard. The pupils in the class were asked to comment on what was different about this task from common mathematics questions. The response sought was that there are many possible answers.
ii) The first two questions were distributed (subtraction, rounding).
iii) The papers were collected and the answers reviewed. Again the possibility of multiple answers was discussed.
iv) The second two questions were distributed (area, fraction as operator)

The responses of the pupils to the tasks were coded. The coding was as follows:

0 meant no correct answers
1 meant only one correct answer
2 meant two or three correct answers
3 meant all or many correct answers
4 meant that a general statement was given

To illustrate the way that this code was applied to the rounding question, the following is the meaning of the codes. Individual correct answers were numbers such as 5.82 or 5.78. A code of "3" was given to a response like "5.75 5.76 5.77 5.78 5.79 5.80 5.81 5.82 5.83 5.84". Examples of responses which were considered to represent a general statement, "4", were "5.75 ... right up to 5.84999..." or "between 5.75 and 5.849".

Key results

Key findings from the "Mathematics Only" phase of the research (from Sullivan, Clarke and Wallbridge, 1991; Clarke and Sullivan, 1991) included:

1. The capability to give multiple and general responses increased with age.

2. Revision of the "Good Question" format to specifically request multiple answers resulted in a significant increase in the proportion of grade 6 students providing multiple responses.

3. Results suggested a task-specific dependence in student levels of response.

4. Students were more likely to provide multiple responses in an interview context than in a test context.

5. The learning outcomes of a seven-lesson teaching program for grade 6 pupils based solely on the use of such open-ended mathematics tasks were not significantly different from the outcomes of a control class instructed to follow the program presented in the most commonly used text. These results render the hypothesised benefits problematic.

It is possible that the period of instruction for the experimental group was insufficient to overcome the many years of schooling during which the provision of a single correct answer has been the accepted student goal. It seemed to the authors that the distinction between the capability to provide multiple answers to an open-ended task and the inclination to do so required further investigation. In particular, it was not known whether the reluctance to provide multiple answers was specific to mathematics or common to other academic disciplines. This is the question which this study sought to address.

THIS STUDY

Mathematics and Other Academic Contexts

This paper is concerned primarily with the question:

> Is the reluctance of pupils to give multiple or general responses to open-ended mathematics tasks replicated with open-ended tasks in other disciplines?

The sample employed also made possible the investigation of the effects of grade level, gender, task-specificity, the presence or absence of explicit cueing, and the interaction of these variables with academic domain. Central to the study was the distinction between student response to open-ended tasks which did not explicitly request multiple answers ("uncued") and student response to open-ended tasks which explicitly requested multiple answers ("explicit cueing"). It was assumed that student responses to the uncued tasks would provide an indication of student "Inclination" to provide multiple or general responses, while student responses to tasks employing explicit cueing would indicate student "Capability" to provide multiple responses.

For this study, students at years 7 and 10 from four secondary schools (one single-sex boys' school, two single-sex girls' schools, and one co-educational school) were asked to respond to open-ended items from a variety of academic contexts. The communities served by the four schools had comparable socio-economic characteristics. Four separate research protocols were employed in the administration of the tasks. The following structural concerns were explicitly addressed in the protocol design: order of task administration; explicit or non-explicit cueing; and, task variation within a domain.

In particular, since the results arising from a single open-ended task in a given discipline may be distorted by task idiosyncracies such as the task context or the cognitive demands of the task, the investigation was repeated with a different task in each academic context. Over all four protocols, two tasks were used in each academic domain:

Social Studies
In a Victorian country town, the population fell by 50% over a period of 5 years.
Why might this have happened?

A 19 year old girl moved out of home.
Why might she have done this?

English
A fragment of an early sonnet by Shakespeare has been found. It includes the incomplete line:
"And ________________________ did mask the sun."
What might be missing?

A game of scrabble had only just started when the board was tipped over. The letters on the floor were: T, S, P, O.
How might the letters have been arranged before the board was tipped over?

Science
A plastic ice-cream container half full of water is left on the front lawn to keep cats and dogs away. One week later the container is empty. How might this have happened?

A farmer lost his entire crop.
Why might this have happened?

Mathematics
A rectangle has a perimeter of 30 units.
What might the area be?

The average of five numbers is 8.
What might the numbers be?

It was intended that the tasks be well within the cognitive capability of most students. The concern of the study was the distinction between student inclination (uncued) and capability (with explicit cueing) to give single or multiple answers, and it was not intended that task difficulty be a factor for the domains of Social Studies, English and Science. It is acknowledged, however, that student lack of familiarity with the context of a specific task might increase the task's cognitive demand and lead to task-specific differences. As will be seen in the following results section, student responses across the two tasks within a domain were sufficiently different as to provide an endorsement of the use of more than one task in each domain. The analysis of student response levels will combine both tasks as a representative sampling of responses in each domain. Both mathematics tasks had been used in earlier studies and provided a basis for comparison with earlier findings arising from the use of only mathematical tasks.

The protocols guiding the administration took a basic form derived from that for the earlier administration of mathematical tasks (see Clarke and Sullivan, 1991), and were administered in academically-neutral classrooms (with respect to the academic domains which formed the focus of this study) by a trained research assistant.

Protocol 1	Introductory discussion employing the example "The sum of two numbers is ten; what might the numbers be?" Distribution of two questions: One mathematics, and one science. Response sheets collected and differences between answers discussed. Distribution of two questions: One language and one social studies, with the direct request that multiple solutions be sought.
Protocol 2	As for 1 with the Mathematics/Science and Language/Social Studies pairs exchanged (that is, Language and Social Studies questions given first).
Protocol 3	As for 1 with different tasks.
Protocol 4	As for 2 with different tasks.

Since the focus of this study is on student inclination to provide multiple answers to open-ended tasks, analysis disregarded the correctness of student response in applying the earlier coding of student responses. The responses of the pupils to the tasks were coded as follows:

0 meant no answer provided
1 meant only one answer provided
2 meant two or three answers provided
3 meant all or many distinct answers provided
4 meant that a general statement was attempted.

With respect to order of task administration, a preliminary administration of the protocol 1 and 2 tasks to a smaller sample (n = 87) without any explicit cueing of multiple responses, established that a different order of administration of the domain-specific tasks did not lead to significant differences in response level. As a consequence, the protocol structures and sampling design anticipated an analysis with respect to year level, gender, degree of cueing, and task specificity, but not task order. The sample construction is shown in Table 1.

Table 1: Sample construction

	Protocol 1		Protocol 2		Protocol 3		Protocol 4		Total Sample	
	Yr 7	Yr 10	Yr 7	Yr 10	Yr 7	Yr 10	Yr 7	Yr 10	Yr 7	Yr 10
Male	70	69	72	67	33	30	27	32	202	198
Female	76	71	52	65	32	25	32	23	192	184

RESULTS

To summarize the data collection from which the following results were derived: two tasks were administered for each domain (Task One and Task Two). Each task was given in both cued and uncued modes. No one student attempted more than one task from a domain, and no student attempted both cued and uncued forms of a given task. Student level of response is reported in two forms: the mean level of response for a given group, and the distribution of student responses among the five response categories employed (0, 1, 2, 3, 4).

A more extensive discussion of results can be found in Clarke, Sullivan and Spandel (1992).

Task-specificity

There appeared to be some dependence of response level on the specific task employed. A two-factor analysis of variance for uncued and cued response level, by task and domain revealed statistically significicant differences attributable to both task, domain, and their combination.

The differences evident in student response level to different tasks within the one academic domain made clear the task-specificity of responses to open-ended tasks. For the remainder of the analyses employed in this paper, responses to both tasks within a specific domain were combined.

Inclination or Capability - the significance of explicit cueing

The results of previous mathematics-only studies had revealed significant differences in student response level dependent on whether or not an explicit request was made for multiple answers. It was postulated as a consequence of these results that the differences between responses to "uncued" and "cued" open-ended tasks reflected a fundamental distinction between student Inclination and Capability to provide multiple and general responses. This study set out to investigate whether the significance of this distinction pertained in other academic contexts. Figure 1 sets out the differences between uncued and cued mean response levels for the four academic domains addressed in this study.

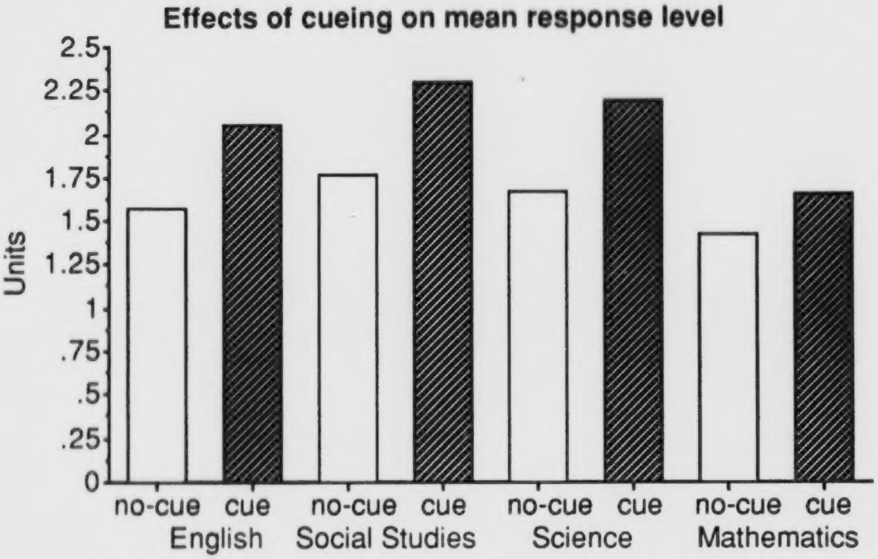

Figure 1: Comparison of the uncued and cued mean response levels for English, Social Studies, Science and Mathematics.

It is evident from Figure 1 that the difference in response level arising from the inclusion or omission of the specific request for multiple responses identified in previous studies was present to an even greater extent in non-mathematical contexts. Effect sizes for each of the non-mathematics domains exceeded 0.5, and separate t-tests comparing uncued and cued response levels for each academic context confirmed the significance of the differences displayed in Figure 1. The detail of the differences in response levels is provided in the following section.

Domain-specificity

The key question addressed by this paper concerned whether the response patterns observed with open-ended mathematics tasks would be replicated with open-ended tasks in other academic domains. In the results which follow the distinction is maintained between Inclination (uncued responses) and Capability (cued responses).

If either inclination or ability to provide sophisticated (predominantly multiple) responses to open-ended tasks is a reflection of some general intellectual capacity, then one would expect a high correlation of response levels across all academic domains. Alternatively, if either the capacity or the inclination to give more sophisticated responses is a consequence of expertise in the specific domain, then one would not expect high correlations for the response levels across different domains. The sample design employed in this study did not permit correlations to be calculated for both uncued and cued responses for all possible paired combinations of academic domain. For the purposes of this study it was felt more important to ensure that all participating students supplied both cued and uncued responses. This maximized the diversity of the sample which formed the basis of the investigation of

Inclination and Capability. No individual was required to respond to both cued and uncued tasks in the one domain in case completion of one task type affected performance in the other task type. As a consequence, this study could not address the two hypotheses outlined above directly.

It is certainly not possible to distinguish between the "Intellectual Capacity" and "Domain-specific Expertise" hypotheses on the basis of this study. Further research is required to adequately investigate these two hypotheses.

Year level

Previous mathematics-only studies employed exclusively uncued task formats and included consideration of the mathematical correctness of the response in the coding, as set out earlier in this report. These mathematics-only studies identified an association between (uncued) response level and year level. The study reported here found that the association between response level and year level differed in character for uncued and cued task administrations. These results are shown in Figures 2 and 3.

Continuing to interpret student response level to uncued open-ended tasks as indicative of the level of inclincation, this study made it clear that the difference between year levels in the inclination to provide multiple answers is only significant for Social Studies and Science. That is, pupils in years 7 and 10 were similarly disinclined to give multiple responses to open-ended tasks in English and Mathematics (when not explicitly requested to do so).

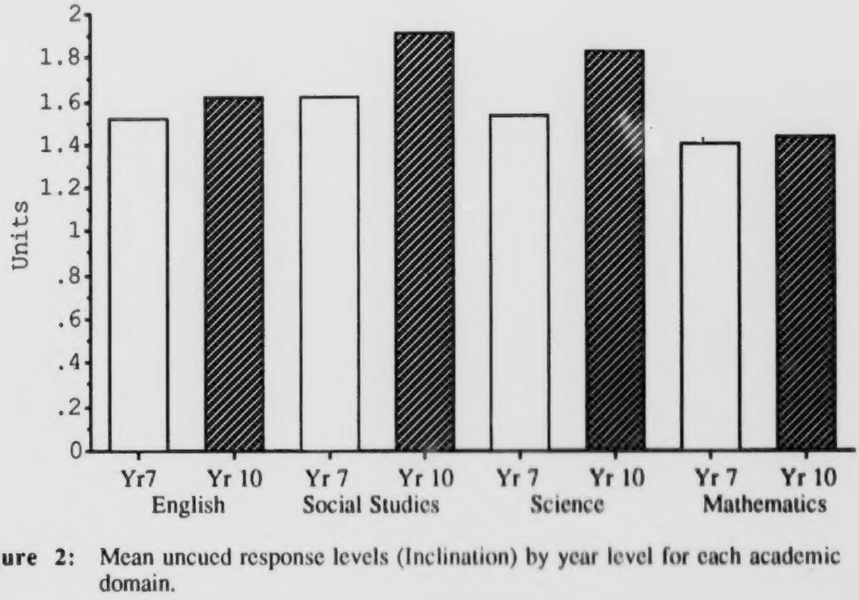

Figure 2: Mean uncued response levels (Inclination) by year level for each academic domain.

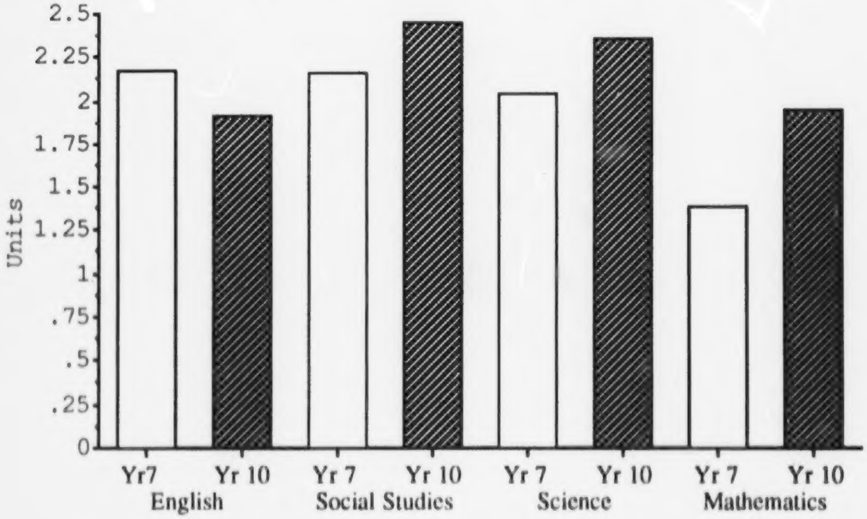

Figure 3: Mean cued response levels (Capability) by year level for each academic domain.

When an explicit request for multiple responses is included in the task, student response levels at year 10 are significantly higher than those for year 7 in each of the academic domains studied, with the notable exception of English. We interpret these results as suggesting that the Capability to respond to an explicit request for multiple answers to an open-ended task increases with year level, except in the case of English.

Gender

In contrast with the findings of Thomas (1989), girls in this study were more likely than boys to provide multiple answers, when specifically requested to do so, in all four domains. However, only in English and Science were these differences statistically significant. Figure 4 sets out the respective Inclinations of girls and boys to provide multiple answers. It is evident that boys and girls do not differ significantly in their Inclination (or reluctance) to provide multiple or general answers to open-ended tasks, except in Social Studies, for which girls display a significantly higher mean response level than do boys.

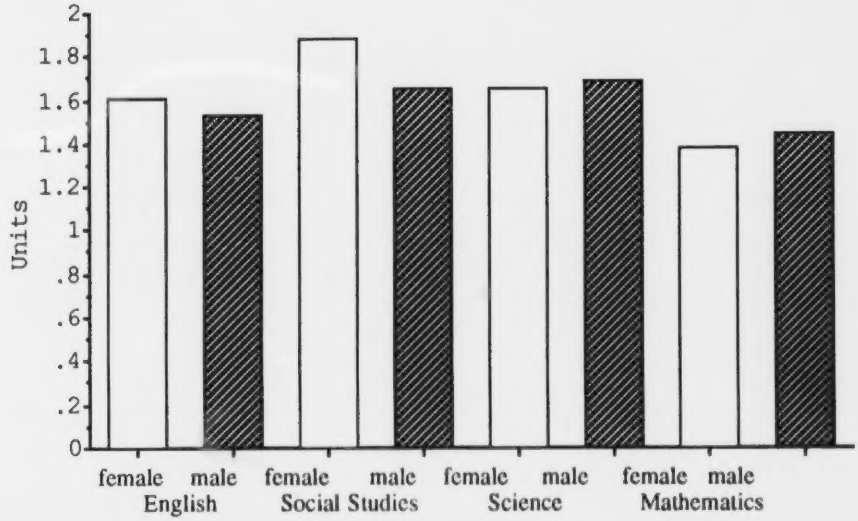

Figure 4: Gender-specific mean uncued response levels (Inclination) for each academic domain.

Figure 5 examines whether a significant difference exists for any of the academic domains studied between the Capability of girls and boys to provide multiple answers when explicitly requested to do so.

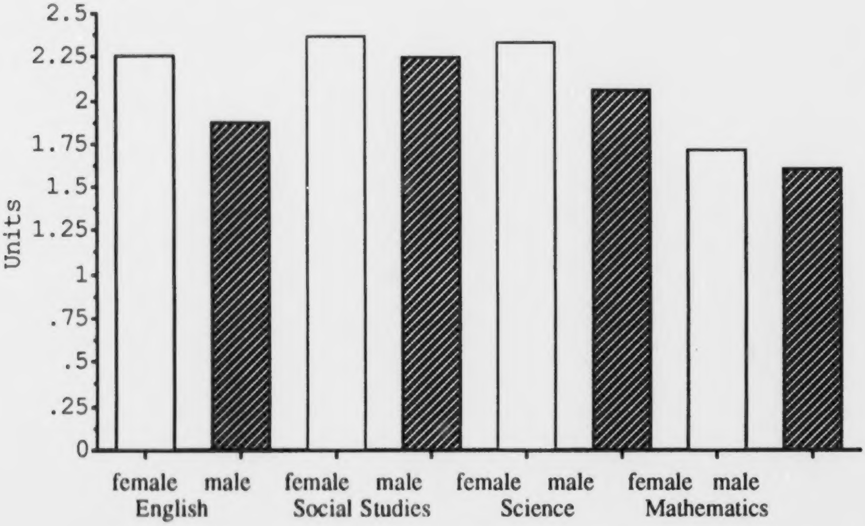

Figure 5: Gender-specific mean cued responses (Capability) for each academic domain.

For the sample of pupils used in this study, girls displayed a greater Capability to provide multiple answers than did boys. This difference was significant in the academic domains of English and Science.

CONCLUSIONS

Conclusions which follow from these results include:

- that the inclination to give single responses (or the reluctance to give multiple responses) is a product of schooling, and not peculiar to mathematics;
- that the explicit request of multiple responses produces a significant increase in the response level in all academic contexts;
- that the capability to give multiple responses increases significantly with year level, except in the context of English;
- that gender-related differences in response level are evident, and where these exist they favour girls.

Substantial additional research is required if we are to understand the significance of the meanings constructed by students in responding to open-ended tasks. Such research must address those student conceptions of legitimate mathematical activity on which their response inclinations are predicated (Clarke, Wallbridge and Fraser, 1991, and this research), and issues of cognitive load or working memory capacity and related developmental theories of learning outcomes which determine student response capability (for example, Collis, 1991; Sweller, 1989). The associated matters of inclination and capability must be understood if we are to employ such tasks with success in mathematics classrooms for the purposes of either instruction or assessment.

Among the various concerns expressed about the use of open-ended tasks for assessment in mathematics, student unfamiliarity with the required mode of response is frequently cited. Student unfamiliarity with the demands of open-ended tasks was also raised as a concern in the context of reading assessment. It should be noted that, for the case of mathematics, even when multiple responses were given the explicit endorsement of an instructional intervention no corresponding increase in response level ensued (Sullivan, Clarke, and Wallbridge, 1991; Sullivan, Clarke, Spandel, and Wallbridge, 1992). This would suggest that student reluctance or incapacity to provide multiple responses to open-ended tasks was attributable to more than simply lack of familiarity with the task format. It is possible that while the brief instructional intervention effectively addressed the issue of familiarity, student inclination to give single responses to academic questions is a consequence of extensive training, and that a much more extended instructional intervention would be required to challenge this 'training' effect. Such a training effect could be attributable to the explicit training of students to provide single responses to academic tasks, or to a lack of explicit training in the provision of multiple or general responses. This study has demonstrated that this training effect is evident in other academic contexts besides mathematics, and may be more appropriately seen as a consequence of schooling rather than just mathematics instruction.

There are other issues besides those raised above, which arise if open-ended items are to be used to assess student learning. The research reported here has raised additional concerns. For instance:

- Task specific differences in pupil performance were evident throughout the study and warrant investigation.
- Inferences concerning student learning drawn from student performance on open-ended tasks must take into account student beliefs concerning appropriate answers in academic contexts or risk mistaking student inclination for student capability.
- It appears that young children find it substantially more difficult than older children to provide multiple answers to mathematics tasks.

The legitimacy of relating student responses to non-routine and open-ended tasks to curricular content currently being studied continues to be the subject of research. It may not be realistic to expect students of any age to access recently-acquired skills in open-ended or problem-solving situations.

Substantial additional research is required if we are to understand the meanings constructed by students in responding to open-ended tasks sufficiently well to employ such tasks with success in mathematics classrooms.

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THE OTHER CONSEQUENCES OF A PROBLEM-BASED MATHEMATICS CURRICULUM

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INTRODUCTION

Among the debates engaging the energies of the mathematics education community, one of the more energetic has concerned the role of problem solving in mathematics instruction. This debate has encompassed issues from what constitutes a problem to whether problem solving should be the medium or the message of the mathematics curriculum (for instance, Clarke and McDonough, 1989; Lawson, 1990; Owen & Sweller, 1989; Schoenfeld, 1985). Claims and counter-claims have been made regarding the advisability and the feasibility of basing a mathematics syllabus on non-routine mathematics tasks. Attempts to evaluate the success of such curricula have typically employed achievement tests to distinguish student outcomes. The authors of this study felt that a problem-based curriculum would be characterised more appropriately by the belief systems which the instructional program engendered in participating students than by the students' achievement on conventional mathematical tasks.

It is a student's belief systems that are likely to influence the student's subsequent participation in the study of mathematics, to structure their consequent learning of mathematics, and to guide and facilitate the application of mathematical skills to everyday contexts. If it could be demonstrated that student achievement on conventional mathematics tasks was enhanced by a problem-based program, and if student performance on non-routine problem-solving tasks was heightened by such a program, the ultimate value of the instruction would depend still on whether the student chooses to continue to study mathematics, develops a set of beliefs which supports and empowers further learning and sees any relevance in the skills acquired in class for situations encountered in the world beyond the classroom. Conventional instruction does little to address such concerns, and research has commonly ignored such outcomes.

THE INSTRUCTIONAL PROGRAM

In 1989, the California Post-Secondary Education Commission (CPEC) released a request for proposals that would drastically revamp the Algebra I-Geometry-Algebra II sequence. The curriculum envisioned in the guidelines would set "problem solving, reasoning and communication as major goals; include such areas as statistics and discrete mathematics; and make important use of technology" (CPEC, 1989, p. 4). The IMP Curriculum Development Project obtained funding to develop and field test three years of problem-based mathematics that would satisfy the a-f University of California requirements for high school mathematics.

Program Goals

The goals of the Interactive Mathematics Program (IMP) were to:

- broaden who learns mathematics, by making the learning of core mathematics accessible to groups previously under-represented in college mathematics classes;
- expand what mathematics was learned, consistent with the recommendations of the Curriculum and Evaluation Standards (NCTM, 1989), emphasizing problem solving and the communication of mathematical ideas;
- change mathematics instruction, by requiring students to be active learners and investigators, by integrating the study of mathematical domains, such as algebra, geometry and statistics, with each other and with areas of application, and by making use of current technology;
- change how teachers perceive their roles, by emphasizing the role of the teacher as guide and model learner and by changing dominant modes of classroom communication from teacher explanation to student interaction;
- change how mathematics learning is assessed, by assessing students' use of mathematical knowledge to solve complex problems, and by diversifying assessment strategies to include student portfolios, self-assessment, teacher observations, oral presentations, and group projects, as well as written homework and tests.

Pupil Selection

Methods of selection of pupils for participation in the IMP classes varied. The principal criterion was student self-nomination. Selection methods varied across the three school sites. One high school collected information on student performance, instructional preferences and academic history, and then selected "60% of the group who would have been placed in Algebra and 40% from those below." It did not appear that the sample of IMP students arising from the various selection criteria were of higher academic standing than those students in conventional Algebra classes.

Teacher Selection

Teachers were also chosen by self-nomination.

The IMP Materials

The IMP materials consist of modular units, each requiring approximately five weeks of instructional time. These units employ historical, literary, scientific and other contexts to provide a thematic coherence to the students' exploration of mathematics. For instance, in one unit the Edgar Allan Poe short story *The Pit and the Pendulum* is used to facilitate student investigation of variation, measurement uncertainty, normal distribution, graphing, mathematical modelling, and non-linear functions. The instructional sequence of each unit addresses mathematical concepts and skills and mathematical problem solving in a context that provides both the rationale for the skills being acquired and a means of integrating newly acquired knowledge within a coherent structure.

An IMP Classroom

Class size averaged around 32. Classroom activities were typified by group work, writing, and oral presentations. Graphing calculators were available at all times. The characteristics of IMP and Algebra classrooms, as perceived by the pupils, were documented in the course of this study, and are detailed in the results which follow.

Assessment Practices

Priority was given in IMP classrooms to a diversity of assessment strategies, consistent with the program goals. For example, in one IMP class, grades were calculated from student performance on homework (30%), classwork and class participation (30%), problems of the week (30%), and unit assessments (10%). It appeared that most assessing of Algebra students was through weekly quizzes and chapter tests.

METHOD

Subjects

The subjects of this study were 182 students at three Californian high schools participating in the IMP program outlined above. In addition, matching data were collected on 74 Algebra 2 students and 143 Algebra 4 students from the same schools. Data on a further 52 Algebra 2 students were collected from a fourth high school to provide a numerically comparable sample of students at the same level as the IMP pupils.

Procedures and Measures

During June, towards the end of the academic year, all students completed two questionnaires:

- a *Mathematics Belief* questionnaire examining student perceptions of their mathematical competence, and student beliefs about mathematical activity, the origins of mathematical ideas, valued activities which assist in learning mathematics, typical classroom activities in mathematics, and attitudes to mathematics;

- a *Mathematics World* questionnaire requiring students to identify the extent to which specific everyday activities were mathematical.

At the time of administration of the questionnaires, IMP students had completed almost one year in the program.

In addition, the next Fall the Mathematics Scholastic Aptitude Test (SAT) was administered to the school populations, facilitating comparison of the mathematics performance of IMP students with their peers in conventional classes.

Mathematics belief. The mathematics belief questionnaire was adapted from an instrument employed to measure the student belief outcomes of an innovative program employing student journals (Clarke, Stephens and Waywood, in press). Every item had been validated through interviews with students. Minor changes in phrasing were made for administration in American schools.

Mathematics world. The mathematics world questionnaire was adapted for American administration from an instrument employed in a study of community perceptions of mathematical activity (Clarke and Wallbridge, 1989; Wallbridge, in progress). In this questionnaire, students were asked to indicate whether they thought specific everyday activities were highly mathematical, quite mathematical, slightly mathematical, barely mathematical, or not mathematical.

RESULTS

The results which follow make reference to three groups of students:
i) 180 IMP students - mean age 15.3 years
ii) 126 Algebra 2 students - mean age 15.4 years
iii) 137 Algebra 4 students - mean age 16.9 years

Comparing the Algebra 2 and Algebra 4 Samples

In all, 48 student measures were generated through the two questionnaires. Algebra 2 and Algebra 4 samples (n = 126 and n = 137, respectively) did not differ significantly on any of the 48 measures except the use of worksheets, for which the Algebra 4 students recorded an even lower incidence than did the Algebra 2 students, and the importance accorded to the teacher's explanations: Algebra 4 students attached lower importance to these than did the Algebra 2 students. It seems reasonable to summarise these findings by observing that, with respect to the beliefs documented here, conventionally-taught students adhere to a specific set of beliefs with a high level of stability over time. These beliefs and the associated perceptions of classroom practice were sufficiently coherent and distinct from those held by IMP students to suggest general belief characteristics of both class types. Results are given as comparisons between IMP and Algebra 2 students since these represent the most similar sample populations. However, the uniformity of view among the Algebra 2 and Algebra 4 samples from four Californian high schools suggests a homogeneity among the "conventionally taught" pupils that supports the representativeness of the Algebra 2 sample and encourages extrapolation of the findings of this study as they relate to text-oriented, conventionally-taught, US and Australian classrooms.

In each table where comparisons are made between groups the corresponding p-value is given. A complete listing of all questionnaire items and a more extensive discussion of findings can be found in Clarke, Wallbridge, and Fraser (1992).

Student Mathematics Achievement on Conventional Tests

Where comparison was possible between IMP and Algebra 2 students at the same school, mean SAT scores for IMP classes were higher than mean SAT scores for traditional Algebra/Geometry classes. At one high school this difference in performance was statistically significant. These results are documented in Table 1.

Table 1: Comparison of mean SAT scores of Algebra and IMP students on two school sites

School	Class type	Mean SAT score	S.D.	p value
A	Algebra (n = 83)	420.48	82.96	.0372
	IMP (n = 74)	443.37	77.21	
B	Algebra (n = 86)	367.56	57.02	.1003
	IMP (n = 67)	373.88	60.95	

Student Perceptions of Their Mathematics Competence

IMP students were significantly more likely to rate themselves highly on how good they were at mathematics than were Algebra 2 students (Table 2).

Sample Item:
If I had to give myself a score out of 10 to show, honestly, how good I think I am at math, the score I would give myself would be:

Table 2: Mean self-rating scores for IMP and Algebra 2 students

Class Type	Mean	S.D.	p value
Algebra 2	6.86	1.2	.0012
IMP	7.5	1.38	

Two comments should be made concerning this higher self-rating by IMP students. First, SAT scores indicated that where comparison was possible IMP students tended to be more capable at conventional mathematics tasks than were their peers in Algebra classes, which suggests that these self-ratings had some basis in fact. Second, the difference in self-ratings can also be interpreted as a difference in confidence. We would suggest that heightened self-confidence in mathematics is likely to lead to increased participation in further mathematics, and a greater likelihood that the student will make use of the mathematical skills acquired. Both are desirable outcomes.

Student Attitude to Mathematics Classes

IMP students were significantly more likely to feel positive about mathematics classes (Table 3).

Sample item:

How do you feel in math classes at the moment?
A. Interested B. Relaxed C. Worried D. Successful
E. Confused F. Clever G. Happy H. Bored
I. Rushed J. (Write one of your own) ___________

A student attitude index was calculated by scoring each positive response +1 and each negative response -1, and summing for each student.

Table 3: Mean student attitude index for IMP and Algebra 2 students

Class Type	Mean	S.D.	p value
Algebra 2	-.52	1.85	.0001
IMP	.97	2.14	

Student Perceptions of Mathematical Activity

The distinguishing characteristic between the problem solving students and the Algebra 2 students was the degree to which they perceived mathematics to be a mental activity (Table 4).

Sample Item

Mathematics is something I do (circle one or more):
A. Every day as a natural part of living
B. Mostly at school
C. With a pencil and paper
D. Mostly in my head
E. With numbers

Table 4: Student perceptions of mathematical activity (proportion of students selecting each alternative)

Response alternatives	Class type	Proportion (%)
Every day as a natural part of living	Algebra 2	49
	IMP	52
Mostly at school	Algebra 2	63
	IMP	64
With a pencil and paper	Algebra 2	41
	IMP	42
Mostly in my head	Algebra 2	27
	IMP	39
With numbers	Algebra 2	51
	IMP	46

Table 4 is significant in the context of this paper in that it was only in these perceptions of mathematical activity that the IMP and Algebra students responded in a similar fashion. The marked differences in beliefs and perceptions reported by the two groups, which constitute the essential findings of this study, are only evident in Table 4 in the greater inclination for IMP students to report mathematics as being a mental activity.

Student Perceptions of Mathematical Ideas

IMP students were more likely to agree that mathematical ideas could be clearly explained using every day words that anyone could understand, than were Algebra 2 students. IMP students were also less likely to view the ideas of mathematics as ones that can only be explained using numbers and language specific to mathematics. The IMP students were more likely to view mathematics as having developed in response to people's needs. The IMP students were also less likely than the Algebra 2 students to view mathematics as having been invented by mathematicians or to hold that the ideas of mathematics have always and will always be true. Table 5 documents these differences.

Sample Item
The ideas of mathematics

A. Have always been true and will always be true.	Agree	Disagree
B. Were invented by mathematicians	Agree	Disagree
C. Were discovered by mathematicians.	Agree	Disagree
D. Developed as people needed them in daily life.	Agree	Disagree
E. Have very little to do with the real world.	Agree	Disagree
F. Are most clearly explained using numbers.	Agree	Disagree
G. Can only be explained using mathematical language and special terms.	Agree	Disagree
H. Can be explained in everyday words that anyone can understand.	Agree	Disagree

Table 5: Students' perceptions of the ideas of mathematics
(Agree/Disagree responses were scored +1/-1, so mean scores represent the proportionate majority view.)

Sub-items	Class type	Mean
Have always been true and will always be true.	Algebra 2	.016
	IMP	-.28
Were invented by mathematicians.	Algebra 2	-.13
	IMP	-.36
Were discovered by mathematicians.	Algebra 2	.18
	IMP	.01
Developed as people needed then in daily life.	Algebra 2	.57
	IMP	.77
Have very little to do with the real world.	Algebra 2	-.72
	IMP	-.82
Are most clearly explained using numbers.	Algebra 2	.26
	IMP	-.02
Can only be explained using mathematical language and special terms.	Algebra 2	-.41
	IMP	-.69
Can be explained in everyday words that anyone can understand.	Algebra 2	.21
	IMP	.63

In summary: IMP students were more likely to hold a socially-oriented view of the origins and character of mathematical ideas rather than a platonist belief in the existence of mathematical absolutes awaiting discovery.

Student perceptions of School Mathematics

The IMP students were significantly more likely to agree that writing was important in helping them to understand mathematics. The IMP students were also more likely to see value in talking to other students than were the Algebra 2 students. The IMP students were significantly less likely than the Algebra 2 students to view drill and practice as the best way to learn mathematics.

Sample Item:
Circle the alternative which best describes how true you think each statement is (SA = Strongly Agree, A = Agree, D = Disagree, and SD = Strongly Disagree):

- Explaining ideas clearly is an important part of mathematics SA A D SD
- Mathematics does not require a person to use very many words SA A D SD
- Writing is an important way for me to sort out my ideas in mathematics SA A D SD
- Talking to other students about the mathematics we are doing helps me to understand SA A D SD
- Drill and practice is the best way to learn mathematics SA A D SD

The distinguishing characteristics between the IMP students and the Algebra 2 students were:

- the importance attached by IMP students to writing in mathematics (p = .04)
- the degree to which IMP students perceived talking to other students as useful in helping them to understand mathematics (p = .0005)
- the relative importance attached to drill and practice by the Algebra 2 students (p = .0001)

Student Perceptions of Mathematical Activity at School

The greatest degree of difference between IMP students and the Algebra 2 students was evident in their perceptions of mathematical activity at school.

Table 6 illustrates the differences in student perceptions of their mathematics classrooms. In these figures, the differences between the two class types are most clearly illustrated. Key differences between IMP and Algebra 2 classes can be summarised as follows:

- IMP students were significantly more likely to be writing words and drawing diagrams, and less likely to be writing numbers.
- IMP students were significantly more likely to be working with a friend or with a group, and less likely to be working on their own.
- While there was no difference between IMP and Algebra 2 classes in the relative frequency of listening to the teacher, IMP students were significantly more likely to be listening to other students than were students in Algebra 2 classes.
- IMP students were significantly more likely to be working from a worksheet and less likely to be copying from the board or working from a textbook.

Students were asked to respond on a five-point scale to the cue "When doing mathematics at school, I am likely to be ..." The mean values in Table 6 should be read as students' perceptions of the relative frequency (on a 5-point scale) with which they engaged in each of the listed activities.

Table 6: Mean relative perceived frequency of student engagement in each activity (5-point scale).

Sub-items	Class type	Mean	S.D	p value
Talking	Algebra 2	2.25	1.0	.002
	IMP	2.59	.88	
Writing numbers	Algebra 2	3.13	.86	.0001
	IMP	2.66	.83	
Writing words	Algebra 2	1.70	.93	.0001
	IMP	2.68	.94	
Drawing diagrams	Algebra 2	1.85	.82	.0001
	IMP	2.70	.83	
Working on my own	Algebra 2	2.59	.90	.0001
	IMP	1.91	.87	
Working with a friend	Algebra 2	2.10	.90	.0001
	IMP	2.69	.80	
Working with a group	Algebra 2	1.96	1.0	.0001
	IMP	3.17	.80	
Listening to the teacher	Algebra 2	2.80	.98	.63
	IMP	2.75	.96	
Listening to other students	Algebra 2	2.19	.96	.0001
	IMP	2.79	.83	
Copying from the board	Algebra 2	2.40	1.02	.0001
	IMP	1.91	1.04	
Working from a textbook	Algebra 2	3.15	.93	.0001
	IMP	0.25	.70	
Working from a worksheet.	Algebra 2	2.05	1.06	.0001
	IMP	3.32	.97	

Student Perceptions of the Relative Importance of Course Components

IMP students valued working with others significantly more highly than did Algebra 2 students (p = .0001). By contrast, Algebra 2 students valued the teacher's explanations (p = .0005), and the textbook (p = .0001) more than did IMP students.

Student Perceptions of Mathematics in Everyday Activity

IMP students were significantly more likely to identify a mathematical component in everyday activities than were Algebra 2 students. This result is evident in Table 7.

Table 7: Mean math world index for students in Algebra and IMP classes

Class type	Mean	S.D.	p value
Algebra 2 (n = 113)	19.292	5.591	.0014
IMP (n = 172)	21.477	5.598	

This greater willingness to recognize mathematical elements in everyday activities may be a consequence of the problem-based and context-specific nature of most of the IMP program.

Gender differences

Comparison was made in this study of the attitudes to mathematics of boys and girls in IMP and Algebra classes, and of the boys' and girls' self-ratings of their mathematics competence. These results are shown in Table 8.

Table 8: Self-ratings and attitude measures for students in Algebra 2 and IMP classes: Comparison by gender.

Class type	Measure	Gender	Mean	S.D.	p value
Algebra 2	Self-rating	Male (n = 58)	7.333	1.875	.0101
		Female (n = 67)	6.433	1.994	
IMP	Self-rating	Male (n = 77)	7.636	1.297	.081
		Female (n = 96)	7.271	1.410	
Algebra 2	Attitude	Male (n = 58)	-0.345	1.821	.3176
		Female (n = 68)	-0.676	1.872	
IMP	Attitude	Male (n = 78)	0.756	2.021	.2344
		Female (n = 96)	1.146	2.234	

Girls in both class types were less likely to rate their own mathematical competence highly than were boys. However this difference was only statistically significant for students in Algebra classes. Both boys and girls in IMP classes had similar positive attitudes to mathematics. In Algebra classes, both male and female students felt negatively towards mathematics, however boys' attitudes were less negative than those of girls. On the basis of these findings, it appears that the IMP program was of particular value to female students.

CONCLUSIONS

The inclusion of the Algebra 4 sample in the study encourages the extrapolation of conclusions from comparisons of class types at a specific grade level to more general conclusions comparing problem-based and conventional instruction for high school mathematics classes. The conclusions which follow, however, relate specifically to the study sample.

A. The students as learners.

1. IMP students rated themselves as significantly more mathematically able than did the Algebra students.
2. IMP students held a significantly more positive attitude towards their mathematics classes than did the Algebra students.
3. On school sites where comparison was possible, IMP students averaged higher SAT scores than did pupils of conventional classes.
4. IMP classes appeared to have less negative outcomes for girls than did conventional Algebra classes.

B. Student perceptions of mathematics

5. IMP students were significantly more likely to perceive mathematics as a mental activity.
6. IMP students held beliefs consistent with a view of mathematics as arising from individual and societal need; while Algebra students were more likely to view mathematical ideas as having an independent, absolute and unvarying existence.
7. The IMP students were significantly more likely to perceive mathematics as having applications in daily use.
8. IMP students were significantly more likely than Algebra students to believe that mathematical ideas can be expressed "in everyday words that anyone can understand".

C. Instructional alternatives

9. IMP students attached significantly more value to interactive learning situations; whereas, Algebra students valued "the teacher's explanations" and "the textbook".
10. IMP students valued writing and talking to other students as assisting their learning. Algebra students were significantly more likely to value "drill and practice".

11a. It is possible to identify a coherent and consistent set of classroom practices which can be associated with conventional instruction (cf. Clarke, 1984). The relative frequency of particular classroom activities reported in Clarke (1984) is consistent with the emphases reported by students in Algebra classes in this study.

11b. IMP students' reports of the frequency of engagement in particular activities suggest a set of classroom practices which would identify, in the students' view, the characteristics of the IMP classroom.

11c. The characteristics of these two instructional models appear sufficiently distinct to represent clear alternatives.

In conclusion, the classroom practices of the IMP program, as reported by the students, placed greater emphasis on a variety of modes of communication and on facilitating student-student interaction than was the case with conventional instruction. By contrast, conventional instruction was perceived as solitary, text-driven, and typically expressed through special terms and numbers.

To what aspect of the IMP experience might we attribute the student beliefs documented in this study? The small-group, interactive classroom and the problem-based mathematics curriculum represent two key characteristics of the Interactive Mathematics Project. Whether such belief systems would arise in interactive classrooms lacking a problem-based emphasis or in more conventionally-taught problem-based classrooms is a matter for further research.

With respect to mathematics achievement outcomes, IMP students were at least as successful as their conventionally-taught peers on conventional tasks, as represented by the SAT. Performance on non-routine tasks will be investigated in a later study.

Certainly the IMP program has provided students with significantly different experiences from those found in conventional mathematics classes, and these experiences appear to have led to demonstrably different beliefs about mathematical activity, mathematics learning, school mathematics, and the mathematics evident in everyday activity.

This research suggests that a problem-based curriculum is capable of developing traditional mathematical skills at least as successfully as conventional instruction, while simultaneously engendering measurably different belief systems in participating students.

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MULTIMODAL FUNCTIONING IN NOVEL MATHEMATICAL PROBLEM SOLVING

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Problem-solving in school mathematics has traditionally been considered as belonging only to the mode of thinking concerned with making logical connections between data and the mathematical model and then teasing out the relationship between the variable in the model and the concrete symbolic mode. Little, if any, attention has been given to the place of the intuitive processes in the context of mathematical problem-solving. The paper will present some of the results obtained in the early stages of the study.

The data presented in this study were collected as part of a larger study which considered in more detail the strategies used in solving different types of mathematical problems (Collis, Watson and Campbell, 1991). These strategies are considered here as is the relationship of strategy selection to performance on tests commonly used to gauge mental imagery and creativity. The selection of strategy used was judged in relation to recent work on multimodal cognitive functioning.

Biggs and Collis (1991) point out that much of our thinking in the area of problem-solving is multi-modal. In particular, students at the early high school level, when presented with a novel mathematical problem, have available three modes of intellectual functioning which they can bring to bear to seek a solution, sensori motor, ikonic and concrete symbolic. The first of these is usually not of great significance in school-based problem solving and will not be discussed here. The second is highly developed by the time students reach the age level of interest here and has reached this high level largely without school help. The third is very dependent on school-based teaching, and the subsequent learning, of the concrete symbolic systems of reading, writing and arithmetic - in this paper the last is the major concern.

The move to the concrete symbolic mode marks a significant shift from a direct imaging of reality to a written, higher order symbolization of reality. These second order symbols form a system with direct referents in the experienced world. The symbols also have a logic and an order between themselves which are represented directly in reality and give us one of our most powerful tools for acting on the environment. These symbol systems include written language, mathematics, maps, musical notations and so on. Collis and Romberg (1991), analysed a sample of the kinds of open-ended problem solving items used in a variety of recent assessment projects and found that all of the projects that they analysed in their study made use of open questions. These consisted of items in which the students had to construct their own responses. Although the items varied on many dimensions, they all set out to test the higher order aims of mathematical problem solving and to reveal the student's reasoning as he or she moves towards a solution. The basic format for this type of item came down to one in which the context was set by a series of propositional statements followed by questions to which the student was expected to construct a response. The student needed to take the given propositions and decide on a course of action which might be schematized as in Figure 1.

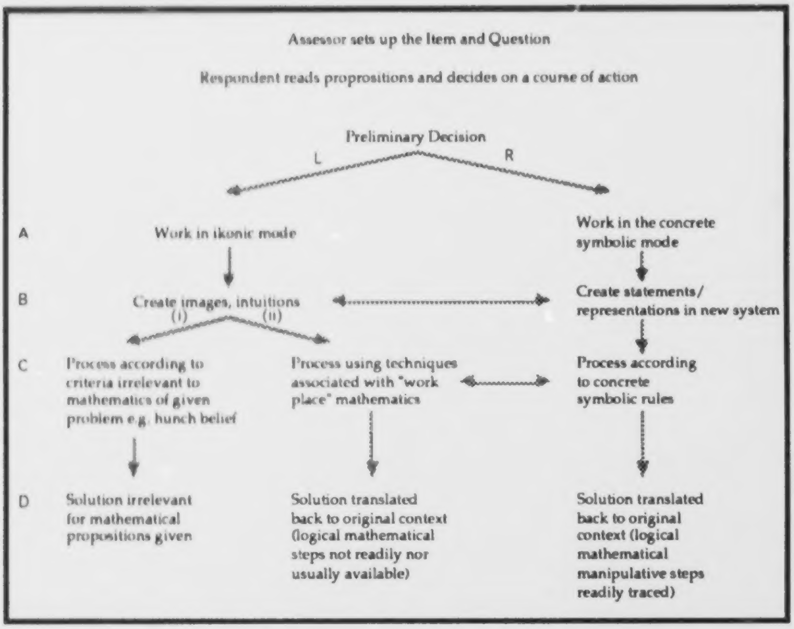
Assessor sets up the Item and Question
Respondent reads proprositions and decides on a course of action
Preliminary Decision
L
R
A
Work in ikonic mode
Work in the concrete symbolic mode
B
Create images, intuitions
(i)
(ii)
Create statements/ representations in new system
C
Process according to criteria irrelevant to mathematics of given problem e.g. hunch belief
Process using techniques associated with "work place" mathematics
Process according to concrete symbolic rules
D
Solution irrelevant for mathematical propositions given
Solution translated back to original context (logical mathematical steps not readily nor usually available)
Solution translated back to original context (logical mathematical manipulative steps readily traced)

Figure 1: The Problem Solving Path (adapted from Collis & Romberg, 1991)

An initial research question in relation to the model in Figure 1 concerns the relationship between (a) the strategy choices made while problem solving and (b) scores obtained on tests traditionally used to assess visualization and intuitive creativity skills hypothesized to be important factors in ikonic functioning. The latter were obtained by using the Betts Questionnaire on Mental Imagery (QMI) (Sheehan, 1967), the Gordon Test of Visual Imagery Control (Gordon, 1949) and the well-known 'Uses of a Brick' Creativity test (see Guilford, 1967).

THE STUDY

The Items

Problems were selected which could be solved by techniques associated with the different courses of action a student might decide to take (see Figure 1). Two categories of problems were identified according to expected solution strategy, those which would suggest the use of concrete-symbolic processes (CS) and those which would suggest ikonic processing (IK). On this basis, two sets of problems were prepared for use in interviews (see Collis et al, 1991) with both categories of problem present in each set.

The following problem is an example of a CS item:

Chickens and Goats Problem. *A farmer is counting the chickens and goats in his yard. He counts a total of 50 heads and 140 feet. How many chickens and how many goats does the farmer have?*

The most likely methods of solution involve symbols and diagrams; that is, methods that are primarily school-taught, based on concrete symbolism and traditionally associated with mathematical problem solving in the school context. An example of a second type of problem, where a different set of solution methods would be expected, is the following:

Cube Painting Problem_ *A cube that is 3 cm by 3 cm by 3 cm was dipped in a bucket of red paint so that all of the outside was covered with paint. After the paint dried, the cube was cut into 27 smaller cubes, each measuring 1 cm on each edge. Some of the smaller cubes had paint on 3 faces, some on 2 faces, some on only 1 face, and some had no paint on them at all. Without drawing the cube, explain how you would find out how many of each kind of smaller cube there are.*

This problem would appear to involve a significant visual or imaging, intuitive component, even if the individual attempts to draw a diagram or picture during the course of seeking a solution.

Sample and Test Administration

Two Advanced Mathematics classes in a Tasmanian High School were selected as the sample. There was a total of thirty-eight students altogether, nineteen in year 9 (mean age: 14 years 10 months; 11 boys, 8 girls) and nineteen in year 10 (mean age: 15 years 10 months; 12 boys, 7 girls). At the first testing session the three group tests noted earlier were administered, two to assess aspects of each student's ability to image situations and one to obtain a measure of each student's creativity.

The Interviews

Sixteen students were selected for interviewing individually on their problem solving strategies. For each interview, the interviewer sat next to the student. Problems, or parts of problems, were presented to the student one at a time on cards. Once the problem was presented students could start solving the problem whenever they were ready; each had a pen and paper, ruler and calculator available if required. The order of presentation of the problems was counterbalanced. In addition, to check on the effect of images associated with each part of a question on the method of solution, some problems were serialised. Students were asked to tell the interviewer everything that came into their minds while reading the problem (or a part of the problem) and while solving it. During the course of the interview, the interviewer prompted the student, if necessary and where appropriate, by asking if they 'saw' any aspects of the 'story', if they were anticipating what the problem was about or what information they thought they still needed to solve the problem. The interviews were tape-recorded.

PLAN OF ANALYSIS

Categorisation of the Interview Data

A variation of Haylock's Think-Board (Haylock, 1984) was used to explore the students' responses in relation to the mode (CS or IK) which each individual statement apparently represented. The Think-Board was adapted so that it could be used to record the problem solver's actual 'moves' during the process (see Figure 2). The broken line divides the board into two regions, the upper region represents concrete symbolic (CS) responses while the lower half represents ikonic (IK) responses. Each of these regions is divided into three smaller regions by the diagonals, these smaller regions represent subjects of the CS and IK regions of which they form a part.

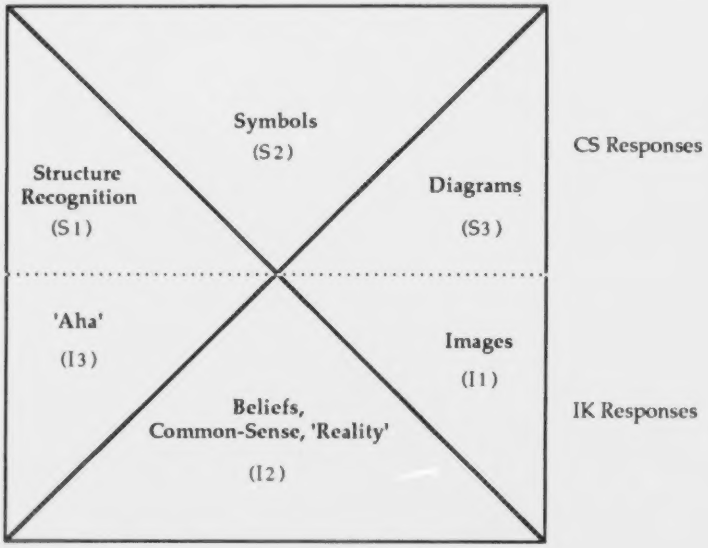

Figure 2: The Adapted Think-Board

Starting at the top left, the smaller regions are defined as follows: *Structure Recognition:* recognition of a CS structure in a problem as similar to a previous example; *Symbols:* use of a CS procedure; *Diagrams:* the use of a CS type diagram, e.g. a graph; *Images:* reporting visual images related to the problem or its context, e.g. "I can see the pieces of the cube with one side painted."; *Reality, Beliefs etc:* use of real world experience which appears to have some 'practical' relationship to the problem, e.g. "Everyone knows that you can't run a car without petrol!"; and *'Aha' Experience:* a sudden, apparently unbidden, 'insight' into the structure of the problem, usually visual.

Interviews were transcribed and each student's responses to each question were mapped onto a Think-Board. The number of responses falling in the upper (CS) half of the board and the number falling in the lower (IK) half of the board for each question were determined and used as the basis for the statistical analyses. Where a response was seen to be a combination of more than one section of the Think Board it was decided that it should contribute to the score for each of those sections. Examples of responses and their classification for the two problems noted earlier are shown in Figure 3. They illustrate two very different approaches in terms of ikonic and concrete symbolic functioning.

Example 1. Cube Problem with resulting scores IK 7, CS 1
I am imagining a whole cube that isn't cut up. (I_1) I can see the cube being cut up. (I_1) Now I can see the cube in slices coming away from each other and I can see how many have 2, and how many have 3... (I_1) It is like a Rubix cube. (I_2) [From this image the student worked out: 3 sides (I_1), no sides (I_1) and 2 sides (I_1).] Then I added the numbers together and found that there were 6 left so there are 6 with one side painted. (S_2)

Example 2. Chickens and Goats problem with resulting scores IK 2, CS 5
[On reading the question] I thought of the farmer bending down, counting the legs on the livestock after counting the heads. (I_1) [Pause] A chicken has two feet. (I_2) .. I'll call the chickens X and the goats Y. (S_1 and S_2) [Set up simultaneous equations (S_2) and solved using the substitution method. (S_2)] There are 20 goats and 30 chickens. (S_2)

Figure 3: Examples of responses and their classification.

RESULTS AND DISCUSSION

The results of the data gathered may be looked at most easily in relation to four key questions.

Question 1: Did students use the same primary strategy (CS or IK) regardless of the type of problem set?

Cochran *Q*. tests revealed that when the data were considered using all ten problems, the primary strategies used by the students were not consistent across the two different types of problems (CS & IK) ($Q = 38.86$, $p<.001$). This general finding also held for each set of 5 problems considered separately ($Q = 26.52$, $p<.001$): $Q = 27.30$, $p<.001$). Further analysis to check whether primarily IK strategies were applied in response to CS problems produced non-significant results. ($Q = 3.00$, $p<.10$). Inspection revealed that the strategies applied in these cases were primarily CS. Likewise it was found that students did not consistently apply CS strategies to IK problems ($Q = 0.00$, $p<.99$); rather IK strategies were primarily used to solve IK problems. These results seem to indicate that over and above any individual differences in the use of ikonic or concrete symbolic strategies, the nature of the problem itself is highly significant in determining which type of strategy will be used.

Question 2 (which supplemented Question 1): Did the strategies used depend upon the obvious characteristics of the item?

The dependent *t*-test performed on the pooled CS and pooled IK responses of the whole sample for all items supported the view that *a priori* CS problems elicit more CS responses than *a priori* IK problems ($t(15) = 5.98$, $p<.001$). It was clear that significantly more CS responses than IK responses were given to CS problem types ($t(15) = 2.40$, $p<.05$), and significantly more IK responses than CS responses were given to IK problems ($t(15) = -3.42$, $p<.02$). These results support the conclusion that it is the apparent nature of the problem itself that determines the strategy to be used.

Table 1: Problem Category and Type of Response: Total Number of Responses of 16 Subjects adjusted for Set Type

	CS-type Problems	IK-type Problems
CS Responses	89.2	26.0
IK Responses	56.4	55.6

Question 3: What was the typical path followed by students in the process of solving each type of problem?

The total number of times that students changed modes per type of question was 35.8 for the CS-type problems and 22.6 for the IK type problems, giving means per student of 2.2 (SD=1.1) and 1.4 (SD=1.4), respectively. The dependent *t*-test on the means for each student of the number of modal changes in each type of question produced a significant result ($t(15) = 3.10$, $p<.02$), the direction of the differences indicating that students tended to change modes significantly more often when answering CS-type problems than when answering IK-type problems. A dependent *t*-test on the pooled CS and IK responses (see Table 1) showed significantly more IK responses to be given to CS problem types than CS responses given to IK problems ($t(15) = 3.97$, $p<.001$). This may suggest that students are more likely to utilize IK strategies if they find some degree of difficulty in solving CS problems and are unable to use CS strategies effectively. In the case of IK problems, however, students are unlikely to use CS strategies to assist in their problem solving pathway. In summary, closer inspection of the data summarised in Table 1 indicates that when students are working on a CS approach and find some difficulty in proceeding, they are more inclined to use an IK approach to make progress, whereas when they use an IK approach as the initial step, this either solves the problem or they discontinue their attempt.

Question 4: How did the specific individual characteristics of the students, as measured by standard assessments of imaging and creativity, relate to the nature of their strategies?

The results from the Mann-Whitney *U* tests performed on the pooled CS and IK responses for CS and IK type questions indicate that the degree of overall Vividness may be important in determining the type of strategy adopted by the student during a problem solving task. This effect was found to be approaching significance even with the small sample tested ($U = 17.5$, $p = 0.06$). Since Vividness is a general measure of vividness of imagery which includes such components as kinaesthetic, tactile and aural imagery, among others, the visual subscale (VisVivid) was treated as a separate variable. Further analyses using the Mann-Whitney *U* test indicated that it is this visual component of imagery that strongly influences the student's selection of the most effective strategy to be used in a problem solving task ($U = 6.5$, $p<.001$). It appears that students with high VisVivid scores are more likely to produce IK responses than their counterparts with low VisVivid scores. Finally, measures of both Controllability and Creativity were found not to contribute significantly to the type of strategy adopted by the student during specific problem solving tasks ($U = 22.5$, $p > .05$ and $U = 27.5$, $p > .05$, respectively). In other words, students with high Control scores in relation to visual imagery, or with high Creativity scores are not significantly different from students with low Control or low Creativity scores in the content considered in this paper.

CONCLUSION

Within the limitations set by the data and the small sample, this study highlights some important factors which must be taken into account in the teaching, learning and assessing of school mathematics especially as it relates to problem solving in novel situations.

1. There seem to be two basic approaches to problem solving at this level, one based on CS, school taught procedures, the other related to IK mode processing. The latter, in this study, means appeals to common sense, everyday life and visualisation, and the use of intuitive reasoning. It is interesting to note that students, who see what they believe to be a solution using ikonic approaches, see little point in backing up their solution by an appeal to mathematics.

2. In the course of solving a problem, students will move from mode to mode but are more likely to move from CS to IK when a school based method fails to satisfy rather than vice versa. They do however, move both ways, in the case of IK to CS procedures the move is most often made only after the problem is really solved as illustrated with the student response to the Cube Painting Problem.

3. The characteristics of the problem and its context appear to be the basic factors in determining the initial strategy, particularly when CS type problems are presented.

4. Students who show heightened overall Vividness and have the ability to obtain clear visual representations of a problem in developing ideas tend to prefer using IK strategies during problem solving tasks.

These findings, tentative as they must be because of the limitations of the study, have two important implications for mathematics teachers in particular. First, those responsible for assessing mathematical problem solving at this level, should take into account the fact that the obvious problem characteristics may determine the basic strategy of the students. Second, all teachers should be aware of the significance of the IK mode in problem solving and adjust their mathematics teaching and assessing accordingly.

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STUDENTS' CERTAINTY AND CHECKING BEHAVIOUR DURING MATHEMATICAL PROBLEM SOLVING

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This study investigates the certainty and uncertainty that students feel as they work on a mathematical problem and how this relates to the checking that they carry out. It is hypothesised that the over-confidence in decisions that characterises reasoning in many fields of human endeavour is also exhibited in mathematical work and that it may partly explain why students generally are reluctant to check their work. Students who feel certain that their work is correct would see little reason to check it. In the problem used in this study, students became uncertain when they moved from a particular case where they could count to a much larger case where a general rule was required. They also became uncertain when the arithmetic became harder - the size of this effect had not been expected. Students with wrong methods that gave easy arithmetic were, in the end, almost as certain that their answers were correct as students with the correct method. Students often did not know how to use extra information to check their answers. About half of the students who were correct became less certain after being given supporting information.

How confident students feel in the correctness of their answers to mathematical questions is a subject that has been little investigated. This paper proposes that it is of interest because how certain one feels in one's decisions and solutions should affect whether one checks or not. A frequent finding of research into the methods and strategies used by students to solve mathematical problems is that they neglect to check (Galbraith, 1986; Stacey, 1989). Several reasons have been given to explain this phenomenon. One is that "reflection" is a neglected phase of problem solving receiving little instructional time. Students do not seem to feel a need to check their work: they respond in an immature way, leaving to the teacher the responsibility of determining whether something is right or wrong. There is also considerable evidence that children do not know strategies for checking their work. Stacey and Groves (1985) noted that many students interpret a verbal instruction to check as only an instruction to repeat. Bell, Costello and Kuchemann (1983) and Galbraith (1986) have also noted that there is widespread misunderstanding about the roles of examples and counter-examples in checking. To check, one needs an adequate understanding of the content involved and also an appreciation of the logical structure of proving things to be true and false. Lee and Wheeler (1987) , for example, point out that students can only use substitution of numbers as a strategy for checking algebra if they have a reasonably clear understanding of the relationship between arithmetic and algebra.

That certainty may also influence checking behaviour was suggested to us by Fischbein's (1987) analysis of the role of intuition in mathematical thinking. Fischbein reviews research from a variety of sources to support his hypothesis that certainty in a reasoning process is produced by reliance on immediate, self-evident, intrinsically certain intuitions. These are necessary for reasoning to be efficient and productive. He sees certitude as a fundamental need of the human mind; without it a reasoning process could not continue. In fact, a substantial body of research into the subjective evaluation of confidence has consistently found that people are overconfident in the accuracy of their own knowledge, decisions , interpretations and solutions (Lichtenstein, Fischoff and Phillips, 1982). This natural tendency to overconfidence may contribute to students' lack of checking, particularly of their reasoning. Although Fischbein's analysis mostly concerns stable conceptual structures and clusters of beliefs, he also recognises "anticipatory" intuitions which are specific to the problem solving process.

Galbraith also noted that when students were asked to select the better of two explanations, many students made their choice of the grounds of simplicity. This observation has also been made by Stacey (to appear) who observed that when groups of students were solving problems together, they often chose a simple incorrect solution, even when a correct solution had been proposed by a group member. The crucial factor in whether the group solution was correct seemed not to be having the ideas, so much as choosing between them. Misplaced confidence in a simple idea and lack of adequate checking strategies were common faults.

AIMS

With the background outlined above in mind, data were collected to explore the following questions:

i) What factors cause certainty and uncertainty in problem solving amongst students?

ii) How does the certainty of students who answer a question correctly compare to the certainty of students who answer it incorrectly?

iii) Is there a relationship between certainty and checking? Are students who do a problem in only one way more certain of their results than students who search for other approaches?

iv) What is the effect on a student's certainty of information supporting and information contradicting an answer?

METHOD

One problem (see Figure 1) was given to 227 Year 8 students (average age 13 years) at two girls' schools during class time. After completing the problem, the students answered a questionnaire. About 15 students (exhibiting different responses) were interviewed shortly after the problem solving to further elucidate the reasons for the responses. The questionnaire asked students to rate their certainty in their answers to the three parts of the problem on scales from 0 to 10. The zero, five and ten positions on the scale were annotated with descriptive comments, such as "completely sure your answer is wrong" at zero. This measurement of certainty was adapted from calibration studies reviewed in Lichenstein, Fischhoff and Phillips (1982). Then students were asked to indicate, separately for the 10x10 and 50x50 blankets, whether they had done the question in only one way or in more than one way and whether they had obtained one or more answers. This was the criterion selected for judging whether students had checked their work. Finally students were asked to re-assess their certainty on each part of the problem after being presented with one of three pieces of additional information. They were randomly given the correct answer of one of the following cases: 20x20 blanket (version 1), 50x50 blanket (version 2), or both 3x3 and 6x6 blankets (version 3). Version 2 gives them direct information about their answer to the 50x50 blanket. Version 3 is intended to draw students' attention to the incorrectness of a method which involves finding the answer for a larger blanket as a whole number multiple of the answer for a smaller blanket. This will be referred to as the "multiples of previous answer" method. A student using this method assumed the amount of sewing was directly proportional to the size of the blanket, so may, for example, have multiplied by the answer for the 10x10 blanket by 5 to get the answer for the 50x50 blanket. The purpose of giving the three different versions was to compare the relative effect of direct and indirect information on re-assessing certainty.

Imagine you are going to make a patchwork doll's blanket by sewing together some tiny squares of material measuring 1cm by 1cm. You want to know how much sewing you will have to do to make the blanket.

To make a square blanket measuring 3cm by 3cm, you need 9 squares of material and it takes 12 cm of sewing. (Diagram given of 3x3 blanket with sewing between the squares, but not along the outside edge of the blanket, clearly shown)

- *How much sewing is needed to make a square blanket measuring 5cm x 5cm?*
- *How much sewing is needed to make a square blanket measuring 10cm x 10cm?*
- *How much sewing is needed to make a square blanket measuring 50cm x 50cm?*

Figure 1: Problem solved by students in study.

RESULTS AND DISCUSSION

About half (55%) of all students and two thirds of the students who were correct checked their work in some way. Of those who did check, 31% were correct whereas only 18% of those who did not check were correct. This difference is statistically significant (chi-squared=5.3, d.f.=1, $p<0.02$). A t-test showed that students who checked their work and found the same answer in two different ways had significantly greater certainty than students who did not check ($p<0.05$). These in turn had higher certainty than students who checked but found two different answers. These results are as we had expected.

In order to relate changes in certainty to mathematical behaviour, responses were classified according to the solution method used. Some students followed one general rule for all three answers. Some began by drawing the 5x5 blanket and counting the number of centimetres of sewing but used a general rule for the 10x10 and 50x50 blankets. Others used the generalisation only for the third blanket. The mean certainty on each part of the problem is shown in Figure 2 for students consistently using the four most popular methods. Data for the whole group of students are given by Stacey and del Beato (to appear). The most common methods were:

i) the perimeter rule (51 students merely gave the perimeter of all the blankets);

ii) the correct rule, in effect $2n(n-1)$ (46 students, some of whom may have counted to obtain correct results for the smaller blankets);

iii) the "multiples of previous answer" method (14 students) explained above;

iv) the formula n^2+n to find the amount of sewing for the n x n blanket (20 students). Algebraic notation is implied here, although no students used it.

The perimeter and the n^2+n rule were probably popular because they fitted the given information that a 3x3 blanket requires 12 cm of sewing. In all cases shown in Figure 2, and all those not shown here but given by Stacey and del Beato (to appear), students were most certain about their answers for the 5 x 5 blanket and then certainty dropped for the 10x10 blanket and again for the 50x50 blanket. The higher certainty for the 5x5 blanket was associated with use of drawing and counting to obtain the answer. The only two groups that did not count (the perimeter and the n^2+n rule) had the lowest certainties for this part. These groups of students seemed to settle immediately on a solution that happens to fit the given data without exploring the situation in any depth, but they did not do this because they were very certain of the answer. In Fischbein's terms, they did not have an anticipatory intuition of which they were very sure. Instead, they seemed merely to accept the uncertainty and not do anything about it - even the few of them who drew the 5x5 blanket did not even count.

Uncertainty on moving to a generalisation and when calculating.

The responses of some students in the interviews indicated that the two important factors contributing to the drop of certainty as the blanket size increased were:

i) the uncertainty students felt when moving to a generalisation; and

ii) the uncertainty they felt about the results of calculations involving large numbers.

One student, for example, said "as the numbers increased I was less certain that the numbers were correct" and another said she became less certain because "I couldn't draw it so I had to work it out in my head."

The effect of the first factor (making a generalisation) was evident in the much larger mean drop in certainty when students moved from counting to use of a general rule (1.47 averaged over 27 clear instances) compared to the drop when students used the same rule on two consecutive parts of the problem (0.78 averaged over 18 instances). (Students who were correct were excluded from this analysis because it was sometimes unclear whether they had counted or used a rule.)

The effect of the second factor (more difficult calculations) was examined by comparing the mean drop in certainty from the 10x10 to the 50x50 blankets for students whose rules required only simple calculations (e.g. the perimeter rule which only required 50 x 4) with the mean drop in certainty for students whose rules required calculations involving a product of two digit numbers (e.g. 50x50). Students using hard calculations dropped their certainty by 0.96 (averaged over 25 clear instances). A t-test showed that this drop was significantly greater at the 5% level than the mean drop of students using easy calculations (0.21 averaged over 58 instances). The size of this effect had not been expected. This may partly explain why the natural checking of students concentrates so much on repeating arithmetic calculations, where teachers see a need to concentrate on checking the reasoning.

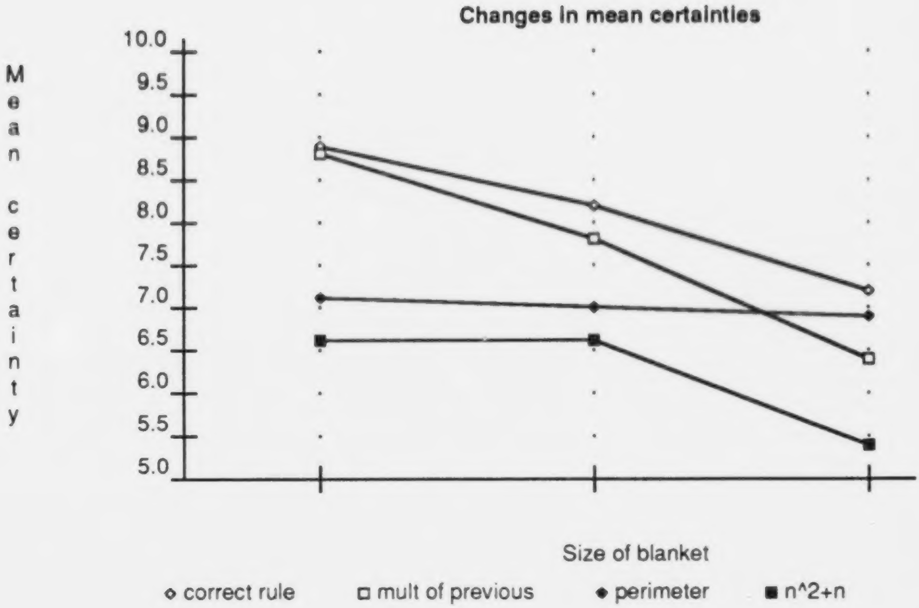

Figure 2: Changes in mean certainty of users of different rules.

Certainties of students with a simple or complex rule.

A comparison of students with the correct rule and the perimeter rule shows the practical implications of the drops in certainty due to generalising and harder calculations. The two factors combine, so that on the challenging part of the problem, the certainties of students who are correct and those who have grabbed the simplest wrong rule apparently without any investigation, are comparable. The perimeter group began with an immediate generalisation, based on little evidence. They did not subsequently have to make the transition to a generalisation and their calculations were at all stages very simple. Thus, although the perimeter group were significantly less certain of their 5x5 blanket answers ($p<0.01$) than the correct group, there was no significant difference between their certainties for the 50x50 blanket and those of the correct group.

Effect on certainty of information supporting or contradicting answers.

The number of students involved in this part of the study was 120 of whom 28 had correct methods and 92 had incorrect methods. Students had been asked to reassess their certainty in their answer for the 50x50 blanket after receiving either Version 1 (correct answer for 20x20 blanket), Version 2 (correct answer for 50x50 blanket) or version 3 (correct answer for 3x3 and 6x6 blankets). The numbers of students in each category are unequal because versions were distributed at random, when students finished solving the problem. For students who had correct methods, Version 2 caused an average increase in certainty of 2.27, whilst the other two versions caused slight decreases (-0.50 for version 1 and -0.14 for version 3) . For students who had incorrect methods, Version 2 caused the greatest average decrease in certainty (-3.46), whilst the other two versions caused smaller decreases (-2.41 for version 1 and -1.65 for version 3). Although Version 2 (the correct information) caused the greatest changes in certainty, only half of the incorrect students who received it dropped their certainty rating to zero. Responses from the interviews suggest that for some students there is a reluctance to completely reject an answer even in the light of contradictory evidence. This, a prediction of Fischbein's theory, has implications for checking. If students are unwilling to admit that an answer is wrong, they are unlikely to check.

The decreases in mean certainty for correct rule students had not been expected. This effect could not be attributed to a few outliers. Of the 17 students in this category, 8 had reduced their certainty. Clearly they had not seen the added information as supporting their (correct) ideas. We postulate that this was because they did not appreciate just how fast a quadratic function such as 2n(n-1) can grow and felt that their (correct) answer of 4900 was much larger than the answers given (760 for the 20x20 blanket). This phenomenon was also observed during the interviews. Some students used the additional information provided in Versions 1 and 3 only as a gross check on the size of their answer, rather than as a test for their own rules. Thus if the answer for the 20x20 blanket was between their own answers for the 10x10 and 50x50 blanket, they were often satisfied that they were correct. This is consistent with other findings that students do not fully appreciate how to check their work.

CONCLUSION

The use of certainty ratings has proved to be a promising research tool for understanding students' thinking during problem solving. When presented with evidence that an answer is incorrect, students are often reluctant to reject their answer completely. Students who jumped quickly to generalisations based on little evidence were just as certain about the correctness of their answers to the final part of the problem as students who were correct. Thus, in a group discussion, both a simple wrong rule and the correct rule may be propounded with equal conviction, leading a group to choose the wrong answer over the right answer. Both making a generalisation and doing arithmetical calculations caused students to lose certainty in their work. Students who have made wrong assumptions at the start of their work which happen to lead to simple processing will be no more likely to check their work at the end than students who are correct. Their simple (wrong) rule has "proved itself" in the ease of calculation of the answers it produces.

When given information which would enable an independent check of a generalisation, many students used it only as a guide to size. They possibly did not see the general rule they had themselves invented and used, as applying in a precise way to the given data. This is one example of the many aspects of checking which students might be taught about.

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LETTING CHILDREN DESIGN THE CURRICULUM

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INTRODUCTION

Since 1989 the Basic Skills Testing Program (BSTP) in New South Wales has gathered data on the performances of some 220 000 year 6 and 120 000 year 3 children on tests of language and mathematics. In mathematics, each of the three strands of Number, Space, and Measurement are defined separately in the reports prepared for teachers and for parents.

This paper describes an exploration of children's responses to problems in the tests in search of any underlying general cognitive performance in mathematics. Some conclusions about order of difficulty within strands and links between strands are discussed. The data used is the responses of about 160 000 students to over 120 problems from the 1989, 1990 and 1991 year 6 tests.

OVERVIEW OF BSTP

'Basic Skills' in Numeracy was defined as that content outlined in the course of study for mathematics (NSW Department of Education, 1989). This syllabus emphasizes problem-solving in real life contexts; the numeracy tests were required to reflect this emphasis. Each year 6 test had separate stimulus material (sale catalogue, magazine, scrapbook) which provided the context for the problems posed in the test.

With such a large number of children to be assessed, the decision was made to use optical mark-sense readers, which allowed children to respond to problems by marking their booklets with an ordinary HB pencil. Within the constraints of machine-readable responses, many questions were developed in as open-ended a form as possible.

Parents received a report on their child. Teachers received a report on each child and a report on the performance of their school group.

The development of the BSTP, including the nature and purpose of the reports provided, is described in the report of the 1989 program (Masters, Lokan, Doig, Khoo, Lindsey, Robinson and Zammit, 1990).

TOWARDS A "CHILDREN'S CURRICULUM"

What an analysis of system-wide assessment can give is a picture of the children's preferred sequence of the curriculum. For example, when considering fractions, many curricula recommend that unit fractions be taught in the order $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and so on. This appears to be a reasonable thing to do. On the other hand, research indicates that children's 'difficulty order' is $\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}$ (Masters, 1987).

In much the same way, Stern proposed a 'child's order' to the teaching of basic number facts in place of the traditional order, which followed common numerical order (i.e., first the 2 times tables, then the 3 times, then the 4 times, etc.) (Stern and Stern, 1971).

In the BSTP context, the children's preferred sequence may not be just a single discrete list of topics but also include interactions between topics described in different strands.

In order to understand what it is that the children are telling us, it is necessary to build a picture from the information they have provided. An analysis of the responses from each strand (Number, Space and Measurement) using an Item Response Theory (IRT) approach makes it possible to build such a picture. (For a full explanation of these techniques see Wright and Stone, 1979. Their use in the BSTP is explained in Masters et al, 1990).

When problems are placed along a continuum defined by the difficulty that children had in answering a problem correctly, it becomes possible to understand what we are being told. Figure 1 shows these continua for each of the three strands for BSTP Numeracy in year 6.

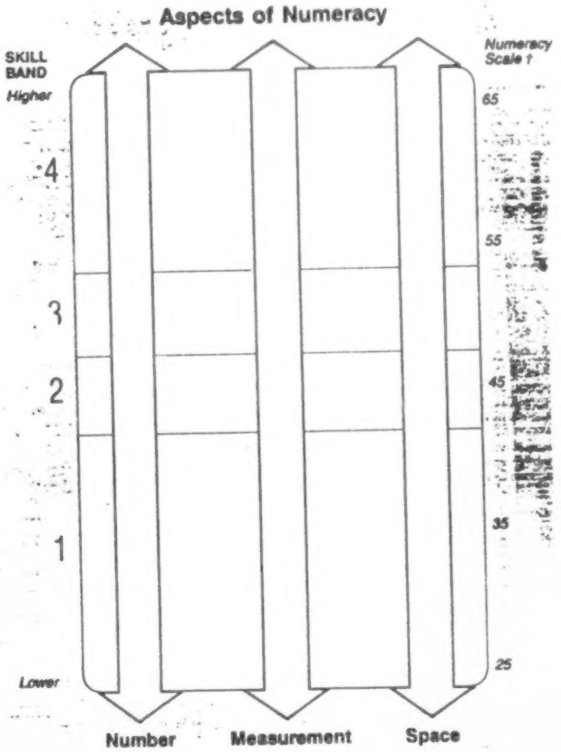


Figure 1: Difficulty continua for the three strands for Numeracy, Year 6.

It is possible now to analyze those problems which cluster at particular points along these continua. This analysis involves, in the first instance, describing the problems in terms of their curriculum content. This procedure leads to the creation of 'skill bands' or 'skill levels' (Masters et al, 1990). In this way "the ... scale developed to measure performance ... reflect[s] the student's level of maturity in that domain [curriculum content]" (Romberg, 1989: page 15). Figure 2 shows the skill band descriptions for year 6 1991.

Number	**Measurement**	**Space**
AT BAND AND THE BANDS LISTED BELOW IT, STUDENTS IN THESE BANDS ARE GENERALLY ABLE TO		
• solve a problem using division when the reverse process is suggested by word clues • recognise square numbers *(eg 4, 9, 16, 25)* • solve problems involving several steps, including 'rounding off' • estimate a percentage *(nearly 50%)* from a pie graph	• solve problems involving a mixture of measurement units *(grams / kilograms* or *millilitres / litres)* • work out length from a scale drawing on which width is shown • mark on a scale the increase in water level when objects *(four marbles, 25 mL each)* are dropped into a container	• interpret information shown on a line graph *(the first 100 kilometres took about ...?)* • work out how many identical 3D objects will fit into a given space *(6 cm tennis balls in a 12 cm x 12 cm x 12 cm box)* • work out the number of objects represented by part of a pie graph • choose a shape which can be used to form a pattern without gaps *(to tile a floor)*
• use several steps to locate information *(postage costs)* in a table • work out a fraction of a quantity *(in a recipe)* • estimate how many items *(at $1.35 each)* can be bought with a given amount of money *($5)*	• choose the best estimate for an area *(car windscreen in square metres)* • recognise the temperature of boiling water • convert 12 hour time to 24 hour time • estimate the masses of everyday objects *(Which object weighs about one kilogram?)*	• identify the top view of a model given a drawing which shows a different viewpoint • recognise the simple shapes which combine to make a given shape • use compass directions and landmarks to find a position on a map
• choose the calculation to give the cost of items sold in sets *(packs of 4 tins cost $1.60; 8 tins cost 2 x $1.60* NOT *$1.60 ÷ 2* or *8 x $1.60* or *$1.60 ÷ 8)* • estimate the cost of one item when the price for a set is given *(6 pens for $1.85)* • work out whether to use +, −, ×, ÷ to solve a problem	• solve a problem involving the conversion of length units *(centimetres to metres)* • work out and compare the areas of shapes drawn on a grid • locate a date on a calendar *(where the days are shown by initial letters)*	• match a box to the cut-out from which it is made • identify the compass direction of one place from another
• add or subtract four-digit numbers where trading is needed *(2053 + 1674; 6125 − 4038)* • use word clues like **altogether** to work out that multiplication is needed to solve a problem • work out that subtraction is needed when diagrams and word clues like **difference** are given	• work out a time which is in the next hour (on a digital clock) *(6 minutes later than 12:57 p.m.)* • estimate the height *(in metres)* of an object visible in the classroom *(doorway)*	• show the reflection of a pattern on a grid • follow instructions to find a position on a simple grid *(gameboard)* • use coordinates to find a place on a simple map *(Colour in the building at position E2.)*

Figure 2: Skill band descriptions (Year 6, 1991).

EXPLORING THE DATA

The method used to explore the 1989,1990 and 1991 year 6 data was as follows:

- all problems relating to a strand were composed into a single list so that number, space and measurement problems could be examined separately;
- the problems on these lists were arranged in difficulty order as determined from the students' responses, using an IRT model;
- problems listed were then grouped to form four levels or 'skill bands' (Masters et al, 1990);
- each problem was examined to decide the type of mathematical skills required to complete it;
- the fundamental conceptual elements being used by children to solve a problem were inferred;
- the conceptual content of each strand for each level was compared within and between strands.

To illustrate the method, its application to the Number strand is described in detail below. The first level/skill band described is of those problems that the children found to be the easiest, and subsequent levels are in order of increasing difficulty.

Band One

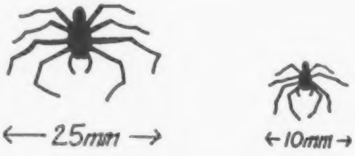
The problems at this level begin to introduce mathematical conventions as well as obvious synonyms for number operations. Key words and phrases replace the symbols of mathematics and it is the ability to translate from the real world to that of symbol and convention that marks those that succeed at this level.

38 6297 people went to the BIG X sale In this number, what does the '2' stand for?

○ 2 ones
○ 2 tens
○ 2 hundreds
○ 2 thousands

Figure 3: Question 38, 1989

2 Here are two spiders.



What is the difference between the widths of the two spiders?

○ 250 mm ○ 15 mm
○ 35 mm ○ 10 mm

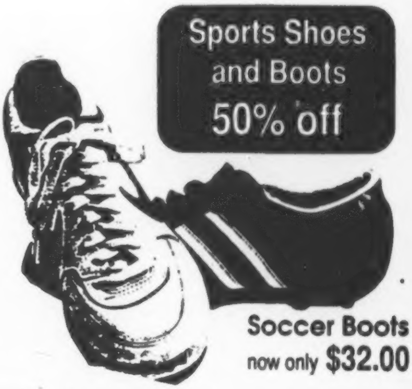
Figure 4: Question 2, 1991

Question 38, 1989: (Figure 3) - This problem is about the conventions of the decimal system of numeration. The differences between a single '2' and the '2' used in the context of a four-digit number can only be memorized like any other convention, for there are no clues to help the children here. The high success rate on this problem demonstrates that this convention is accessible to most children.

Question 2, 1991: (Figure 4) - If the word clue 'difference' is recognised as a cue to subtract, the problem is solved. Again, the high success rate demonstrates that most children do recognise this clue.

Band Two

Many problems have specific clues, such as 'altogether' indicating that some combining process like addition is required. Contextually based problems have their solution processes 'hidden' within the context of the problem, with no such obvious word clues. Although most children can perform arithmetic operations correctly, the ability to generalize the meaning of, say, addition, to all additive situations is less common. Extracting enough information in order to choose the correct operation to use is the key to success at this level. In real life this may be the most utilized facet of problem solving.



25 Mike chose some soccer boots at the BIG X sale.
His mother paid for them with a $50 note.
Colour in the notes and coins that would make up the change they got.

$10 $10 $5 $5 $2 $2 $1 $1

Figure 5: Question 25, 1989

Question 25, 1989: (Figure 5) - To calculate the amount of change due in a particular transaction may well be straight-forward, but to actually 'make' the change from a selection of notes and coins involves an additional degree of competency. The skills necessary to manipulate the multiple data, together with correctly performing the chosen processes in combination is typical of problems at this level.

8 Alex is doing some French Knitting. Each centimetre of knitting has five rows of stitches.

How many rows of stitches are there in 10 centimetres of knitting?

- ○ 2 rows of stitches
- ○ 5 rows of stitches
- ○ 15 rows of stitches
- ○ 50 rows of stitches

Figure 6: Question 8, 1989

Question 8, 1989: (Figure 6) - A thorough understanding of the concept of multiplication is necessary to see that it is the correct interpretation of this problem. Either the use of a 'one-to-five' ratio can be used (the function view of multiplication) or a 'ten lots of five' approach (an additive view). In either case a good grasp of the concept is required. The most common error at this level is to confuse multiplication with division, an indication of an uncertain grasp of the concepts involved.

Band Three

Children who have developed the necessary abilities and skills to master the problems at this level are able to extract pertinent information from competing, irrelevant information. Being able to ignore irrelevant data is a sophisticated and valuable skill especially necessary to solve those real life problems involving tables. At one level the table may simply be a list of prices, at another, more complex level, the table may be a bus or train time-table which involves relating at least two pieces of information. To simultaneously vary two aspects of tabular information to gain a solution is an even more difficult exercise in relationship building.

Delivery charges

	Up to 50 km	50 to 100 km	Over 100 km
Under 10 kg	free	$2.00	$8.00
Over 10 kg	$5.00	$10.00	$15.00

42 Sofia lives 83 km from the store. How much will it cost her to have a 3 kg Moulding Set delivered?

- ○ Free ○ $2 ○ $8
- ○ $5 ○ $10 ○ $15

Figure 7: Question 42, 1989

Question 42, 1989: (Figure 7) - To enable a solution to be found for this problem, it is necessary to first determine which distance category is to be used. Next, it must be understood that the (3kg) parcel belongs to the 'under 10kg' class of parcels. The difficulty of combining and relating such pieces of information is evidenced by the proportion of students unable to deal successfully with this problem.

11 The best estimate for the cost of **one** sticker is

○ about 30 cents. ○ about $1.00.

○ about 60 cents. ○ about $15.00.

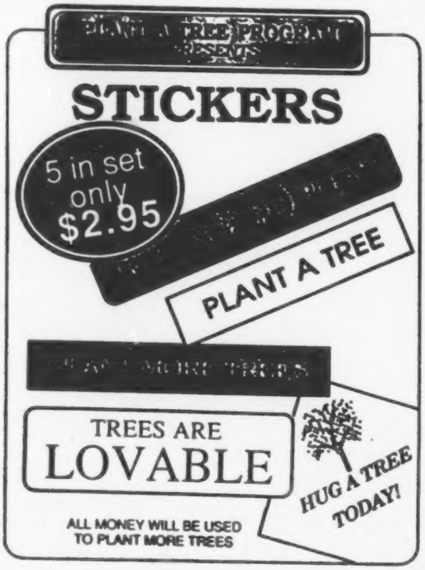


Figure 8: Question 11, 1990

Question 11, 1990: (Figure 8) - The relevant information here is the price of a set and the number in each set. The correct process must then be chosen and a rounding performed either before (a better strategy) or after the division.

Band Four

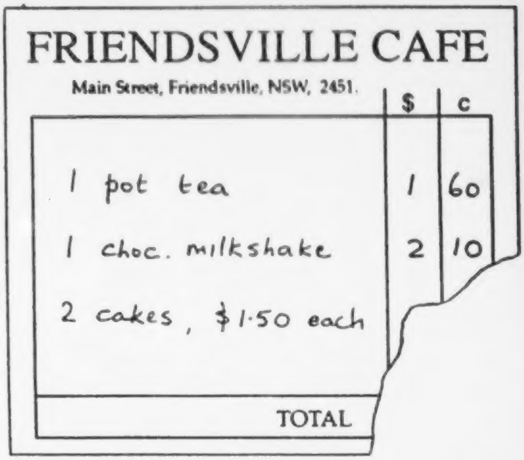
Children who have fully developed understandings of concepts, together with the appropriate related skills are able to solve quite complex problems. The ability to round prices to whole dollars is very important in the daily life of everyone; whether the need is to select the correct notes and coins to pay for goods, or whether it is simply to check the accuracy of the bill itself, working with approximate costs derived from such rounding is without doubt an essential skill. Further, in real life there is often no language clue to which arithmetic processes one should use to solve a problem. Solving these problems would be impossible without a deep understanding of the fundamental concepts of mathematics.

BEST COPY AVAILABLE

Sam and her mother went to the café. *Their bill is on page 13 of the scrapbook.*

What was the total cost, to the nearest dollar?

○ $4 ○ $6
○ $5 ○ $7



FRIENDSVILLE CAFE

Main Street, Friendsville, NSW, 2451.

	$	c
1 pot tea	1	60
1 choc. milkshake	2	10
2 cakes, $1·50 each		
TOTAL		

Figure 9: Question 33, 1991

If you become a subscriber to the magazine (see back page), how much do you save in a year?

○35 cents ○$4.15

○75 cents ○$4.50

Each issue of
THE COMET
costs 75 cents,
but

WHY NOT SUBSCRIBE?

With a subscription you get **THE COMET** delivered six times a year for only **$4.15!**

Figure 10: Question 17, 1990

Question 33, 1991: (Figure 9) and Question 17, 1990: (Figure 10) - Both these complex problems involve the use of several operations in combination. In the first, doubling, adding and rounding are required. In the second, multiplication, dollars/cents conversion then subtraction are required.

The same procedure was applied to problems in the space and measurement strands.

INTERPRETING THE RESULTS

What can we learn from all this information that the children have provided through their responses to these problems?

The most obvious information concerns the order of the assessed content, the increasing complexity of the tasks, or more interestingly, the increasing maturity and sophistication of the children's thinking. This is precisely the notion underpinning Principle 4 for new models of assessment explicated by Romberg (1989), which says: 'based on the tasks administered to a student in a domain, their complexity, and the student's responses to those tasks, the information should ... yield a score for that domain ... Note that this score is not just the number of the correct answers a student has found. ... The intent of the score is that it reflect the degree of maturity the student has achieved with respect to that domain.' (page 14). It would be a mistake though to think only in terms of discrete domains. As Romberg (1989) points out 'there are several different aspects of doing mathematics within any mathematical domain.' (page 14).

Table 1 below summarises all levels for the three strands of the Year 6 assessment. The most complex level is first, with complexity decreasing going down the table. Thus Table 1, read vertically within each strand, summarises the children's order of difficulty of concepts contained in the NSW mathematics syllabus.

NUMBER	SPACE	MEASUREMENT
Solve problems involving several steps	Picture a complicated object from a drawing in which parts are hidden.	Solve problems involving a mixture of measurements
Select information from a table.	Picture how shapes would look when reflected.	Work out sizes on scale drawings.
Know which operation to use.	Use knowledge of East-West compass directions.	Estimate measurements in everyday use.
Understand four figure numbers.	Compare the lengths of paths.	Read everyday instruments.

Table 1: Strand Content for Numeracy

In developing Table 1, typical problems that illustrate the response required have been used to facilitate the uncovering of any links between the strands.

A further simplification/generalisation of the level descriptors, focussing upon fundamental elements within the strands, makes even clearer any possible inter-relationships between strands, as can be seen in Table 2 below.

NUMBER	SPACE	MEASUREMENT
Multiple step problems.	Work with three dimensions.	Measure with several magnitudes.
Read data from two scales simultaneously.	Work with two dimensions.	Work with two different scales at the same time.
Know which operation to use.	Work with one dimension.	Estimate single measures.
Understand conventions.	Compare one dimensional attributes.	Read everyday instruments.

Table 2: Simplified Strand Content

Information about the types of cognitive functioning, and their generalizability across what are usually accepted as being different aspects of Numeracy, are clearly visible in Table 2. This information, provided by the children's responses, parallels the ideas of Collis and Biggs (1980), and provides a possible framework for a new curriculum, based upon children's 'preferences', as well as having implications for future curriculum planning and research.

Tables 1 and 2 are not complete: some possible areas for further investigation are

extend the levels downwards to include the year 3 data
try to identify other links between strands.

In curriculum planning

should the order in which topics, or parts of topics, are tackled in the present syllabus be revised?

is it possible to form a curriculum which deals, in order of complexity (from Table 2), with
simple facts and conventions,
application of basic knowledge,
problems with only two variables,
problems involving three or more variables?

In classroom activities

can the order of presentation within topics take account of children's preferences?
can topics making similar 'cognitive demands' be integrated?
can the frequency with which number, measurement, space topics are presented in isolation be reduced?
can assessment more often use tasks which require skills from several (sub)strands?

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SOME PLUSES AND MINUSES OF RADICAL CONSTRUCTIVISM IN MATHEMATICS EDUCATION

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RADICAL CONSTRUCTIVISM IN MATHEMATICS EDUCATION: AN OVERVIEW

The veteran, widely respected American mathematics educator, Robert Davis, recently stated that anyone "who observes mathematics education has to be impressed by the quite sudden eruption of 'constructivism' as a central concern of so many researchers" (Davis, 1990, p. 114). As we have observed elsewhere (Ellerton and Clements, 1991, pp. 53-56), there can be no doubt that since the mid-1980s many mathematics educators in different parts of the world have identified themselves with the so-called "constructivist" movement and with radical constructivism in particular.

Each of the keynote addresses at the 1987 annual International Conference of the Group for the Psychology of Mathematics Education, for example, was dedicated to the theme, and since then three major edited collections have been prepared in which the theoretical bases and implications of radical constructivism for mathematics education have been explored. In 1990 a collection of 12 papers was published by the National Council of Teachers of Mathematics in the United States (Davis, Maher, and Noddings, 1990), and in 1991 a book edited by Ernst von Glasersfeld and containing chapters written by leading American and European writers, appeared, carrying the title *Radical constructivism in mathematics education*. Another collection of papers with strong constructivist leanings (entitled *Epistemological foundations of mathematical experience*, edited by Les Steffe, and with an international writing team) is, at the time of writing, in press.

Ackermann (in press) has pointed out that not only are there as many definitions of constructivism as there are minds to construct them, but also that attempts by teachers to develop learning environments in accord with constructivist principles have not always met with success. As Duckworth (1987, p. 31) put it, in discussing the difficulties of applying Piagetian theory in the classroom, "either we [the teachers] are too early and they can't learn it, or we're too late and they know it already." The dilemma of constructivist teaching is that while educators do not wish to impose "right" answers, they still feel compelled to teach the subject matter.

The science educator, Strike (1987), made a useful comparison of the effectiveness of the terms "democracy" and "constructivism," however loosely defined these might be, to motivate apparently disparate groups of people to coordinated and productive activity. After noting that agreement has not been reached on the meaning of constructivism in education, Strike (1987) went on to say that just as the banner of "democracy" can unite people who in reality do not agree on many social and political issues, so, too, in education can the term "constructivism" serve as a unifying theme. He wrote:

> Perhaps it [i.e. the word "democracy"] at least serves to remind people that they share a common political position, or that they share common antagonists.

> "Constructivists" may function in a similar way. Whatever it means, if we are constructivists, at least we know that it is important to be interested in children's misconceptions, to describe how they think about science, that we have Piaget as part of our heritage, and that behaviorists are the bad guys. Loyalty and group identification is made of such stuff. Clarity and intellectual progress are not. (p. 481)

Duit (in press), in commenting on Strike's analogy, pointed to the danger of constructivism being only superficially adopted in education. This paper is concerned to identify some of the strengths and weaknesses of the present-day constructivist movement in mathematics education.

RADICAL CONSTRUCTIVISM

The fundamental tenet of the *radical* constructivist position is that mathematics is not an "out there" pre-existing body of knowledge waiting to be discovered, but rather is something which is personally constructed by individuals in an active way, inwardly and idiosyncratically, as they seek to give meaning to socially accepted notions of what can be regarded as "taken-to-be shared mathematical knowledge." As von Glasersfeld (1990), probably the best known advocate of the radical constructivist position in mathematics education, has stated:

> . . . knowledge is the result of an individual subject's constructive activity, not a commodity that somehow resides outside the knower and can be conveyed or instilled by diligent perception or linguistic communication. (p. 37)

According to von Glasersfeld (1990, p. 37), all good teachers know that guidance which they give to students "necessarily remains tentative and cannot ever approach absolute determination," because, from the constructivist point of view, there is always more than one solution to a problem, and problem solvers must approach problem situations from different perspectives.

The cornerstone of radical constructivist theory is Piaget's emphasis on action (that is to say, all behaviour that changes the knower-known relationship) as the basis of all knowledge. An individual gets to know the real world *only* through action (Sinclair, 1990; von Glasersfeld, in press). Hermine Sinclair, Piaget's successor in Geneva, described von Glasersfeld, who is regarded as the high priest of radical constructivism, as perhaps "an even more radical constructivist than Piaget" (Sinclair, 1987, p. 29).

For radical constructivists working in the field of mathematics education, the crucial issue is *not* whether mathematics teachers should allow students to construct their mathematical knowledge, "for the simple reason that to learn is to actively construct" (Cobb, 1990a). "Rather," Cobb (1990a) says, "the issue concerns the social and physical characteristics of settings in which students can productively construct mathematical knowledge." (For further commentary on constructivism, and on radical constructivism in particular, see for example, Cobb, 1986; Confrey, 1987; Dorfler, 1987; Kilpatrick, 1987; Labinowicz, 1985; von Glasersfeld, 1983).

During the past ten years the impact of radical constructivist thinking on mathematics education researchers and, increasingly, on teachers of mathematics in schools, has been considerable. Notwithstanding certain tensions between theory and practice (Cobb, 1988), there has been a wave of research aimed at identifying the roles of teachers of mathematics who wish to adopt radical constructivist approaches (see, for example, Wood and Yackel, 1990). It is appropriate, therefore, that we pause for a moment not only to reflect on whether the claims of radical constructivism with respect to mathematics education are justifiable, but also to assess which aspects of the application of radical constructivist thinking in school mathematics have been beneficial and which have not. We shall deal with the pluses first.

RADICAL CONSTRUCTIVISM IN MATHEMATICS EDUCATION: SOME PLUSES

1. "Ownership" of Mathematics by the Learner

Probably the simplest way of summarising the radical contructivist position in mathematics education is in terms of the notion of "ownership." Radical constructivists believe that, ultimately, mathematical knowledge is not something that is acquired by listening to teachers or reading textbooks, but is something that learners themselves construct through actively seeking out, and making, mental connections. When someone actively links aspects of his or her physical and social environments with certain numerical, spatial, and logical concepts a feeling of "ownership" is often generated. In such cases a learner is likely to make comments such as "I *know* this because *I* worked it out *myself."*

Radical constructivists believe that this notion of "ownership" is powerfully relevant to school mathematics. They maintain that in the past *too much emphasis* has been placed on a linguistic communication pattern by which the teacher explained to students what they had to do, and how they could do it. Very few mathematics educators would deny that traditionally, school mathematics has been regarded by students as a fixed body of knowledge, owned by teachers, textbook and worksheet writers, external examiners, and by great, mysterious figures of the past (like Pythagoras and Euclid). During the 1980s, however, constructivist mathematics educators around the world (for example, Confrey 1987; Dorfler, 1987, 1989; Kamii, 1985; Labinowicz, 1985; Steffe, 1987, 1990) called for teachers to establish teaching/learning environments in which students, as a matter of course, created mathematics themselves and therefore came to feel that they "owned" the mathematics that they were learning.

2. Quality Social Interaction as Basis for Quality Mathematics Learning

Pateman and Johnson (1990) have claimed that it has been "constructivist" teachers of mathematics who have led the recent important movement towards establishing mathematics learning environments that nurture interest and understanding through co-operation and high quality social interaction. Such environments are likely to foster a kind of socio-cognitive conflict and challenge that stimulate learning. Pateman and Johnson maintained, as did Steffe (1990), that the belief that children construct their own mathematics out of their own actions and their reflections on those actions (in social settings) provides a new framework for those responsible for devising mathematics curriculum and school mathematics programs. According to Pateman and Johnson (1990), three aspects of curriculum need to be considered - content, methodology and assessment:

> content (which can hardly be rigidly prescribed in advance by the constructivist teacher, methodology (which probably needs to be idiosyncratic to children and

> context), and assessment (particularly difficult for those so used to competitive ratings). The constructivist teacher will need to be somewhat of an opportunist, and also an able elementary mathematician willing to continue to learn both about mathematics and children in the attempt to develop them as autonomous creators of their own mathematics. (p. 351)

We believe that this call for richer, more expressive interaction patterns in mathematics classrooms is one of the most important outcomes of the radical constructivist movement.

Steffe (1990) has elaborated ten principles for mathematics curriculum design that are in keeping with the main radical constructivist thrusts, and Cobb (1990b, p. 208) has identified the following social norms for worthwhile whole-class discussion in mathematics classrooms:

1. Explaining how an instructional activity that a small group has completed was interpreted and solved;

2. Listening and trying to make sense of explanations given by others;

3. Indicating agreement, disagreement, or failure to understand the interpretations and solutions of others;

4. Attempting to justify a solution and questioning alternatives in situations where a conflict between interpretations or solutions has become apparent.

Most experienced teachers would like to think that these norms already apply in whole-class discussions that occur in their own classrooms, but mathematics classroom discourse analyses indicate that this is rarely the case.

Cobb (1990b, pp. 209-210) called for constructivist mathematics educators to develop a new context - a "mathematico-anthropological context" - that will assist coherent discussion on the specifics of learning and teaching mathematics. According to Cobb there is research support for moving to establish mathematics classroom environments that incorporate the following qualities:

1. Learning should be an interactive as well as a constructive activity - that is to say, there should always be ample opportunity for creative discussion, in which each learner has a genuine voice;

2. Presentation and discussion of conflicting points of view should be encouraged;

3. Reconstructions and verbalisation of mathematical ideas and solutions should be commonplace;

4. Students and teachers should learn to distance themselves from ongoing activities in order to understand alternative interpretations or solutions;

5. The need to work towards consensus in which various mathematical ideas are co-ordinated is recognised.

Many teachers of mathematics would accept all five of these points. But much still needs to be done, for all too often in school mathematics rhetoric and classroom realities do not bear much resemblance to each other (Desforges, 1989). Nonetheless radical constructivists are determined to refine and apply their ideas to mathematics classrooms however difficult and time-consuming this process might prove to be (Ackermann, in press; Steffe, 1990).

3. Principles for Improved Mathematical Discourse

We would wholeheartedly endorse the radical constructivist educator's desire to establish learning environments that result in students owning the mathematics they learn. There have been other positive features arising from the application of radical constructivist theories to mathematics education, though these are all related to the "ownership" issue. These pluses have been well summarised by Paul Cobb, whose writings have attempted to draw together the threads of the constructivist movement in mathematics education. Cobb and his co-workers at Purdue University sought to explore how the theoretical positions held by radical constructivists might most advantageously be interpreted in mathematics classrooms (see, for example, Cobb, Yackel, and Wood, 1992).

In the second half of the 1980s, there was a veritable flood of articles in the mathematics education literature on the nature of discourse in mathematics classrooms that built on earlier theoretical works (for example, Chomsky, 1957; Mehan & Wood, 1975; Bauersfeld, 1980). Cobb (1990a), in a paper entitled *Reconstructing elementary school mathematics*, summarised research which attempted to assess the effectiveness of the application of radical constructivist ideas to mathematics teaching and learning. He made five main points:

1. To claim that students can discover mathematics on their own is an absurdity.

2. Students do not learn mathematics by internalising it from objects, pictures, or the like. Mathematics is not a property of learning materials, structured or otherwise.

3. The pedagogical wisdom of the traditional pattern of first teaching mathematical rules and skills, and then providing opportunities to apply these in real life situations, is questionable. An alternative approach takes seriously the observation that from a historical perspective, pragmatic informal mathematical problem solving constituted the basis from which formal, codified mathematics evolved.

4. The teacher should not legitimise just any conceptual action that a student might construct to resolve a personal mathematical problem. This is because mathematics is, from an anthropological perspective, a normative conceptual activity (see Shweder, 1983), and learning mathematics can be seen as a process of acculturation into that practice. (Cobb (1990a) notes that certain other societies and social groups have developed routine arithmetical practices that differ from those taught in Western schools.)

5. Mathematical thought is a process by which we act on conceptual objects that are themselves a product of our prior conceptual actions, and from the very beginning of primary schooling, students should participate in and contribute to a communal mathematical practice that has as its focus the existence, nature of, and relationships between mathematical objects. For Cobb (1990a), understanding mathematics is constructing and acting on what he calls "taken-to-be-shared" mathematical objects.

The pluses identified in this section of the paper refer to important developments in mathematics education that can be linked, at least partly, to the radical constructivist movement. We turn now to some of the minuses.

SOME MINUSES OF RADICAL CONSTRUCTIVISM IN MATHEMATICS EDUCATION

1. The Constructivist Bandwagon

Despite the fact that much good has derived from the radical constructivist movement in mathematics education there is a possibility that the movement has questionable philosophical and linguistic foundations (Matthews, 1992; Suchting 1992), and that therefore the drawing of clear pedagogical implications at this stage is premature. Certainly, there has been considerable debate over how "constructivism" should be defined (see Cobb, 1986; Pateman and Johnson, 1990, p. 353). Furthermore, despite the detailed investigations of the group at Purdue University, led by Cobb (see Cobb, 1990b; Cobb, Yackel, and Wood, 1992; Krummheuer and Yackel, 1990; Wood, 1990), into what a constructivist mathematics classroom might look like, it is still the case that "constructivism remains relatively untried in the everyday classroom setting" (Pateman and Johnson, 1990, p. 352).

The eleventh annual conference of the International Group for the Psychology of Mathematics Education, held in Montreal in 1987, had, as its theme, the theory of constructivism, and one of the plenary speakers, Jeremy Kilpatrick (1987) began his paper by comparing the constructivist movement in mathematics education in the United States with the waves of religious fundamentalism that had swept that nation over the past three-and-a-half-centuries:

> A siege mentality that seeks to spread the word to an uncomprehending fallen world; a band of true believers whose credo demands absolute faith and unquestioning commitment, whose tolerance for debate is minimal, and who view compromise as sin; an apocalyptic vision that governs all of life, answers all questions, and puts an end to doubt - these are some of the parallels that might be drawn. (p. 4)

Such comment raises the question of the extent to which research findings support constructivist theories of how mathematics is learnt. It is also pertinent to ask whether modern analytic philosophers are united in supporting the major radical constructivist contentions, and we shall turn to that issue later in this paper.

Both Kilpatrick (1987) and Matthews (1992) have called some, though not all, radical constructivists to task for claiming that their theories represent something new in mathematics education. After drawing attention to many of the laudable, insightful, and progressive aspects of constructivist theory, Matthews went on to say that nonetheless one does not need to be a constructivist to agree with most of the pedagogical claims of the constructivists. He stated:

> Socrates' teaching of Pythagoras' theorem to the slave boy is an early and enduring model of teaching for understanding, Aquinas and the medievals stressed personal engagement with and attachment to the subject matter being taught, they spoke of a 'love of learning', Montaigne's essays on education advocate dialectical rather than didactic teaching, Dewey and the progressives emphasised class discussion, debate,

> the testing and scrutiny of teacher and pupil opinions, as of course has Paulo Fréire in his many publications. Recently, Richard Peters, Paul Hirst, Israel Scheffler, Jane Martin and other philosophers of education stressed the importance of pupil comprehension and understanding for education.

Unfortunately, there is a tendency for those who march under the banner of radical constructivism to accuse all others of advocating and practising transmission modes of education. As Kilpatrick (1987, p. 22) has noted, radical constructivists "ought to be more tolerant towards competing theories" because "people who claim there are many possible ways to construct knowledge ought to be more friendly and understanding toward people who have failed to construct their theory."

2. *Confusion on the Communication Issue*

The hardline radical constructivist position on communication. Radical constructivist mathematics educators maintain that there is an inherent and inescapable indeterminancy in linguistic communication, and that the notion of "understanding" is more concerned with fit, rather than match. According to von Glasersfeld (1990, p. 36), "once we have come to see this essential and inescapable subjectivity of linguistic meaning, we can no longer maintain the preconceived notion that words convey ideas or knowledge and that the listener who apparently 'understands' what we say must necessarily have conceptual structures identical with ours." This line of argument has been maintained, without modification, by von Glasersfeld for almost a decade (see, for example, von Glasersfeld, 1983; 1988; 1989; 1990; 1991).

On this basis, conceptual schemas - and indeed concepts in general - cannot be conveyed or transported from one to another by words, but can only be abstracted from experience. That is to say, "language is not a means of transporting conceptual structures from teacher to student, but rather a means of interacting that allows the teacher here and there to constrain and thus to guide the cognitive structure of the student" (von Glasersfeld, 1990, p. 37).

Or, as Wheatley (1991, p. 10) puts it, "knowledge is not passively received, but is actively built up by the cognizing subject," and ideas and thoughts "cannot be communicated in the sense that meaning is packaged into words and 'sent' to another who unpacks the meaning from the sentences." Thus, Wheatley states, "our attempts at communication do not result in conveying meaning but rather our expression *evoke* [Wheatley's emphasis] meanings in another, different meanings for each person."

Opposition to the hardline position on communication. von Glasersfeld (1990) summarised radical constructivist beliefs on communication in a paper that was originally presented at the Sixth International Congress on Mathematical Education, held in Budapest. The official report of the discussion that followed the paper (see Steffe, 1988, pp. 106-108) does not reflect the considerable controversy that arose at the Congress with respect to the hardline radical constructivist position on communication. Many of those who took part in the discussion (one of the present authors, Clements, was present) believed that mathematical concepts can, in fact, be conveyed from one person to another by words.

There is a sense in which it is professionally dangerous to be seen to challenge radical constructivist dogmas on knowledge, understanding, and communication, for those who hold this point of view are usually so convinced that they are right that they cannot imagine the possibility that they could be wrong; furthermore, there is a tendency among radical constructivists in the field of mathematics education to view the unconverted as conservatives who hold to a transmission view of education. Despite this possibility, in a recent publication (Ellerton & Clements, 1991) we opposed the hard-core positions on language put forward by von Glasersfeld and Wheatley and, in particular, disputed Wheatley's (1991, p. 10) claim that "ideas and thoughts cannot be communicated in the sense that meaning is packaged into words and 'sent' to another who unpacks the meaning from the sentences."

To support our contention we invited readers to consider (see Ellerton and Clements, 1991, p. 55) the following statement, which would not cause too much surprise if it were found in a secondary mathematics textbook:

> I want you to imagine two regular polygons, whose sides are each 3 cm long. One of the polygons (Polygon A) has 10 sides, and the other (Polygon B), has 12. What is the size, in degrees, of each interior angle of Polygon A? What is the size, in degrees, of each interior angle of Polygon B? Would the interior angles be different if each side length had been 6 cm (rather than 3 cm)?

We contended that the above package of words *does* have the power to communicate unambiguously, *provided* the reader has prior knowledge of the terminology and concepts that are mentioned. Having received the "message" it is the reader's task to "unpack the meaning from the sentences." Certainly, different learners will do the unpacking in different ways, but that is not really the point being debated.

We recognise, of course, that many school children, on being asked to read the above paragraph (on polygons), would not construct the same meaning as that intended by the writers of the paragraph. Furthermore, we recognise that it is difficult (if not impossible) to get to know the meaning that another person would ascribe to that paragraph when reading it. But this does not contradict our claim that paragraphs such as the one on polygons *can and do* communicate shared meanings in a relatively unambiguous way to those who have the necessary background understandings.

Radical constructivists assert that with any communication, the person receiving the communication only makes sense of what is being communicated by actively and inwardly constructing an idiosyncratic interpretation of the input. We agree with that view, but would add that this is nothing more than expressing the meaning of the word "communication."

The many thousands of people around the world whose task it is to develop questions for external written mathematics examinations are constantly faced with the challenge of writing questions that define situations that demand the application of relatively complex mathematical operations in an unambiguous way. Rightly or wrongly, questions on such external examinations that can be interpreted in various ways, and therefore have more than one possible correct answer, are regarded as poor questions (unless, of course, open-ended questions, with many possible answers, are deliberately set - see Sullivan and Clarke, 1991). Clearly, the prevalence of written examinations in mathematics around the world over the past 150 years demonstrates that most educators believe that mathematical meaning can be communicated in unambiguous ways through words (and associated diagrams).

Like Cobb (1990a), we believe that "the hard-core Piagetian perspective downplays the role of linguistic activity in the development of abstract thought," and would support his contention that this "is a major weakness of current constructivist theorising."

Criticisms by Suchting and Matthews of the radical constructivist position on communication. We do not stand by ourselves in opposing the radical constructivist viewpoint on communication. Suchting (1992, pp. 224-226), for example, maintains that von Glasersfeld does not make sense in his utterances on the matter. After quoting a passage that appeared in a paper by von Glasersfeld in 1989 which asserted that constructivism denies "the existence of objective knowledge and the possibility of communicating it by means of language," Suchting (1992) then proceeded, in the manner of the analytic philosopher, to subject the meaning of the passage to intensive analysis:

> Firstly, what is the point of the last four words? Taken by themselves they would naturally be understood to state or imply a qualification of some sort. But there is no hint as to what this might be intended to be, and it seems impossible to see how 'objective knowledge', were it to exist, could be communicated ***except*** [Suchting's emphasis] by means of language of some sort . . . Secondly, consider the five words immediately preceding those just commented upon. They are also very puzzling in the context of the cited passage in which they appear. For if 'it', that is, 'objective knowledge', is asserted not to exist, and does not exist, then there can be no question of the possibility of communicating 'it' (linguistically or any other way), as distinct from the possibility of merely *seeming* [Suchting's emphasis] to communicate it. (p. 224)

Suchting (1992, p. 247) concluded that much of von Glasersfeld's doctrine of constructivism "is simply unintelligible" or, "to the extent that it is intelligible enough to provide some foothold for understanding and criticism it is simply confused."

3. *Oversimplication of the Ontological Question: What is Mathematical Knowledge?*

Another characteristic feature of radical constructivist theory is its rejection of the idea that some kind of absolute mathematical reality exists independent of the physical world (Labinowicz, 1985, p. 5). For most radical constructivists, mathematical knowledge is constructed as a result of mental activities that precipitate accommodation (that is to say, a change in an individual's mental model). From this perspective, the teacher's main task is to infer models of students' conceptual constructs, and then to generate hypotheses on how the students might best be given the opportunity to modify their mental structures in desirable ways (von Glasersfeld, 1990, pp. 33-34).

Mathematical knowledge, seen from this perspective, becomes highly personal and individualistic. But advocates of radical constructivism also say that the conceptions of the world acquired by humans are unlikely to be totally idiosyncratic, because contemporary constructivism argues in favour of social interactions which will effectively minimise the number of solely idiosyncratic constructions made by learners (Duit, in press). It is the teachers' role to guide learners towards the construction of "taken-to-be-shared" mathematical knowledge, that is to say, towards forms of knowledge that are regarded as mathematical, both in the domain of professional mathematicians and in the domain of everyday living where "mathematical" skills are needed to survive with dignity.

It should not be imagined, however, that this constructivist view of mathematical knowledge is supported by all contemporary philosophers. Matthews (1992), for example, maintains that the relativist conclusions of constructivists follow from an epistemological error. He draws attention to Lerman's (1989) suggestion that the core epistemological theses of constructivism are:

1. Knowledge is actively constructed by the cognising subject, not passively received from the environment.

2. Coming to know is an adaptive process that organises one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.

After commenting that the first claim above is in the realm of psychology, and the second in the realm of epistemology, Matthews (1992) argues that despite Lerman's (1989) assertion to the contrary, (1) does not necessarily imply (2).

An alternative ontological analysis of the nature of mathematical knowledge is provided by Bigelow and Pargetter (1990), whose theory of Realism argues that mathematics can be understood realistically if it is seen to be the study of universals, of properties and relations of patterns and structures, the sorts of things that can be in several places at once. This is a kind of scientific Platonism, and it suggests, for example, that the theorem attributed to Pythagoras concerning the areas of triangles is a logical truth that would summarise properties of our world whether or not humans recognise this to be the case.

While it would appear to be impossible to prove whether an ontological theory such as the one argued by Bigelow and Pargetter is superior to the radical constructivist view of knowledge, it is important to note here that the relativism of radical constructivism is being seriously questioned by contemporary professional philosophers - a point that certainly does not emerge in current mathematics education literature. Furthermore, there are data to suggest that the radical constructivists' relativist view of knowledge is counter-intuitive. Del Campo and Clements (1988) reported data that indicated that although a high proportion of a sample that included university staff professed to hold social constructivist views, they found themselves admitting that they thought that Pythagoras' Theorem was true even before the standard Pythagorean relationships concerning right-angled triangles were ever recognised by humans.

Interestingly, von Glasersfeld (in press) admits that he has never denied that an "absolute" reality exists. He states:

> I only claim, as the sceptics do, that we have no way of knowing it. And as a constructivist, I go one step further: I claim that we can define the meaning of "to exist" only within the realm of experiential world and not ontologically. When the word "existence" is applied to the world that is supposed to be independent of our experiencing (i.e. an "ontological" world), it loses its meaning and cannot make any sense.

This kind of argument is rejected by contemporary philosophers such as Bigelow and Pargetter (1990), Matthews (1992), and Suchting (1992), who assert that there are no logical grounds for the claim that reality derives solely from the experiential world.

Indeed, Suchting (1992, p. 247), after a detailed analysis of a recent paper by von Glasersfeld (1989), concludes that with von Glasersfeld "there is a complete absence of any argument for whatever positions can be made out." Suchting goes on to assert that von Glasersfeld repeats certain words and combinations of words like *mantras*, "and while this procedure may well eventually produce in some what chanting is often designed to do, namely, produce a certain feeling of enlightenment without the tiresome business of intellectual effort, this feeling nearly always disappears with the immersion of the head in the cold water of critical interrogation."

SUMMARY

This paper, after drawing attention to the sudden rise in interest by mathematics educators in the notion of radical constructivism and its application in mathematics classrooms, and to the lack of clear definitions of the terms "constructivism" and "radical constructivism," identifies a number of strengths and weaknesses associated with the radical constructivist movement in mathematics education.

Strengths

It is our view that the most important strength of radical constructivism in mathematics education is the way it seeks to make the notion of "ownership" powerfully relevant to learners of mathematics. Radical constructivist educators seek to establish teaching and learning environments in which students, as a matter of course, create mathematics themselves, and therefore come to feel that they "own" what they learn.

A related strength is the desire to create quality learning environments in which rich social and linguistic interaction provides the basis for socio-cognitive challenge.

The paper also summarises a number of important conclusions that constructivist researchers have reached. Among these conclusions are (a) to claim that students can discover mathematics on their own is an absurdity; (b) students do not learn mathematics by internalising it from objects, pictures, or the like (that is to say, mathematics is not a property of learning materials, structured or otherwise); and (c) understanding mathematics is constructing and acting on "taken-to-be-shared" mathematical objects.

Weaknesses

The tendency for mathematics educators to accept radical constructivist notions in superficial ways has meant that there has not been adequate discussion at all levels of some of the questionable philosophical and linguistic foundations. We believe that the hard-line radical constructivist position on communication, that is to say the view that mathematical concepts and principles cannot be communicated by language, is at best an exaggeration and at worse false. More seriously, any association of the pejorative term "transmission" with teachers who prefer to adopt largely verbal styles of teaching is to be deplored.

At the philosophical level, we have argued that the refusal of many radical constructivists to acknowledge the possibility that mathematical knowledge is not entirely relativist, is short-sighted. Recent philosophical critiques of the writings of radical constructivists by Matthews (1992) and Suchting (1992) have been summarised, and implications of the theory of Realism by Bigelow and Pargetter (1990) discussed.

It was argued that the two fundamental theses of radical constructivism, namely (a) knowledge is actively constructed by the cognising subject, and not passively received from the environment; and (b) coming to know is an adaptive process that organises one's experiential world, and does not discover a pre-existing world outside the mind of the knower; are such that (a) does not imply (b), despite assertions to the contrary by radical constructivists.

A Final Comment

We realise that this paper might generate more heat than light. If indeed that proves to be the case, then that would merely bear out one of the contentions in this paper that it is professionally dangerous to be seen to be challenging constructivist dogmas. Mathematics educators need to be aware of the danger of polarising discussion with terms such as "behaviourist," "transmission" and "Platonist" at one pole, and "dialogue," "constructivist" and "generative" at the other.

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IMPLICATIONS OF NEWMAN RESEARCH FOR THE ISSUE OF "WHAT IS BASIC IN SCHOOL MATHEMATICS?"

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In many countries there is a strongly held public view that the most important goal of elementary school mathematics is for young children to acquire a working knowledge of the four operations and, in particular, to be able to obtain correct answers to pencil-and-paper number questions that require the application of standard algorithmic procedures (such as vertical addition and subtraction, long multiplication, short and long division).

The main purpose of this paper is to challenge this view by summarising data from Newman error analysis studies carried out in several countries which suggest that society in general, and teachers of mathematics and mathematics teacher educators in particular, urgently need to revise the traditional view of what constitutes "basic skills" in mathematics education.

The Newman data indicate that an understanding of the vocabulary and the semantics of elementary school mathematics is fundamentally important. What is the point of children being able to carry out the mechanics of standard algorithms for the four arithmetic operations if, given a mathematics problem (verbally or in writing), they either cannot understand the problem, or they cannot work out an appropriate sequence of operations?

THE NEWMAN HIERARCHY OF ERROR CAUSES FOR WRITTEN MATHEMATICAL TASKS

Since 1977 when Newman (1977a,b) first published data based on a system she had developed for analysing errors made on written tasks, a steady stream of research papers reporting studies from many countries has appeared, in which her data collection and data analysis methods have been used (see, for example, Casey, 1978; Clarkson, 1980, 1983, 1991; Clements, 1980, 1982; Marinas and Clements, 1990; Watson, 1980).

The findings of these studies have been sufficiently different from those produced by other error analysis procedures (for example, Hollander, 1978; Lankford, 1974; Radatz, 1979), to attract considerable attention from both the international body of mathematics education researchers (see, for example, Dickson, Brown and Gibson, 1984; Mellin-Olsen, 1987; Zepp, 1989) and teachers of mathematics. In particular, analyses of data based on the Newman procedure have drawn special attention to (a) the influence of language factors on mathematics learning; and (b) the inappropriateness of many "remedial" mathematics programs in schools in which there is an over-emphasis on the revision of standard algorithms (Clarke, 1989).

The Newman Procedure

According to Newman (1977a,b; 1983), a person wishing to obtain a correct solution to an arithmetic word problem such as "The marked price of a book was $20. However, at a sale, 20% discount was given. How much discount was this?", must ultimately proceed according to the following hierarchy:

1. Read the problem;
2. Comprehend what is read;
3. Carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy;
4. Apply the process skills demanded by the selected strategy; and
5. Encode the answer in an acceptable written form.

Newman used the word "hierarchy" because she reasoned that failure at any level of the above sequence prevents problem solvers from obtaining satisfactory solutions (unless by chance they arrive at correct solutions by faulty reasoning).

Of course, as Casey (1978) pointed out, problem solvers often return to lower stages of the hierarchy when attempting to solve problems, especially those of a multi-step variety. (For example, in the middle of a complicated calculation someone might decide to reread the question to check whether all relevant information has been taken into account.) However, even if some of the steps are revisited during the problem-solving process, the Newman hierarchy provides a fundamental framework for the sequencing of essential steps.

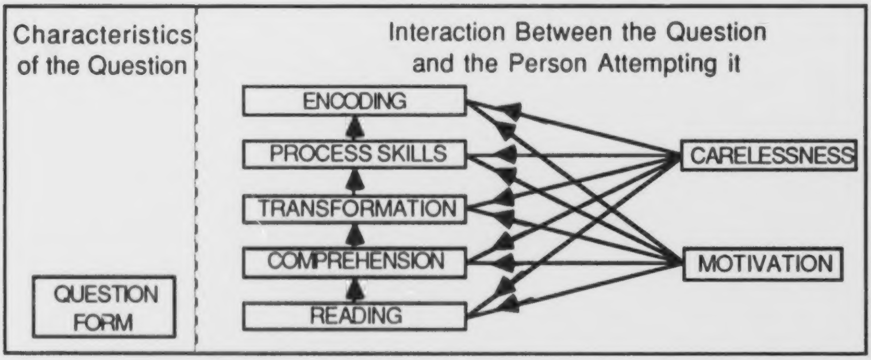
Characteristics of the Question
Interaction Between the Question and the Person Attempting it
ENCODING
PROCESS SKILLS
TRANSFORMATION
COMPREHENSION
READING
CARELESSNESS
MOTIVATION
QUESTION FORM

Figure 1: The Newman hierarchy of error causes (from Clements, 1980, p. 4).

Clements (1980) illustrated the Newman technique with the diagram shown in Figure 1. According to Clements (1980, p. 4), errors due to the form of the question are essentially different from those in the other categories shown in Figure 1 because the source of difficulty resides fundamentally in the question itself rather than in the interaction between the problem solver and the question. This distinction is represented in Figure 1 by the category labelled "Question Form" being placed beside the five-stage hierarchy. Two other categories, *Carelessness* and *Motivation*, have also been shown as separate from the hierarchy although, as indicated, these types of errors can occur at any stage of the problem-solving process. A careless error, for example, could be a reading error, a comprehension error, and so on. Similarly, someone who had read, comprehended and worked out an appropriate strategy for solving a problem might decline to proceed further in the hierarchy because of a lack of motivation. (For example, a problem-solver might exclaim: "What a trivial problem. It's not worth going any further.")

Newman (1983, p. 11) recommended that the following "questions" or requests be used in interviews carried out in order to classify students' errors on written mathematical tasks:

1. Please read the question to me. *(Reading)*

2. Tell me what the question is asking you to do. *(Comprehension)*

3. Tell me a method you can use to find and answer to the question. *(Transformation)*

4. Show me how you worked out the answer to the question. Explain to me what you are doing as you do it. *(Process Skills)*

5. Now write down your answer to the question. *(Encoding)*

If pupils who originally gave an incorrect answer to a word problem gave a correct answer when asked by an interviewer to do it once again, the interviewer should still make the five requests in order to investigate whether the original error was due to carelessness or motivational factors.

Example of a Newman Interview

Mellin-Olsen (1987, p. 150) suggested that although the Newman hierarchy was helpful for the teacher, it could conflict with an educator's aspiration "that the learner ought to experience her own capability by developing her own methods and ways." We would maintain that there is no conflict as the Newman hierarchy is not a learning hierarchy in the strict Gagné (1967) sense of that expression. Newman's framework for the analysis of errors was not put forward as a rigid information processing model of problem solving. The framework was meant to complement rather than to challenge descriptions of problem-solving processes such as those offered by Polya (1973). With the Newman approach the researcher is attempting to stand back and observe an individual's problem-solving efforts from a coordinated perspective; Polya (1973) on the other hand, was most interested in elaborating the richness of what Newman termed *Comprehension* and *Transformation*.

The versatility of the Newman procedure can be seen in the following interview reported by Ferrer (1991). The student interviewed was an 11-year-old Malaysian primary school girl who had given the response "All" to the question "My brother and I ate a pizza today. I ate only one quarter of the pizza, but my brother ate two-thirds. How much of the pizza did we eat?" After the student had read the question correctly to the interviewer, the following dialogue took place. ("I" stands for Interviewer, and "S" for Student.)

I: What is the question asking you to do?
S: Uhmm . . . It's asking you how many . . . how much of the pizza we ate in total?
I: Alright. How did you work that out?
S: By drawing a pizza out ... and by drawing a quarter of it and then make a two-thirds.
I: What sort of sum is it?
S: A problem sum!

I: Is it adding or subtracting or multiplying or dividing?
S: Adding.
I: Could you show me how you worked it out? You said you did a diagram. Could you show me how you did it and what the diagram was?
S: (Draws the diagram in Figure 2A.) I ate one-quarter of the pizza (draws a quarter*).

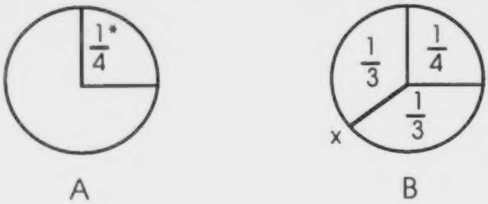

Figure 2: Diagrammatic representations of the pizza problem.

I: Which is the quarter?
S: This one. (Points to the appropriate region and labels it 1/4.)
I: How do you know that's a quarter?
S: Because it's one-fourth of the pizza. Then I drew up two-thirds, which my brother ate. (Draws line *x* - see Figure 2B - and labels each part 1/3)
I: And that's 1/3 and that's 1/3. How do you know it's 1/3.
S: Because it's a third of a pizza.

(From Ferrer, 1991, p. 2)

The interview continued beyond this point, but it was clear from what had been said that the original error should be classified as a *Transformation* error - the student comprehended the question, but did not succeed in developing an appropriate strategy. Although the interview was conducted according to the Newman procedure, some of the student's difficulties were identified without forcing her along a solution path she had not chosen.

DATA FROM STUDIES BASED ON THE NEWMAN APPROACH

Data From Early Australian Studies

In her initial study, Newman (1977a) found that *Reading*, *Comprehension*, and *Transformation* errors made by 124 low-achieving Grade 6 pupils accounted for 13%, 22% and 12% respectively of all errors made. Thus, almost half the errors made occurred before the application of process skills. Studies carried out with primary and junior secondary school children in Melbourne, Australia, by Casey (1978), Clements (1980), Watson (1980), and Clarkson (1980) obtained similar results, with about 50% of errors first occurring at the *Reading*, *Comprehension* or *Transformation* stages. Casey's study involved 116 Grade 7 students, Clements's sample included over 700 children in Grades 5 to 7, Watson's study was confined to a preparatory grade, and Clarkson's sample contained 13 low-achieving Grade 7 students. In each study all students were individually interviewed and with the exception of Casey, who helped interviewees over early break-down points to see if they were then able to proceed towards satisfactory solutions, error classification was based on the first break-down point on the Newman hierarchy.

The consistency of the findings of these Melbourne studies involving primary and junior secondary students contrasted with another finding, also from Melbourne data, by Clarkson (1980) that only about 15% of initial errors made by 10th and 11th Grade students occurred at any one of the *Reading*, *Comprehension* or *Transformation* stages. This contrast raised the question of whether the application of the Newman procedure at different grade levels, and in different cultural contexts, would produce different error profiles.

Newman Data From Studies in Asia and Papua New Guinea

Since the early 1980s the Newman approach to error analysis has increasingly been used outside Australia. Clements (1982) and Clarkson (1983) used the approach in error analysis research carried out in Papua New Guinea, and more recently the methods have been applied to mathematics and science education research studies in Brunei (Mohidin, 1991), India (Kaushil, Sajjin Singh and Clements, 1985), Indonesia (Ora, 1992), Malaysia (Clements and Ellerton, 1992; Kownan, 1992; Marinas and Clements, 1990; Teoh, 1991), Papua New Guinea (Clarkson, 1991), the Philippines (Jimenez, 1992), and Thailand (Singhatat, 1991; Sobhachit, 1991).

Rather than attempt to summarise the data from all of these Asian studies, the results of five studies which focused on errors made by children on written mathematical tasks will be given special attention here. The five studies, which have been selected as typical of Newman studies conducted outside Australia, are those by Clarkson (1983), Kaushil et al. (1985), Marinas and Clements (1990), Singhatat (1991), and Clements & Ellerton (1992). Pertinent features of these studies, conducted in Papua New Guinea (PNG), India, Malaysia, Thailand, and Malaysia respectively, have been summarised in Table 1.

The percentage of errors classified in each of the major Newman categories in these five studies is shown in Table 2. The last column of this table shows the percentage of errors in the categories when the data from the five studies are combined.

From Table 2 it can be seen that, in each of the studies, over 50% of the initial errors made were in one of the *Reading*, *Comprehension*, and *Transformation* categories. The right-hand column of Table 2 shows that over 60% of students' initial breakdown points in the five studies were in one of the *Reading*, *Comprehension*, and *Transformation* categories. This means that, for most errors, students had either not been able to understand the word problems or, if understanding had been present, they had not devised appropriate strategies for solving the given problems.

Table 1: *Background Details of the Asian and PNG Studies*

Study	Country	Grade level	Sample size	Number of errors analysed	Language of test & of Newman interview	Was interview in student's language of instruction?
Clarkson (1983)	*PNG*	6	95	1851	English	Yes
Kaushil et al. (1985)	*India*	5	23	327	English	Yes
Marinas & Clements (1990)	*Malaysia*	7	18	382	Bahasa Malaysia	Yes
Singhatat (1991)	*Thailand*	9	72	220*	Thai	Yes
Clements & Ellerton (1992)	*Malaysia*	5	44	497	Bahasa Malaysia	Yes

* Note that the 38 errors attributed by Singhatat to "lack of motivation" have not been taken into account for the purposes of this Table.

Table 2: *Percentage of Initial Errors in Different Newman Categories in the Four Studies*

Error Type Study	Clarkson (1983) (1851 errors) %	Kaushil et al (1985) (329 errors) %	Marinas & Clements (1990) (382 errors) %	Singhatat (1991) (220 errors) %	Clements & Ellerton (1992) (497 errors) %	Overall %
Reading	12	0	0	0	2	7
Comprehension	21	24	45	60	26	28
Transformation	23	35	26	8	40	26
Process Skills	31	16	8	15	7	22
Encoding	1	6	0	0	6	2
Careless	12	18	21	16	20	15

DISCUSSION

The high proportion of *Comprehension* and *Transformation* errors in Table 2 suggests that many Asian and Papua New Guinea children have considerable difficulty in understanding and developing appropriate representations of word problems. This raises the question of whether *too much* emphasis is placed in their schools on basic arithmetic skills, and not enough on the peculiarities of the language of mathematics.

Table 3: *Percentage of Indian and Australian Grade 5 Children Correct on Selected Problems (from Kaushil et al., 1985).*

Question	% Indian sample correct	% Australian sample correct
940 - 586 = q	96	75
273 ÷ 7 = q	76	55
A shop is open from 1 pm to 4 pm. For how many hours is it open?	44	87
It is now 5 o'clock. What time was it 3 hours ago?	47	88
Suniti has 3 less shells than Aarthi. If Suniti has 5 shells, how many shells does Aarthi have?	42	73

Further evidence for a possible over-emphasis on algorithmic skills was obtained in the Indian study (Kaushil et al., 1985) when the performances of the Delhi Grade 5 sample on a range of mathematical problems where compared with those of Australian fifth-grade children on the same problems. It was found that the Indian children consistently and significantly outperformed a large sample of Australian children on tasks requiring straightforward applications of algorithms for the four arithmetic operations (for example, 940 - 586 = q). However, on word problems, the Australian children invariably performed significantly better (see Table 3). Clements and Lean (1981) reported similar patterns when the performances of Papua New Guinea and Australian primary school students were compared on tasks similar to those shown in Table 3.

Interestingly, Faulkner (1992), who used Newman techniques in research investigating the errors made by nurses undergoing a calculation audit, also found that the majority of errors the nurses made were of the *Comprehension* or *Transformation* type.

SOME CONCLUDING COMMENTS

The high percentage of *Comprehension* and *Transformation* errors found in studies using the Newman procedure in the widely differing contexts in which the above studies took place has provided strong evidence for the fundamental importance of language factors in the development of mathematical concepts.

In many countries there is a strongly held public view that the most important goal of elementary school mathematics is for young children to acquire a working knowledge of the four number operations and, in particular, to be able to obtain correct answers to pencil-and-paper number questions that require the application of standard algorithmic procedures (such as vertical addition and subtraction, long multiplication and short and long division). The main purpose of this paper has been to challenge this view by summarising data from Newman error analysis studies carried out in several countries that suggest that society in general, and teachers of mathematics and mathematics teacher educators in particular, urgently need to revise the traditional view of what constitutes "basic skills" in mathematics education. Clearly, language factors should be incorporated into the public definition of what constitutes "basic skills" in school mathematics.

The Newman data indicate that an understanding of the vocabulary and the semantics of elementary school mathematics should be a central feature of mathematics curricula. What is the point of children being able to carry out the mechanics of standard algorithms for the four arithmetic operations if, given a mathematics problem (verbally or in writing), they either cannot understand the meaning of the problem or they cannot work out the appropriate sequence of operations?

The research reported in this paper raises the difficult issue of what educators can do to improve a learner's comprehension of mathematical text or ability to transform, that is to say, to identify an appropriate way to assist learners to construct sequences of operations that will solve a given word problem. At present, little progress has been made on this issue, and it should be an important focus of the mathematics education research agenda during the 1990s.

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GENDER AND PERCEPTIONS OF MATHEMATICS ACHIEVEMENT AMONGST YEAR 2 STUDENTS

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This paper reports on the unexpected findings when a sample of year 2 students' perceptions of their abilities in mathematics were compared with their teachers' perceptions. No significant differences by student gender were found for teachers' assessments, but gender differences in favour of males were found for students' own perceptions. Notable was the finding that in all five classes, including some in which the teachers had rated the females' achievements higher than the males', male students rated themselves higher than the females. Another interesting trend noted was that where the number of males and females in a class was approximately the same, females generally rated themselves more highly than in classes where the females were greatly outnumbered.

INTRODUCTION

For some years concern has been expressed about females' lower participation rates in higher level mathematics courses and in their lower achievements in high cognitive level mathematics tasks. In addition to a range of broader societal factors (see Leder, 1990a, for an overview), a number of affective variables have been identified as possible contributors to both. Confidence in learning mathematics, or mathematical self-concept, is one of the most important of these affective variables (Reyes, 1984).

The literature reveals that relatively strong correlations have been found between confidence in learning mathematics and higher achievement levels (Reyes, 1984). When teacher comments about students' mathematics achievement were used, Leder (1984) found that "students who were considered to be good at mathematics had a higher self-concept than students considered to be poor at mathematics" (p.22). Fennema and Sherman (1977, 1978) found that at both middle and high school levels, gender differences in confidence levels were apparent whether or not gender differences in achievement were evident. Leder (1990b) and Leder and Forgasz (1991) found that, in the absence of gender differences in teachers' assessments of achievement in mathematics, gender differences in perceptions of achievement were evident amongst year 7 students; Leder (1989a) reports similar findings following interviews with year 3 and year 6 students. In co-educational classes, Gill (1992) reported that the vast majority of males ranked themselves in the top half of the class whilst an equal proportion of females ranked themselves in the bottom half. When compared to teacher rankings, girls ranked high by the teacher "seemed unaware of this, while boys so named by the teacher were sure of their position" (Gill, 1992, 4). Amongst year 10 students, Foon (1988) found that "males rated their achievements in traditional male dominated subjects (mathematics and science) higher than did females" (p.52) and that females' self-esteem was significantly lower than that of males.

Confidence in learning mathematics has also been linked to classroom processes (Reyes, 1984). Teachers have been found to interact more frequently with males in year 7 mathematics classes than with females (Leder, 1989b, 1990b); in this same study, a greater percentage of the boys than girls rated their achievements in mathematics above average. Koehler (1990), assuming "that differential treatment led to differential achievement" (p.131), outlined several studies at primary level in the United States which indicated that differential treatment was accorded students in mathematics classrooms according to their gender; boys received more attention than girls, received more higher level questions, initiated more contact with teachers and volunteered or called out more frequently in class. Parsons, Kaczala and Meece (1982) found that certain patterns of teacher/student interactions and past performance in mathematics were predictors of high confidence in mathematics.

Since mathematics self-concept would appear to be related to students' decisions about their future mathematics education, school-based factors contributing to any differences in confidence need to be identified. In Australia, there has been little work in this field with very young students.

The main purpose of the study reported here was to try out two instruments for later use in a larger study. Both instruments had been used in previous research studies but under different conditions. One had been used in the United States with year 2 students (Nicholls, Cobb, Yackel, Wood and Wheatley, 1990); the other was used with year 7 students in Australia (Leder and Forgasz, 1991). This article reports the findings of the trial of an adapted version of the latter of these two instruments. Students' perceptions of mathematics achievement were obtained and compared with teachers' perceptions of student achievements in mathematics.

THE STUDY

The sample

Eighty eight year 2 students in five classes at one large independent co-educational school situated on two campuses in the metropolitan area of Melbourne participated in the study. The students came from similar middle level socio-economic backgrounds. Their five teachers, all female, also participated. Table 1 shows the numbers of students and the gender composition of the five classes.

Table 1: Numbers of students and gender composition of year 2 classes

CLASS No	MALES (N=51)	FEMALES (N=37)	TOTAL (N=88)
1	7	6	13
2	11	6	17
3	10	10	20
4	13	6	19
5	10	9	19

The instrument and method

Students were asked to indicate their perceived rank in mathematics achievement within their own class. A vertical column of twenty faces (the anticipated number of students in each class), divided into five sections, was used; the top section was labelled "best in class at maths", the middle section was labelled "middle of the class" and the bottom section was labelled "worst in class at maths". By colouring in one face only, the students indicated their perceptions of their own achievement levels within the class. A 1-5 rating scale was used with a top rating of five being assigned to a response for which any of the four faces within the section labelled "best in class at maths" was coloured in.

The teachers were also asked to indicate each child's achievement level in mathematics on a five point scale.

Students were required to write their names on the questionnaires since comparisons were to be made between students' and teachers' perceptions of achievement levels in mathematics. With students also indicating their gender, analyses by student gender, as well as by class number, could be undertaken.

Results

The means and standard deviations of both teachers' and students' perceptions of mathematics achievement are shown in Table 2; also shown are the means and standard deviations by student gender.

Table 2: Means and standard deviations of teachers' and students' perceptions of mathematics achievement.

	TEACHERS' PERCEPTIONS			STUDENTS' PERCEPTIONS		
	All students	**Males**	**Females**	**All students**	**Males**	**Females**
MEAN	3.33	3.33	3.32	3.95	4.23	3.54
S.D.	1.15	1.18	1.12	1.13	1.03	1.15

Figure 1a. shows the frequency distribution of teachers' perceptions of all students' mathematics achievement. Figures 1b. and 1c. show the frequency distributions of teachers' perceptions of mathematics achievement for males and females respectively. Similarly, Figures 2a., b. and c. show the frequency distributions of all students', males' and females' perceptions of their achievement levels respectively. The similarity of the distributions for teachers' perceptions of females' achievement levels and the females' self-perceptions are noteworthy; the disparity between the comparable distributions for males is particularly striking.

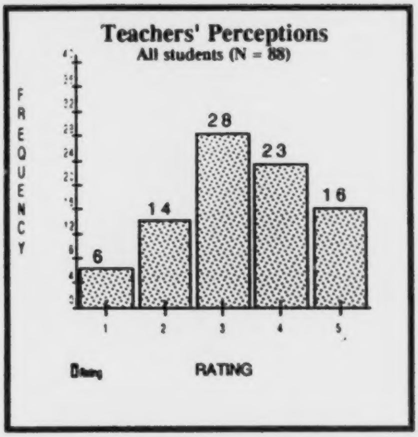

Figure 1a. Frequency distribution of teachers' perceptions of all students' mathematics achievement levels

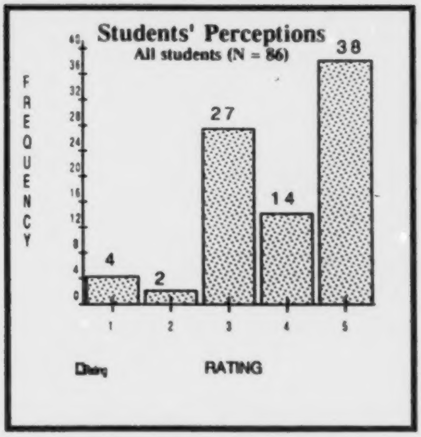

Figure 2a. Frequency distribution of all students' perceptions of their mathematics achievement levels

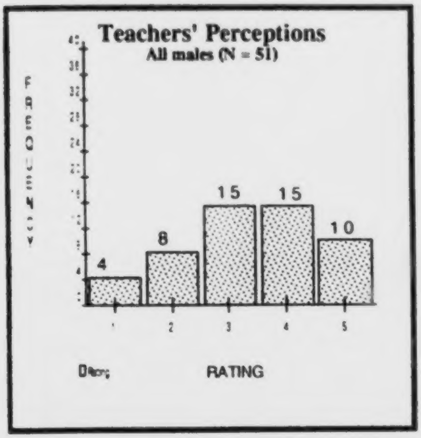

Figure 1b. Frequency distribution of teachers' perceptions of male students' mathematics achievement levels

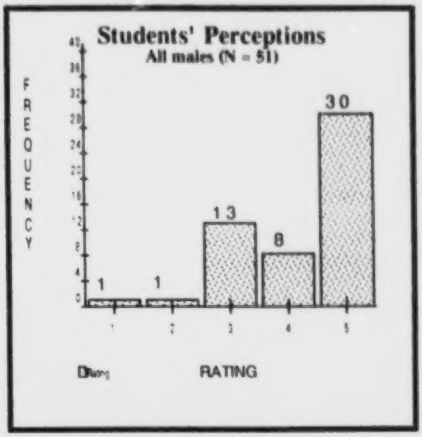

Figure 2b. Frequency distribution of male students' perceptions of their mathematics achievement levels

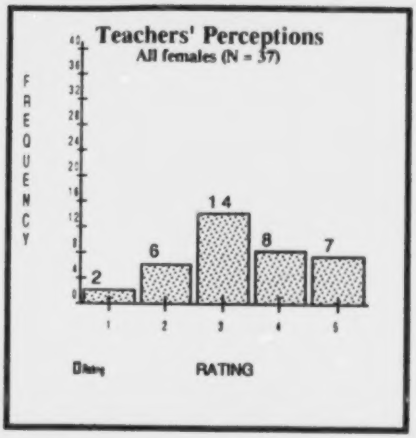

Figure 1c. Frequency distribution of teachers' perceptions of female students' mathematics achievement levels

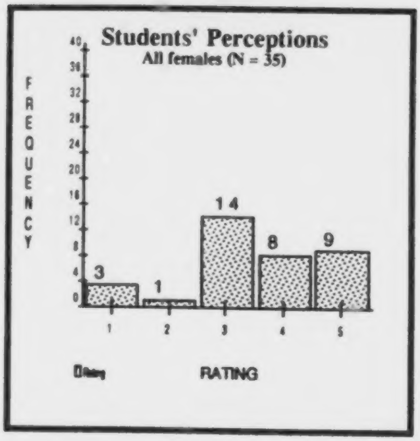

Figure 2c. Frequency distribution of female students' perceptions of their mathematics achievement levels

Pearson product-moment correlation co-efficients between teachers' and students' perceptions of achievement levels were also found. For the entire sample and for males the correlations were only moderate (r=.32 and r=.44 respectively); for females the correlation was low (r=.20). These are particularly interesting considering the frequency distribution patterns described above. It would appear that although, on average, females more closely paralleled their teachers' achievement perceptions, there was not close one-to-one correspondence between them.

Table 3: Pearson product-moment correlations for teachers' and students' perceptions of mathematical achievement levels by gender within class grouping.

Class Number				
1	2	3	4	5
.56	.72	.22	.45	.09
.65	.71	.11	.67	.26
.87	.73	.00	-.29	.05

When Pearson correlations were considered by class and gender within class (see Table 3), it can be seen that for Classes 1 and 2 teachers' and students' perceptions correlated highly whereas in Classes 3 and 5 the correlations were very low. In Class 4, however, the correlation is high for males but moderately negative for females. The reasons for these and the overall correlational differences can only be speculated. Are some teachers more aware of students' abilities than other teachers? Are students receiving differential feedback with regard to their abilities? What roles do societal expectations play? The small samples and the lack of further detail on the classes preclude any conclusions being drawn.

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Paired t-tests indicated that the difference in means between teachers' and students' perceptions of achievement was significant for all students (3.33 compared with 3.95, $n = 88$, $t = -4.19$, $p<.0001$), and for males (3.33 compared with 4.23, $n = 51$, $t = -5.50$, $p<.001$) but not for females (3.32 compared with 3.54, $n = 35$ (two females did not indicate their achievement level), $t = -.69$). Overall, students have significantly over-estimated their achievement levels compared with their teachers' assessments; male students being the main contributors to this effect. Female students, on the other hand, have assessed themselves in line with their teachers' assessments of their achievement levels.

The mean scores for both teachers' and students' perceptions of achievement level by class number and student gender are shown on Table 4.

Table 4: Mean scores of teachers' and students' perceived achievement levels by class number and student gender

	TEACHERS' PERCEPTIONS		STUDENTS' PERCEPTIONS	
CLASS NUMBER	Males	Females	Males	Females
1 (M=7, F=6)	3.43	4.00	3.86	3.33
2 (M=11, F=6)	3.37	2.83	3.82	2.60 (N=5)
3 (M=10, F=10)	3.40	2.70	4.80	4.00 (N=9)
4 (M=13, F=6)	3.85	3.50	4.46	3.33
5 (M=10, F=10)	2.50	3.78	4.10	3.89

From Table 4 it can be seen that, on average, the teachers of classes 1 and 5 scored the females higher; in the other classes the males were scored higher. A two-way ANOVA for teachers' perceived student achievement levels by class number and student gender indicated that there were no significant differences in mean scores by either class number or student gender but there was a significant interaction effect ($p<.05$). That is, overall, the teachers did not rate male and female achievements differently, nor were ratings for all students significantly different for each class. However, the different pattern in mean scores for males and females across classes was significant.

Table 4 also reveals that, on average, males scored themselves higher than did the females in each of the five classes. A two-way ANOVA for students' perceived achievement levels by class number and student gender indicated that the differences in mean scores were significant by student gender ($p<.05$) and by class number ($p<.05$) but there was no significant interaction effect. That is, males and females rated themselves significantly differently, and students in various classes rated themselves significantly differently.

It is interesting to note that in classes 2 and 4, where the ratio of males to females is greatest (approx. 2:1 in each), the teachers have scored the males higher than the females and the differences in the students' own scores are largest in favour of males. In classes 1 and 5, where numbers of males and females are approximately the same, the teachers have scored the females higher than the males; despite the males scoring themselves higher than the females, the differences are smaller than in classes 2 and 4. These data might be interpreted as suggesting that the ratio of male to female students may have a bearing on the relative perceptions of both teachers and students on achievement levels.

It should be noted that class 3 lends only partial support to the previous suggestion. Here, as in Classes 1 and 5, there were equal numbers of males and females in the class. The female students have rated their own achievements higher than in any other class as too have the males, and the difference in male and female students' own ratings is again lower than in Classes 2 and 4. Yet, the teacher of this class has rated the females' achievements lower than the males and, in fact, lower than the teachers in other classes have of their female students. This would seem to imply that the females in this class are perhaps the weakest amongst all the five classes yet their perceptions of their own achievements are higher than those of the females in the other classes. Again, these data may imply that, irrespective of their actual achievement levels, when females are not in a minority their perceptions of their own achievement levels are not as greatly affected as when the males are in a majority. The higher self-perceptions of achievement levels, compared to others, of both males and females in this class may, in some part, be due to the teacher, perceived teacher quality or to the learning environment in that teacher's classroom.

DISCUSSION AND IMPLICATIONS OF FINDINGS

It is to be remembered that the intent of this study was to trial two instruments and to compare the findings with those reported when they had been used previously under different conditions. However, the somewhat unexpected nature of the results are noteworthy; their implications invite further in-depth investigation.

The lack of gender differences in the teachers' assessments of mathematics achievement at the year 2 level was an expected finding. Previous research has indicated that differences in achievement relate to high cognitive level mathematical tasks and, if evident, emerge in the later years of schooling.

The significant gender differences found in perceived mathematics achievement amongst these year 2 students replicate those reported by Leder and Forgasz (1991) for year 7 students and were not anticipated at the year 2 level. Perceived levels of mathematics achievement are reflections of students' beliefs about themselves, that is their mathematical self-concepts. From the significant differences found between the males and the females across all five classes, it would appear that these year 2 male students were more confident of their ability to learn mathematics than were the females. The invariability of this pattern, even in classes where the teachers considered the females' achievements to be higher than the males' achievements, is notable. A single measure of confidence, however, is insufficient to warrant an explicit conclusion; the consistent pattern across all five classes invites further investigation.

Maccoby and Jacklin (1975) point out that "girls are somewhat more willing than boys to disclose their weaknesses" (p.151) on self-esteem inventories, and that boys score higher on scales measuring how much "an individual disguises his actual evaluation of himself and attempts to present an entirely favorable picture of himself to the researcher" (p.151). The strongest gender differences in attention-seeking behaviour, including boasting, have been found for boys aged 7-11 years (Maccoby and Jacklin, 1975). The gender differences in perceived achievement levels may be partially attributable to the novelty of the research situation for the students or the lack of anonymity of student responses. Boys, more than the girls, may have been prompted to `show off' and disguise their true evaluations of themselves.

FINAL COMMENTS

The young age group for which gender differences in mathematics self-concept are reported here is of specific interest. The consistent pattern of the males' higher self-perception across all five classes is a particularly poignant finding. The smaller differences between males' and females' self-perceptions of mathematics achievement when the gender composition of classes is more evenly balanced is also worthy of note. For more definitive conclusions to be drawn, these issues need be explored in greater depth. A much richer source of information about the students, their teachers and the learning environment would be needed.

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CALCULUS STUDENTS' SOURCES OF CONVICTION*

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Research findings from extensive task-based interviews with introductory calculus students are discussed. The research was conducted from a constructivist perspective. Students' sources of conviction were the focus of analysis, with sources of conviction referring to how one determines mathematical truth and validity. The existence and characteristics of three groups of calculus learners were revealed: Collectors, Technicians, and Connectors. The groups differed in the nature and role of their sources of conviction and manner of construction of calculus conceptualizations.

Few studies have been done to examine student learning in calculus and many that have been done have focused on student errors, misconceptions or inability to perform certain tasks (for example, Davis and Vinner, 1986; Orton, 1983a, 1983b; Seldon et al. 1989). These investigations have given insight into students' misunderstandings in calculus. What is needed now is research into how instruction can better guide and support student learning in calculus. This is particularly important when one considers that the drop-out and failure rates in calculus are high compared to other undergraduate courses. Figures between 30% and 50% are reported in the literature (Peterson, 1987; Cipra, 1988). Even students passing a calculus course tend to perform at low levels with respect to both skills and the use of calculus ideas (Peterson, 1987; Cipra, 1988).

For improved student learning in calculus more research is needed into various instructional emphases and formats and their subsequent effects on learning. One perspective by which such research might be formulated is that of constructivism. Constructivism has emerged as an important influence in mathematics education research. This is because constructivism has provided a valuable perspective from which to understand mathematics learning (Ernest, 1989). A constructivist model of knowledge views mathematics learning as an individual, evolutionary process (Kilpatrick, 1987; von Glasersfeld, 1984, 1987). This is in contrast to a view of learning that sees concepts as transferable, "ready-made" from teachers to learners. That is, constructivism views mathematics learning as an active, constructive process in which individuals build up knowledge for themselves.

Ernest (1991) discusses the above notions through discussion of mathematics as a social construction. This philosophy, known as social constructivism, takes the view that "human language, rules and agreement play a key role in establishing and justifying the truths of mathematics" (p.42). Mathematical knowledge is thereby seen to be grounded in the following: (1) "linguistic knowledge, conventions and rules" (p.42), (2) social processes by which an individual's internal, subjective knowledge is turned into external, objective knowledge, and (3) objectivity viewed as public, social acceptance rather than an inherent property of the content of knowledge. These features imply that mathematical knowledge is dependent upon social sharing of language and decisions pertaining to truth and validity. Further, as a consequence for mathematics education researchers, these points imply that ways of determining truth and validity are likely to be important components of mathematics learning.

This is a report of a portion of a study of student learning in calculus from a constructivist perspective. The parts of the investigation presented here focus on the nature and role of students' convictions regarding the validity or truth of calculus interpretations and problem responses and the ways students construct their calculus conceptualizations. The term *sources of conviction* is used to refer to how one determines mathematical truth and validity, or more specifically, how one determines facts, accordance with accepted mathematical principles and standards, legitimacy, consistency and logicality. Responsibility for the determination of truth or validity could lie within various sources, including the teacher's knowledge, the statements of a textbook or other instructional materials, the inherent physical structure of the world, a student's knowledge of the structure and rules of mathematics, or a student's own personal beliefs.

METHOD

The research was a naturalistic study involving three undergraduate calculus classes located at three different post-secondary institutions in Western Canada. Included were a large university and two small private colleges. Task-based and personal interviews with 17 students were the method of inquiry into the nature and role of students' *sources of conviction* and manner of construction of calculus conceptualizations. The interviews were conducted in the last three weeks of a 13 week school term, and each lasted one to two hours. A twenty to thirty minute follow-up interview was conducted 2 to 3 weeks later to have a student clarify or expand upon responses from the first interview. The problems given to students asked them to identify, describe, interpret, explain or apply limit and derivative concepts and they included open-ended as well as relatively focused tasks. The initial written problems were standardized (i.e. the same for all students) and subsequent written and oral questions asked by the researcher were contingent upon a student's previous responses. The interviews also incorporated relevant personal questions related to students' perceptions of calculus and the learning of calculus, study practices, ways of determining "correctness" and attitudes towards calculus. The overall manner of proceeding with analysis of the interview transcripts was similar to that used by Belenky et al. (1986) to analyze interviews conducted "to explore with women their experience and problems as learners and knowers" (p.11).

RESULTS

Student interview data revealed the existence of three groups of students who differed in their *sources of conviction*. These groups were named Collectors, Technicians, and Connectors. The names reflect the nature and role of the groups' *sources of conviction*. For the 17 interview students, 8 were classified as Collectors, 4 as Technicians and 5 as Connectors. Each of the three groups are discussed in upcoming sections. The groups differ from each other in the degree to which their *sources of conviction* are external or internal in nature. Collectors exhibit the highest degree of externalized *sources of conviction*, while Connectors exhibit the highest degree of internalized *sources of conviction*. Technicians fall somewhere in between these two other groups, exhibiting a mixture of external and internal *sources of conviction* .

Collectors

Students who from their *sources of conviction* are classified as Collectors display *sources of conviction* that are generally external in nature. These *sources of conviction* are external in that they reside in statements, rules and procedures presented by the teacher or textbook. They do not generally reside in what students have construed for themselves. The students construct their mathematical knowledge by assembling isolated, relatively unconnected mathematical statements, rules and procedures. Thus, a Collector's calculus conceptualizations can be said to be a "collection" of statements, rules and procedures. More specifically, the external nature of a Collector student's *sources of conviction* guides the student to approach calculus learning as recall or rote memorization of statements, rules and procedures. In this way the role of a Collector's *sources of conviction* is as a validation to the student that he or she makes statements and performs procedures that will be recognized as valid or correct by other individuals. Although the student might validly apply calculus knowledge, the student does not claim to know personally whether particular pieces of mathematics are valid or correct. Rather, the student relies on others to determine validity or correctness. These other individuals are perceived by the student to be people for whom calculus is understandable and meaningful.

A distinctive feature of Collectors' *sources of conviction* was their external nature. They said such things as:

Doug: Like I could say I remember in class that if you have this situation there's no derivative, but I don't know why there's no derivative.

Ellen: I don't know if this is right, but I think I remember something like that from the textbook.

Gordon: I don't know why. I just remembered something about there's not a derivative.

In all the above interview extracts the students make reference to what they remember from their class or textbook, using the teacher or the textbook as a *source of conviction.* These *sources of conviction* are external in nature in that the students employ them to reproduce what they remember from class or the textbook. More explicitly, the students use the teacher or textbook as a means of validation, while they concurrently state they "don't know" (Doug) why a particular piece of mathematics is as it is. That is, the students do not claim any ownership of the calculus concepts, rules or procedures they use.

Another feature of Collectors' interviews is that Collectors were often unsuccessful in completing the interview problems. Collectors frequently made errors, displayed misconceptions, were unable to remember particular rules or procedures or were unable to explain concepts. For example, they frequently did not completely or correctly remember such things as the product rule, the quotient rule or the chain rule. They also displayed a lack of ability to explain derivative or limit concepts.

Collectors displayed beliefs that mathematics is a collection of definite, correct formulas, rules and procedures. They expressed views that mathematics is "black and white" (Doug) and has "definite" answers (Betty) and "correct" (Daniel) ways that problems are to be solved. These views further reveal the external nature of Collector students' *sources of conviction* in that they show how Collectors perceive truth and validity decisions in mathematics as pre-determined, external entities. They do not generally see it possible that these decisions might be influenced by one's own perceptions or interpretations. Rather, they must be remembered or memorized. In fact, Collectors explicitly state they approach their calculus learning by memorization of what they believe will be needed to pass an exam. The following interview extracts provide evidence of how Collectors view their calculus learning as memorization:

Gordon: Yeah. I didn't fully understand it. I was just memorizing. Like memorizing what to do with the formulas, but not understanding why you have to do it. Like if you get the slope, like the chain law, I know what to do to find the derivative. But I don't understand how that works out.

Doug: Well I just plain don't understand calculus. Like I get the questions, but it's not because I understand them. It's because I just memorized them.

Cindy: I can kind of work out formulas and work out a way, kind of memorize almost a way that he tells us to do it. But I don't really understand it. I can understand, like I can memorize like derivatives.

In the above excerpts the students state they use memorization as a learning technique in calculus. They speak of "just memorized" (Doug) formulas and examples and they make it clear they do not feel they personally understand calculus in terms of why one uses particular procedures or how procedures function in reaching a solution.

From the discussion thus far it can be concluded that Collectors' externally oriented *sources of conviction* do not promote a sense of personal understanding or ownership of calculus skills and conceptualizations. In fact, Collectors often spoke of calculus as being separated from their reality and ways of understanding. Comments on their impressions of calculus included:

Cindy: Well I might be able to write it down, but it probably wouldn't be right. I probably wouldn't do it the correct way. But I would, if I was to go back and read it I would understand what I meant. But it wouldn't be the right way so anybody else would understand it.

Doug: It just doesn't come to me easily. So I have to really work at it. Whereas something like English I can just do it. Political science ... There I can actually use, like I can just do it with my own mind. I can give my own interpretations of something. But in math it's either right or wrong.

Ned: ... like with me I have to look through someone else's eyes. Like a foreign kind of viewpoint. That's very hard for me to do.

These excerpts demonstrate that Cindy, Doug and Ned view mathematics as a "foreign kind of viewpoint" (Ned) from which one does not use one's "own mind" or "own interpretations" (Doug). These students do not allow internal *sources of conviction* to play a prominent role in the building of their calculus conceptualizations. In particular, they do not see as valid their personal ways of interpreting or expressing mathematics. This devaluing of one's own mathematical interpretations is particularly clear in the following comments made by Daniel:

Daniel: And right now it is a matter of being able to produce it on a test. And whether or not my interpretation is correct doesn't matter. Because my interpretation isn't going to be counted on the test.

Well in most anything else I could feel confident my views are um maybe not necessarily correct, but that they're feasible, or that I can show how my views and somebody else's views correlate or something. Like you know. In math I don't feel that I have got any basis to say that I'm right and I'm wrong. Because if they, they referring to math people, come up with all this stuff, or how do I say it. I'm just not confident that my way of viewing it, like I could so easily be wrong. Like I just don't feel I have it.

At this point it must be noted that Daniel's feeling of a lack of confidence in his calculus abilities was not an isolated feeling amongst Collectors. All the Collectors except Betty and Ned explicitly expressed a lack of confidence in their abilities to personally understand calculus.

In summary, Collectors generally display *sources of conviction* that are external in nature. Their *sources of conviction* reside predominantly in statements, rules and procedures presented by a teacher or textbook. The role of these *sources of conviction* is as a validation to the student that his or her calculus statements and problem solutions will be recognized as valid or correct by mathematicians or mathematics teachers. By way of this role Collectors' calculus conceptualizations are constructed as an assemblage or collection of relatively unconnected mathematical statements, rules and procedures. Collectors generally believe mathematics has definite rules and procedures and is a dichotomy of right and wrong problem solutions. They generally speak of mathematics as separate from their own reality and they sometimes explicitly devalue their personal interpretations of calculus. In addition, most of the Collectors explicitly expressed a lack of confidence in their calculus abilities.

Technicians

Technicians display a mixture of internal and external *sources of conviction*. Their external *sources of conviction* are similar to Collectors' in that they are based on knowledge of calculus statements, rules and procedures. However, Technicians differ from Collectors in their perceptions and use of these statements, rules and procedures. Technicians see calculus as a logical organization of statements, rules and procedures and they employ this organization as a technique for thinking about and applying calculus concepts. What therefore most distinguishes Technicians from Collectors is that Technicians display personal knowledge of how calculus statements, rules and procedures fit together into a logical whole. This logical whole thereby becomes a calculus "technology" in that it is a science or method for thinking about and applying calculus. Technicians can therefore be viewed as skilled users of the application of calculus techniques. Thus, the role of a Technician's *sources of conviction* is as a set of tools that the technician employs to apply calculus concepts.

However, although Technicians make use of calculus statements, rules and procedures as external *sources of conviction*, a complete examination of their interviews reveals their calculus conceptualizations are more organized than Collectors'. Their knowledge of calculus statements, rules and procedures is more than the "collections" displayed by Collector students. More specifically, instead of a collection of relatively unconnected mathematical statements, rules and procedures, Technicians' *sources of conviction* are based upon statements, rules and procedures organized into a coherent, structured set. This set is then employed as a logical technique to think about and apply calculus concepts. The existence of structured calculus conceptualizations and related *sources of conviction*, rather than an unorganized "collection", is partially evidenced by the fact Technicians generally displayed more extensive calculus knowledge and skills than Collectors. An example of a technician's perceptions of calculus as an organized structure are displayed in the following extracts from the interview with Jennifer:

Jennifer: The derivative. I don't know. It seems like it is a base that you can work from. And everything seems to rely on it. Like plugging values back into it. The points of inflection and critical points and stuff like that.

Understand something? To take that tiny basis of logic and be able to build on it. Like using that maybe as a cornerstone. But if you understand that, then you can understand things more ... Then you can continue onto a higher level ... By applying to another concept. How can I say it? Through practical application.

I think calculus, if you get into a method of thinking it's just a process. It seems to be the same sort of process and you just get into that method of thinking and it's all very logical.

In these interview extracts Jennifer speaks of calculus as a building and application of a structure of statements, rules and procedures. She speaks of knowing how to apply calculus ideas to solve a problem. It is this sense of knowledge of how calculus is structured as an applicable technology that most distinguishes Technicians from Collectors. For example, Jennifer is a Technician rather than a Collector in that she sees calculus ideas and procedures as something one can "build from" and employ as a "process" or "method of thinking" to "work through" and solve problems. These perceptions of calculus as a "method of thinking" (Jennifer), a "pattern" (Richard) or a logical "step by step" (Sally) problem solving process were not present in Collectors' interviews.

Thus, a prominent aspect of Technicians' *sources of conviction* is they are based upon knowledge of calculus as an applicable process or technology for solving problems. What is not clear thus far is whether or not *sources of conviction* that reside in the technology of calculus are internal or external in nature. When one examines comments on Technicians' own calculus knowledge, a mixture of external and internal *sources of conviction* appear. As an example of this aspect of their *sources of conviction* Sally's impressions of and experiences in calculus will be outlined. Sally spoke of both memorizing rules and working through calculus problems for herself. She said such things as:

Sally: Understanding is applying the ideas to get a right answer ... And you need to know the ideas in order to apply them. And know what ideas apply in what circumstances.

When I do a question and I look at the book and the answers match. I guess that the whole way of understanding calculus for me is getting it right. But I guess for a lot of people, you know, it shouldn't be that. It should be just knowing the ideas. But for me it isn't.

I'm trying to get the ideas into my head and how to use those ideas. Like the examples especially. I follow them closely, step by step, what he's doing, how he's applying the ideas to a problem. And then I try to do them myself. And the exercises from the book. And then also, I guess it's just memorizing rules. Though that's not what math should be. But it is a bit because you have to remember the rules in order to use them.

I: Do you feel a need to convince yourself?

S: Yeah. Which is why I study. Why I do questions and assignments. To make sure.

Sally speaks of her calculus learning in terms that reflect both external and internal *sources of conviction*. The external nature of her *sources of conviction* are seen in her references to "just memorizing rules" and getting answers that "match" the textbook answers. Her internal *sources of conviction* are seen in the fact she tries to do problems and exercises for herself, trying to learn "how to use" the ideas. Sally speaks of calculus as learning "ideas in order to apply them." To her, calculus is a technology of knowing "what ideas apply in what circumstances." Although she does problems for herself to "make sure" she is convinced of the validity or correctness of her work, she sees calculus understanding as getting the right answers. In other words, Sally's *sources of conviction* simultaneously reside in externally oriented statements of the textbook and personal, internal knowledge of the application of ideas. This display of a mixture of external and internal *sources of conviction* was also evident in the interviews with the other three Technicians, Jennifer, Nadine and Richard.

In summary, Technicians generally display a combination of external and internal *sources of conviction*. Their external *sources of conviction* reside in knowledge of calculus statements, rules and procedures, while their internal *sources of conviction* arise from knowledge of how to use these rules and procedures. Thus, the role of a Technician's *sources of conviction* is as a means to organize and structure calculus statements, rules and procedures. The resultant structures thereby become a calculus technology that guides and informs a student in the application of calculus ideas and techniques. It is through a sense of mastery of the technology of calculus that Technicians' *sources of conviction* are more internal in nature than Collectors'.

Connectors

Students who from their *sources of conviction* are classified as Connectors display *sources of conviction* that are generally internal in nature. These *sources of conviction* are internal in that Connectors display a sense of being able to interpret calculus for themselves. Similarly to Technicians, Connectors display knowledge of calculus as a technology. They organize their calculus experiences so as to be able to logically and consistently apply calculus ideas and techniques. However, Connectors differ from Technicians in that they display a stronger sense of personal understanding of their calculus conceptualizations. They also display a higher degree of competence in both explanation and application of calculus. Their conceptualizations are displayed as a network of "connections" between various aspects of calculus and between calculus and themselves. In this way the role of a Connector's *sources of conviction* is as a validation to the student that he or she makes statements, performs procedures or creates problem responses that are valid, correct and meaningful to the student as well as other individuals. Thus, Connectors are able to both apply calculus knowledge and make personal sense of this knowledge.

Connectors frequently spoke of approaching their learning as trying to understand for themselves and trying to connect together ideas, statements, rules and procedures. Examples of what they said in relation to these two features are:

Annabel: I'm trying to fit it all together. Not memorizing.

Neil: I definitely try to recreate things and think it through ... Seeing how everything is linked together. And not just this idea, and this idea over here. And if they are connected then one should know it. Even if it's a little more complex. But, I think the connections are important.

Mike: I like, like I said, I like to know how it works for myself. And figure things out for myself. You have more control that way.

Tanya: Because you can't learn from memorizing everything. Because you have to interpret it. You have to understand the theory behind a certain form. The theory behind a certain something, and then apply it to something else. ... Cause you need to, you need to imagine it in your head. What goes on. You can't, you can't see infinity. You have to imagine infinity. You can't see infinitely, or infinitesimally small. You have to imagine it.

In these excerpts the students speak of understanding rather than memorizing calculus. In comparison, none of the Collectors said calculus made sense to them. Technicians expressed some sense of understanding of calculus, but they did not speak of their learning of calculus in the same way as did Connectors. Connectors spoke more of calculus as something one learns through personal involvement with and subsequent flexible application of ideas. This aspect of their learning is particularly clear in Tanya's comments on her learning. She speaks of her "imagination" as an essential component of her calculus learning and notes how she must "interpret" rather than memorize in order to learn how to apply calculus theory. Through these words Tanya expresses a sense of personal understanding or ownership of her calculus knowledge. That is, as *sources of conviction* she uses knowledge and thought processes that she conceives of as her own. This sense of oneself and one's own thought processes and interaction with material as *sources of conviction* by which to learn and use calculus is also seen in the other Connectors' words.

The sense of personal control and involvement of one's own thought processes that is demonstrated by Connectors reveals the internal nature of their *sources of conviction*. More specifically, Connectors' *sources of conviction* are internal in nature in that they reside in a sense of personal comprehension and control of calculus ideas and applications. In this way, the role of Connectors' *sources of conviction* is as both a guide and a confirmation for students that they state and use calculus ideas and applications in ways meaningful to themselves as well as others knowledgeable in calculus.

Another prominent feature of Connectors' interviews was they displayed a higher level of competence with calculus concepts and skills than the other interview students. This fact points to a relationship between high competency in calculus and approaching calculus learning as a Connector, but it is not clear if one causes the other. Connectors' problem responses were often more detailed than the other students', using more symbolic representations and more complete explanations of ideas or procedures. In comparison to the problem responses given by Collectors and Technicians, Connectors' problem responses showed more facility with calculus ideas and techniques.

In summary, Connectors generally display *sources of conviction* that are internal in nature. Their *sources of conviction* reside largely in ideas and techniques they perceive to make sense. That is, Connectors view calculus knowledge as something of which they can gain personal understanding and use. They speak of approaching their calculus learning in terms of aiming to understand, make sense of and flexibly think through and apply ideas and techniques. In this way Connectors use their internal *sources of conviction* to construct calculus conceptualizations of which they feel personal understanding. The role of a Connector's *sources of conviction* is therefore as a guide and a confirmation to the student that she or he makes statements and performs procedures that are meaningful and useful to the student as well as other individuals. Connectors see their own interpretations and thought processes as components of their calculus learning. Their calculus conceptualizations are thereby constructed as a network of personally meaningful, interconnected statements, rules and procedures.

SUMMARY, DISCUSSION AND IMPLICATIONS

Student interview data revealed the existence of three groups of students who differed in their *sources of conviction.* These groups were named Collectors, Technicians, and Connectors. Collectors exhibited the highest degree of external *sources of conviction,* using teacher or textbook presentations as means by which to determine truth or validity. Their calculus conceptualizations were constructed as a collection of isolated, relatively unconnected statements, rules and procedures. Technicians based truth and validity upon their knowledge of the logical, organized structure of calculus and constructed their conceptualizations as a logical organization of statements, rules and procedures. Connectors exhibited the highest degree of internal *sources of conviction,* displaying a sense of personal understanding of calculus. Their conceptualizations were constructed as a network of connections between various aspects of calculus and between calculus and themselves.

Since few of the students in this study were Connectors, it is apparent that the search for effective ways to guide students to personal understandings of calculus must continue. Regardless of whether or not students apply calculus or study calculus beyond an introductory level, it is desirable that they pursue their calculus learning as a meaningful endeavour. What is noteworthy here is that students who saw their calculus learning as personally understandable displayed more competence, confidence and satisfaction in their abilities to do calculus. Technicians used knowledge of calculus language as a technology by which to apply calculus, but their related conceptualizations were not necessarily perceived by them to be personally meaningful. In other words, mastery of the use of calculus language can help students attain competence with calculus skills and basic ideas, but it does not necessarily guide them to personal understandings of calculus conceptualizations. Thus, it appears that language use is an important vehicle by which calculus students might be better guided to calculus learning as a meaningful endeavour.

Also of particular note is the fact that over half the interview students were classified as Collectors. These Collectors, although engaged in conceptual constructions, did not claim any personal understanding of their calculus conceptualizations. The fact that they conceived of calculus learning as replication of teacher or textbook presentations implies calculus instruction might be more successful for them if methods were developed that encourage them to take more personal involvement in the construction of their calculus conceptualizations.

Two other features which emerged during analysis of student interviews are worthy of further investigation aimed at clarification, refinement and generalizability of notions. These features are:

(1) A primary area of examination in this study was students' *sources of conviction*. This concept raises a number of issues in need of further research. First, studies should be undertaken to investigate whether the three groups of students, Collectors, Technicians and Connectors, are present in other groups of calculus students. Whether these groups are present in students studying mathematics at other levels or studying other subjects also needs to be determined. Such studies would contribute to the generalizability of this study and would aid further application of constructivism to mathematics education and other areas of education.

(2) Research needs to be done to determine if the nature of Collectors', Technicians' and Connectors' approaches to calculus learning form a series of transitional learning phases. For example, it is not known if being a Technician might be a transitional phase between being a Collector and being a Connector.

Given the above points, it would be advantageous to continue research into mathematics students' *sources of conviction*. Researchers and teachers would then be better guided in the development of mathematics instruction that facilitates meaningful mathematics learning.

Postscript: For a more complete report of the research on which this paper is based see Frid (1992) or contact the author.

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ASSESSMENT IN MATHEMATICS - SOME ALTERNATIVES

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> As we need standards for curricula, so we need standards for assessment. We must ensure that tests measure what is of value, not just what is easy to test. If we want students to investigate, explore, and discover, assessment must not measure just mimicry mathematics. By confusing ends and means, by making testing more important than learning, present practice holds today's students hostage to yesterday's mistakes.
>
> (Everybody Counts,1989 p. 70)

Improvement in the teaching and learning of mathematics at the school level has been the goal of mathematics educators for many decades. In response to a great deal of research, mathematics education is currently in an exciting state of review and renewal and teachers are participating in a major restructuring of the goals and practices of mathematics education. With the recent work of the Australian Mathematics Curriculum and Teaching Program (Clarke, D. and Lovitt, C., 1989) and the release of the new Curriculum and Evaluation Standards for School Mathematics (NCTM 1989) in the United States, together with the release of the National Mathematics Curriculum in England (Department of Education and Science, 1990) and the National Statement on Mathematics in Australia (Australian Education Council, 1990) there has developed a growing agreement amongst educators about desired directions for change, and the sorts of mathematical experiences likely to prepare students for the challenges of the twenty-first century.

The types of mathematical skills which are being advocated in these important documents are significantly different from the narrow objectives which typified the mathematical goals and attitudes of yesteryear. One such change is the increased emphasis upon mathematical process, as distinct from content, which has resulted in the developing importance of problem solving and its applications in school mathematics. Our vision of an adequate mathematical education has been changed rapidly and permanently by the advent of calculators and computers for tools in both learning and doing mathematics. Similarly it is advocated that school mathematics should become more broadly defined to include the development of co-operative group processes and the fostering of associated communication and learning skills. Underpinning many of these changes is an increased use of "active" participation by pupils in the learning process. Associated with this model of learning are a variety of classroom teaching ideas which may involve pupils in manipulative materials and an increased emphasis on the central role that language plays in the learning of mathematics. This includes the advocacy of the use of extended and relatively more realistic problem situations in the classroom and the use of co-operative learning groups.

Unfortunately, however, with each step forward in our understanding of how to teach mathematics and how children learn mathematics, timed pencil and paper tests have remained the primary means of assessing the pupil's understanding of mathematics. The identification of the subject with a single form of assessment has shaped pupil attitudes, teaching practices and even the content of the subject. The results of such assessment have been used solely to grade and rank students, to regulate entry into courses, to award credentials and to determine selection for employment and further study. This has tended to have a distorting effect upon the aims and goals of mathematics education. It has conferred an inordinate importance upon those aspects of mathematics which can be tested in an unambiguous and straightforward way through student's performance on routine skills and algorithms. As a result, there has been a great deal of resistance to broadening the goals of school mathematics and to exploring assessment strategies which better reflect these goals.

As an example of this resistance, the recently released National Statement on Mathematics for Australian Schools (Australian Education Council, 1990) comments that:

> In the past, mathematics has been assessed mostly by pencil-and-paper timed tests. This form of testing has remained dominant in mathematics, partly because it was perceived to be objective and hence more fair than other forms of assessment.

Australian secondary mathematics teachers are particularly resistant to alternative forms of assessment and have always relied on the age-old traditional method of timed pencil and paper tests. This is what they themselves experienced when they were at school and they consider that these practices are fair and equable. Clarke (1987, p. 9) comments

> schools continue to succeed in the teaching of routine computation and to fail in the teaching of such skills as problem solving....the maintenance of current assessment procedures serves only to maintain the illusion that significant learning is taking place.

Clarke (ibid., p. 8) continues:

> it is through our assessment that we communicate most clearly to students which activities and learning activities we value

so that it is essential that our assessment be comprehensive and give recognition to all valued learning experiences.

In a recent presentation to the Mathematics Association of Victoria, Mousley (1991) relates a story as follows:

> One of my off-campus students, a secondary teacher from New South Wales, recently claimed that assessment is the tail that wags the dog in her school: it seems that this is also true in Victorian schools too. As I observe a growing number of teachers and students starting to view mathematics as a field of enquiry rather than a pre-existing set of knowledge and skills, I glimpse the power of the tail.

That secondary mathematics teachers, in New South Wales anyway, are resistant to any change in the traditional manner of assessment in school mathematics, was borne out to the author in a recent in-service activity involving about 60 government and non-government secondary school mathematics teachers which was called to discuss possible changes in the mathematics curriculum, Years 7 - 12. These teachers were the keen ones who responded to a call from their professional association to discuss these issues, and the meeting was held on an evening in the teachers' own time. An observer would think that the recent changes to mathematics education in England, U.S.A. and in the other states of Australia, especially Victoria had never occurred as they were very dubious of any seeking of a new direction as to what and how mathematics was being taught, let alone the method of how mathematics was to assessed. The Board of Studies had conducted a survey of mathematics teachers' desire for change in Years 9 - 12 and found that most only wanted some tinkering with content at the edges. There was certainly no groundswell of advocacy for changes in assessment procedures, and when pressured, a real opposition to assessment of group work, discussion, assignments, etc. on the grounds that it would be too subjective and students might cheat! The examination of prescribed content remained paramount in Years 9 - 12.

It seems that some acceptance for changes in assessment procedures is countenanced in Years K - 6 and even within the largely mixed-ability classes in Years 7-8 in N.S.W., but once the "real" mathematics begins in Years 9 - 12, the test dominates. Mousley (ibid, p. 150) comments in a similar vein when she says that infants and primary teachers over the past twenty years have tried to gulf the divide between real and school maths, but secondary teachers have felt too constrained by the syllabus in order to meet the traditional assessment requirements in upper years. She suggests that, in her state, changes are occurring because the assessment methods in upper secondary school have changed allowing the students to think in mathematical ways and gain a sense of both the purposes of mathematics and what real mathematicians do.

In 1987, within NSW, the assessment guidelines for courses, including mathematics, in Years 11 and 12 were issued by the old Board of Secondary School Studies and these are still in force (Board of Secondary School Studies, 1987). These included a school-based assessment for each subject which carries a weighting of 50% but is moderated by performance in the Higher School Certificate examination in that subject. Advocacy did occur on assessment by other means than written tests especially in the lower prestige mathematics courses, (e.g. Mathematics in Society and Mathematics in Practice) , but in fact this rarely occurs and the timed paper and pencil test reigns supreme. Admittedly component A and B types of assessment tasks were suggested with component B being concerned with the student's reasoning, interpretive, explanatory and communicative abilities and this led to some experimentation with untimed problem solving tasks. But it seems that for the bulk of the senior students following the more prestigious mathematics courses in N.S.W. (Units 2,3 and 4), the method of assessment remains very traditional. In the most recently developed course, Mathematics in Practice (1989), alternate assessment tasks are suggested especially for the Mathematics of Early Childhood module. However this course currently attracts only 2000 out of 60 000 Higher School Certificate candidates.

In Years 9, 10 mathematics there exists, within NSW, three levels - Advanced, Intermediate and General. The Mathematics Syllabus Committee 7 -12 is currently justifying this division of students before the Board of Studies. It seems likely that it will succeed. The School Certificate is awarded at the end of Year 10 but includes a moderated externally set and marked examination in July of Year 10 in the three distinct courses in Mathematics, in Science and English. These examination grades are then forwarded to the schools but without the names of the candidates. After further assessment, the school then awards the results to appropriate students at the end of the year - Advanced Grades 1-5, Intermediate Grades 1 -5, General Grades 1-5. Again the assessment is almost always by the timed pencil and paper method and little real attention is currently being paid in these courses to alternative forms of assessment. The courses were first developed and approved in 1983, and reflect a most traditional approach to mathematics education.

It seems that the only way to force teachers to use different teaching styles and less traditional methods of testing is to actually change the method of assessment. A number of Examining Authorities tried this in England in the mid 1980s and I am familiar particularly with the Joint Matriculation Board in conjunction with the Shell Centre at the University of Nottingham which progressively added a real problem solving question and questions into the General Certificate of Education (equivalent to our Year 10 School Certificate). This forced teachers in secondary schools presenting candidates for exams set by those authorities to teach problem solving, and teaching packages were produced to in-service teachers to do just this.

This paper really reports just the beginning of a preliminary study designed to evaluate the desire of secondary mathematics teachers in NSW to change their assessment techniques. Currently the author and six third year Bachelor of Education (Honours) students (including two primary and four secondary students) are conducting such an investigation in some twelve Sydney metropolitan High Schools, involving about 100 practising teachers. It is envisaged that a questionnaire will be designed and analysed and a number of follow-up interviews will be conducted in an attempt to learn what alternative assessment methods are being used or at least will be possibly used in the future. The result of this research will be reported at subsequent MERGA Conferences.

Recent research has resulted in a great deal of general criticism being levelled at conventional assessment procedures such as the written test. Most researchers agree that achievement tests are created within the framework of an obsolete concept of school mathematics. According to the old conception, mathematics is a set of symbols and rules, or algorithms to internalise. Therefore achievement tests are merely testing the extent to which pupils have learned symbols to produce answers. Assessment of a student's ability to replicate neatly a learned procedure applied to a routine task in a familiar context is therefore not sufficient. Tests, however, cannot assess any of the richer areas of mathematics which are fundamental to good practice. A test, for example, cannot assess the way that an individual carries out mathematical activity, their mathematical attitudes or their ability to evaluate the work that they do. Kamii and Lewis (1991, p.4) believe that tests tend to become ends in themselves, not a means to assess educational objectives. Knowing this, teachers often teach to the test, not to the curriculum or to the students. Ollerton (1991, p.5) asserts that tests undermine confidence in all but the most able student and therefore reinforce failure amongst the majority. In fact, he believes that a test can prevent students from demonstrating their ability to function as a mathematician. Furthermore, if students are actively encouraged to perceive that the most important aspects of mathematics are narrow skills that are immediately testable, it will inhibit their confidence to recognise the value of and therefore participate more actively in the broader aspects of mathematics.

Many suggestions for change in assessment techniques have been made in the past four years and these new methods are currently being used in the United Kingdom, USA, Victoria and in other Australian states and even in K - 8 classrooms within N.S.W., but not in Years 9 - 12 within this state. Back in 1988, Max Stephens, (Stephens, 1988, p.1) on behalf of AAMT, talked about the formative aspect of assessment as well as the summative. The formative aspect was concerned with the continuing appraisal of students' work and the summative approach is related to giving a clear record of achievement across the full range of learning in mathematics. Summative and comparative assessment become significant at those points where schooling connects with the world of work and the requirements of tertiary studies, when there is a need to provide clearly interpretable records of student achievement.

Keeping these two aspects of assessment in mind, researchers advocate that assessment should occur along a continuum from the least to the most formal, with a range of modes being used in a blend that is appropriate for the age of the student and to the teaching-learning situation. Writing very recently Clarke (1992, p.25) asserts that the distinction between formal and informal assessment is a critical one, and that it is vital that both forms of assessment are seen as being of value in the classroom situation. He describes formal assessment as that which involves the cessation of instruction, typically for the whole class, while an "assessment" event is held. Informal assessment, on the other hand, involves the collection about students' learning coincident with instruction and without disrupting the learning process. He and his colleagues (Clarke, Clarke and Lovitt, 1990) believe very strongly that informal assessment, particularly in the form of direct teacher observations, offers the teacher a greater wealth of information than can be provided by more formal assessment using tests, which often does little more than legitimise and quantify the assessment made through extended classroom contact. However, informal assessment generally lacks structure, and the information it produces is not systematically recorded and lacks the status accorded to a test score. They recommend that, by introducing some structure into their informal procedures, teachers can maximise the information they collect and minimise the time squandered on redundant assessment.

With the recognition of the great value that lies in informal assessment procedures has come an attempt by researchers to structure these in such a way that they can be of great benefit as assessment tools. It is suggested that all of these techniques can be applied in the secondary classroom:

INFORMAL METHODS OF ASSESSMENT IN MATHEMATICS

1. **Observing students.** As Lester and Kroll (1991, p.278) note:

   > observing students while they are engaged in mathematics activities can yield invaluable information not only about their skills abut also about their thinking processes, their attitudes, and their beliefs.....teachers can learn a great deal about students by circulating unobtrusively as students work in small groups and by interjecting questions to clarify their observations.

   Observation, to be effective, should frequently be sharply focussed. Observation should be both individual and in small groups. Stenmark (1989, p. 22) suggests that observation could include student learning styles, student ideas, communication, co-operation and use of manipulatives.

2. **Annotated Records**
   This strategy focuses on recording significant events that are observed by the teacher. Teachers can draw up a checklist of the skills, behaviours or attitudes that they wish to encourage in their students and then record those significant moments that either extend or challenge their image of the particular student. The original checklist helps teachers to know what to look for and also provides a means to summarise the significant moments into a systematic record for each student.

3. **Asking Questions**
   Classroom questions offers probably the best chance to monitor the development of meaningful mathematical understanding. However as Thompson and Briars (1989, p. 22) point out most questions asked by a teacher in a typical lesson require only simple rote responses which fail to give insights into students' understanding. To discover the extent to which these students are making sense of the material, questions which require more thoughtful, elaborate answers are required. Clarke and Sullivan (1990) refer to these as "good" questions and characterise them as requiring, firstly, more than simple recall of a fact or replication of a procedure, secondly, having several acceptable answers and thirdly, as allowing the students to learn by doing them. It is important to allow plenty of wait time to allow the students to give thoughtful answers. (Stenmark, 1989, p. 24)

4. **Interviews**
   These interviews allow the teacher to assess mathematical understanding by probing the depths of that understanding. Is the student merely parroting back memorised responses, or has the student interacted with the ideas and built them in to his or her own conceptual structures? (Stenmark 1989, p. 23) The opportunity for positive rapport is increased while allowing the teacher to assess the level and accuracy of student understanding. These interviews could be seen by teachers to be very time consuming and quantification methods will have to further developed.

5. **Students' Work Folios**
   One of the most effective ways to document progress is to collect representative examples of student work. These are well used in other curriculum areas, but until now have rarely been used for assessment purposes in mathematics. (Stenmark, 1989, p. 8) Teachers and students should be allowed to choose most of the items to include in the portfolio.

6. **Student Journals**
   These enable students to write in mathematics and enable them to record what they are thinking, to summarize key topics and to internally engage in a dialogue through which they can reflect on and explore the mathematics that they have encountered. This a brand new activity in the mathematics classroom and secondary teachers will need to be convinced of its worth.

As well as these six informal methods of assessment, there are at least another three formal methods of assessment which are alternatives to the more traditional timed pencil and paper test. It is suggested that these techniques are also applicable to the secondary mathematics classroom:

ALTERNATIVE FORMAL TECHNIQUES IN ASSESSMENT

1. **Investigations in Mathematics**
   These activities are very appropriate for some topics in the curriculum and provide an opportunity for the student to demonstrate their ability to use the skills that they have learned in solving a problem. The Nottingham Shell Centre investigational material is a good example of this technique. (Problems with Patterns and Number, 1984 and the Language of Functions and Graphs, 1985)

2. **Performance Testing**
This involves giving a group of students, or an individual, a mathematical task that could take from half an hour to a couple of days to complete or solve. An assessor could videotape or tape record the activity and the object of the assessment would be to look at how students are working as well as the completed tasks or products.

3. **Student Self-Assessment**
This is a brand new form of formal assessment in the secondary classroom. Students are certainly capable of self-assessment and mathematical power comes from knowing how much we know and what to do to learn more. Students can also obtain feedback from their peers by making constructive comments on one another's work. (Vale, 1987, p. 7)

By developing a range of assessment alternatives in secondary mathematics - formal and informal - a much wider picture will be built of the students' performance in mathematics. Teachers will oppose these changes vehemently because they represent a huge change in direction from past practice. Many trials of the new techniques will need to occur to convince them of the worth of change. However, over time, teachers will become persuaded that the bigger picture of their students' performance gives a more true account of the students' grasp of mathematics.

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FROM PROCEDURAL ANALOGY TO UNDERSTANDING

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BACKGROUND

In surveying the literature in mathematics education there seems to be a significant lack of success in explaining the relationship between the use of concrete embodiments in teaching mathematics and learning mathematical concepts and procedures. A colleague and I have developed a theory which has the potential to explain the circumstances under which the use of concrete materials produce efficient learning of mathematical concepts and skills (Ohlsson and Hall, 1990). The theory relies heavily on recent developments in mathematics education and cognitive science.

PROCEDURAL ANALOGY THEORY

The procedural analogy theory described briefly here is a theory of instruction in arithmetic. In addition to the original publication concerning this theory (Ohlsson and Hall, 1990), aspects of the theory have been presented elsewhere (Hall, 1990 and 1991). This theory is applicable to the use of concrete teaching materials by teachers who aim to increase learners' arithmetic skills and understandings beyond what would be the case without learners using such materials. That the use of these concrete materials may be seen as an attempt to better represent the abstractness of arithmetic facts and operations is hardly a new idea, but the empirical data supporting such notions is inconclusive.

The *raison d'etre* for concrete materials is to assist learners in internalising the mathematical concepts and skills represented by these materials. In cognitive science terminology, the materials assist the learning of declarative and procedural knowledge. The procedural analogy theory describes how concrete materials allow this declarative and procedural knowledge to be developed to the required target behaviour. Simplification, procedural analogy and symbolism lead finally to automatic responses.

The theory predicts that the pedagogical usefulness of an embodiment is a function of the degree of similarity of the procedure for the embodiment to the procedure for the initial symbolic representation, and argues that this relationship can be quantified. That is, this theory describes specific ways in which teachers may use concrete materials so as to increase the effectiveness of intended learning outcomes.

The four operations in algorithm format constitute a large portion of school children's mathematical experiences, certainly up to the beginning of secondary school. Yet we know that the completion of whole number algorithms from the learners' points of view is problematic. For example, subtracting one number from another is deceptively simple, provided you know how to do it. We will all be familiar with the range of incorrect approaches and the inventiveness of learners in creating incorrect solutions to algorithms.

If we consider the subtraction shown below on the left hand side of Figure 1, we expect learners to do something similar to what is written on the right hand side of the table.

5 4 2 - 2 6 3 2 7 9	3 from 2, cannot. Cross out 4, write 3. Write 1 beside 2. 3 from 12 is 9. 6 from 3, cannot. Cross out 5, write 4. Write 1 beside 3. 6 from 13 is 7. 2 from 4 is 2.

Figure 1: Simple subtraction

Table 1 shows that there are three or perhaps five steps in solving this algorithm. And this is true for an expert in subtraction algorithms. Unfortunately it is not true of learners, and may barely be true of many primary school teachers. For the novice, this simple subtraction is more likely to involve the steps shown in Figure 2.

5 4 2 - 2 6 3 2 7 9	0.0	542 - 263 (recognise question)		
	1.0	Process units		
		1.1	Take away 3 from 2 (cannot)	
			1.1.1	Trade for more units
			1.1.2	Recall 4 - 1 = 3
			1.1.3	Cross out 4, write 3
			1.1.4	Write 1 next to 2
			1.1.5	Recall this is 12
		1.2	Take 3 from 12	
		1.3	Recall 12 - 3 = 9	
		1.4	Record 9 in answer space	
	2.0	Process tens		
		2.1	Take away 6 from 3 (cannot)	
			2.1.1	Trade for more tens
			2.1.2	Recall 5 - 1 = 4
			2.1.3	Cross out 5, write 4
			2.1.4	Write 1 next to 3
			2.1.5	Recall this is 13
		2.2	Take 6 from 13	
		2.3	Recall 13 - 6 = 7	
		2.4	Record 7 in answer space	
	3.0	Process hundreds		
		3.1	Take 2 from 4	
		3.2	Recall 4 - 2 = 2	
		3.3	Record 2 in answer space	
	4.0	Read answer		

Figure 2: 'simple' subtraction

Figure 2 shows that for the novice there may be 26 steps in solving this algorithm. Certainly over time chunking of steps would take place, as would the development of a more automatic response, so the actual number of steps is likely to decrease. On the other hand, those learners whose recall of number facts requires some calculation beyond quick reference to short term memory would increase the initial number of steps. There are two points that I want to make here. The first is that for novices, simple subtractions are anything but simple, they are complex multi-step operations with numerous opportunities for error. The second point relates to the value of the procedural analogy theory in drawing the teacher's attention to the need to cover every aspect of the algorithm to be taught.

MAB procedure	**Target procedure**
0.0 542 - 263	0.0 542 - 263
0.1 Subtract 263 from 5H, 4T, 2U	
1.0 Process units	1.0 Process units
1.1 Take 3U from 2U (cannot)	1.1 Take 3 from 2 (cannot)
1.1.1 Trade for more units	1.1.1 Trade for more units
1.1.2 Move 1L from 4L to bank bring back 10U	1.1.2 Recall 4 - 1 = 3
1.1.3 Join 10U and 2U	1.1.3 Cross out 4, write 3
1.1.4 Recall 10U + 2U = 12U	1.1.4 Write 1 next to 2
	1.1.5 Recall this is 12
1.2 Take 3U from 12U	1.2 Take 3 from 12
1.3 Recall 12U - 3U = 9U	1.3 Recall 12 - 3 = 9
1.4 Record answer, 9U in answer space	1.4 Record 9 in answer space
2.0 Process tens	2.0 Process tens
2.1 Take 6T from 3T (cannot)	2.1 Take 6 from 3 (cannot)
2.1.1 Trade for more longs	2.1.1 Trade for more tens
2.1.2 Move 1H from 5H to bank bring back 10T	2.1.2 Recall 5 - 1 = 4
2.1.3 Join 10T and 3T	2.1.3 Cross out 5, write 4
2.1.4 Recall 10T + 3T = 13T	2.1.4 Write 1 next to 3
	2.1.5 Recall this is 13
2.2 Take 6T from 13T	2.2 Take 6 from 13
2.3 Recall 13T - 6T = 7T	2.3 Recall 13 - 6 = 7
2.4 Record answer, 7T in answer space	2.4 Record 7 in answer space
3.0 Process hundreds	3.0 Process hundreds
3.1 Take 2H from 4H	3.1 Take 2 from 4
3.2 Recall 4H - 2H = 2H	3.2 Recall 4 - 2 = 2
3.3 Record answer, 2H in answer space	3.3 Record 2 in answer space
4.0 Read answer (2H 7T 9U)	4.0 Read answer (279)

Figure 3: procedural analogy

The basis of the procedural analogy theory rests on the use of concrete materials to draw a parallel between the numbers represented by these materials and the arithmetic processes represented by operations on these materials. That is, while materials may be used in a wide range of ways, there are ways that are more effective than others because they more closely resemble the target behaviour that we want learners to use in the solving of algorithms.

Figure 3 shows one use of Multibase Arithmetic Block (MAB) materials and the target algorithm that is developed from this material. The steps emphasised both in the use of MAB materials and in the target algorithm are not unique, and must be developed by the teacher. Once the teacher has decided on the target behaviour, an effective sequence can be developed for the concrete material.

The procedural analogy theory argues that the closer the relationship between the concrete procedure and the target procedure, the higher the analogy between the two sets of steps, and so the more effective will be the learning outcomes. The procedural analogy theory uses an isomorphism index ($I_{1,2}$) as a measure of analogy between the two sets of steps. The index is given by the formula

$$I_{1,2} = \frac{(N_1 + N_2 - 2) - (D_1 + D_2)}{N_1 + N_2 - 2}$$

where N_1 is the number of steps in the first procedure, N_2 the number of steps in the second procedure, D_1 the number of steps in the first procedure but not in the second, and D_2 the number in the second procedure but not in the first. In Table 3, $N_1 = 25$, $N_2 = 26$, $D_1 = 5$ and $D_2 = 6$ giving a value for I of 0.78. This is a relatively high value which is quite difficult to increase in this algorithm. This simply reflects the reality that there are differences between using concrete materials and in writing algorithms. At the same time, slight variations in the steps will lead to a lower isomorphism index. The theory maintains that the higher the I value the more effective will be the value of the concrete materials, and the greater the learning outcomes. That is, the procedural analogy theory provides a method of measuring likely pedagogical success, and one allowing alterations in teaching steps to be assessed prior to teaching.

Application of the theory requires the following steps:

(a) select the arithmetic topic;

(b) identify a procedure which an expert would use to answer questions in this topic;

(c) develop a detailed procedure as target behaviour for learners;

(d) identify suitable concrete materials to support learning this topic;

(e) develop a detailed procedure for the use of these materials;

(f) use the Isomorphism Index to contrast alternatives with concrete materials, and with the target behaviour;

(g) use the Isomorphism Index to identify the best set of teaching procedures;

(h) teach the topic.

RESEARCH DATA

A small scale research project was designed to test this procedural analogy theory. This pilot study involved teaching subtraction to small groups (n=5) of primary school children, selected by virtue of one school's accessibility. Three groups of children were established: experimental group 1 used MAB materials, expanded notation and the target subtraction algorithm. Experimental group 2 used MAB materials and the target algorithm, while the control group used the target algorithm only. The results provided some support for this theory, but none were statistically significant (Kelly, 1990).

I am presently involved in a larger scale research project involving two schools, 120 children, five teachers and three teaching approaches to subtraction using MAB materials. Data gathering is still underway, so no meaningful statistical analyses can be reported here.

DISCUSSION

The present research seeks to verify the procedural analogy theory described here. There are clearly many aspects that could be discussed: the theory itself, the research approach, the data I have collected and ways in which they will be analysed. But I have a particular question that needs answering.

Even though the larger part of this paper has been given over to describing the theory, the description is in the way of a precursor to this discussion. I have some ideological difficulties with this procedural theory: mathematics has to be more than procedures. In particular, assuming that the present theory has some veracity, I want to extend the present theory so as to investigate the extent to which 'understanding' takes place in the learner, and to gain some measure of the efficiency of learners' cognitive networks of mathematical skills and concepts. That is, it is certainly educationally worthwhile to be able to help learners solve algorithms more efficiently, and that is no mean feat. But contemporary mathematics education is less about correct answers and more about understanding and constructing meanings. If this procedural theory is correct, and for argument's sake let me assume it is, then some important questions arise. For example, which steps in the concrete materials sequence or in the target procedure lead to understanding? And how can we gain insights into and prove the existence of such understandings? Further, how does this increased understanding influence learners' networks of cognitive structures, and how can we prove it?

This paper ends then with a list of research questions generated from the procedural analogy theory described here: questions about the procedural analogy theory, about pedagogy, about understanding and about cognitive structures. Such questions include

> How generalisable is the procedural analogy theory, in terms of mathematics topics and ages of learners?
>
> Does the procedural analogy theory lead to increases in understandings?
>
> Do teaching approaches developed through the procedural analogy theory, in contrast to other teaching approaches, increase the likelihood of transfer between topics, and increase learners' problem solving abilities?

Does application of this procedural analogy theory encourage learners to develop a richly connected network of cognitive structures?

Is the cognitive development taking place through the application of this procedural analogy theory superior to the developments using other teaching approaches?

What instruments will give researchers a clearer picture of changes in learners' understandings and cognitive structures?

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MATHEMATICS, COMPUTERS AND "AT-RISK" PRESCHOOLERS

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This paper summarises our earlier research in the field of computers and young children, presents data and analyses from our current research, particularly in the context of mathematics education, and discusses possible future directions for this research.

BACKGROUND

Our work with computers and preschool children dates from an Australian Apple Education Foundation grant awarded in 1984. About this time, we produced a number of papers focussing on the role of computer-based activities in early learning (Elliott, 1985; Hall, 1984, 1985; Hall and Elliott, 1985). At that time there was little research data available as to how to use computers with very young children.

Our first empirical research was exploratory in nature, and addressed questions relating to young children's learning through computers, and the nature of young learners' interactions with computers and peers. A large quantity of data was gathered, partly about young children using mathematics and pre-mathematics software, but also on a wider number of variables such as the frequency with which children used the computer, how long they stayed at the computer, how they used and interacted with the robot attached to the computer, the language used while at the computer, and the extent of independent, parallel and cooperative play (Elliott and Hall, 1985a, 1985b, 1986a, 1986b; Hall and Elliott, 1986). Some of the findings from this research are presented here in Tables 1 and 2.

One of the early concerns of preschool teachers about computer access for young children was that it would become a socially isolating experience. That is, there were preconceived notions that a child would work alone at the computer, and so spend part of the day in isolation. Our data indicated quite the opposite: we had many observations of social interaction, of emerging leadership skills and of cooperative behaviour.

Table 1: Social interaction and cooperative behaviour*

	Total	Mean
Children observed in computer area	190	3.3
Children observed interacting socially in computer area	148	2.6

* data based on 57 observations

The data in Table 1 illustrate some of our findings: typically there were three children at the computer, and typically they were interacting socially and working cooperatively.

Table 2: Sex differences in computer usage

Average number of girls in computer area at any given time	1.2 (10%)
Average number of boys in computer area at any given time	2.2 (24%)
Average time on task	
- girls	8.2 minutes
- boys	18.8 minutes

Data from Table 2 show that in this group of children there are clear sex differences in the number of boys and girls at the computer at any given time, and there are differences between boys and girls in the time they remain working in the computer area. More boys came to the computer than girls, and they remained longer.

More recently we decided to use as our subjects those preschoolers considered to be 'at-risk'. Typically this meant children who were as 'average' in all respects as other children except they were in community and home settings where there was a constant high level of stress, which often led to slower physical, social and academic development. Further, we selected mathematics to be the discipline because it is an important and neglected area in preschools, and because these children's mathematical skills and understandings were likely to be below that of their peers in other preschools. Another consideration in choosing mathematics was our view that teachers and other care givers in early childhood settings frequently lack confidence in mathematics: we saw the computer as giving these people support and a structure in which to encourage young children's mathematical learning. In this research we established that 'at-risk' children, working in pairs at a computer together with an adult, and using commercially available mathematical software, were able to make significant gains in a range of mathematical skills (Elliott and Hall, 1990a, 1990b, 1990c).

Data were collected from two groups of children who worked at a computer in pairs with adult assistance. Children spent twenty-minute sessions at the computer, and totalled two to four hours in all over the six weeks of the research program. The control group used a range of non-mathematical software, the experimental group used commercially available mathematics software designed for young children. In each case the adult supported children in their learning, gave assistance when necessary, and encouraged children to remain on task. Table 3 shows some of the findings from this research with 'at-risk' preschoolers.

Table 3: 'At-risk' preschoolers pretest and posttest scores

	Pretest	Posttest
	(maximum score is 80)	
Control group (n = 4)	47.5	43.0
Experimental group (n = 6)	46.5	65.2

Parents and other care givers frequently do not expect these children to 'learn': they have low expectations of these children in terms of academic achievement. So even though a Wilcoxon two sample test showed the gains made by the experimental group to be significant at the 2% level, from a more general educational perspective an equally significant finding is simply that some mathematical learning did indeed take place.

THE PRESENT RESEARCH

Our original research with preschoolers and computers had as its theoretical base children's social and cognitive development, and we gave more emphasis to the cognitive aspect as we moved from project to project. In trying to further our theoretical base we looked firstly to Information Processing Theory. Information Processing Theory may be seen as a process where sensors receive data which are transferred to short term memory, and may then be called from or sent to long term memory. Directing the processing of data in short term memory, and its flow in and out of long term memory is a control mechanism. This control mechanism varies considerably between people, and between contexts. An individual's awareness and ability to manipulate this control function consciously is referred to as *metacognition*. Its importance lies in the increasingly accepted view that high levels of metacognitive awareness are associated with high levels of performance on a range of cognitive tasks. Applying this to our own research raises questions such as 'can adult intervention as young children use mathematical software at computers lead to increases in children's metacognitive awareness?' and 'will this result in higher levels of mathematical achievement?'. This theoretical position is not as well developed as the one described below.

A second theoretical base for the present research derives from Vygotskian notions of the zone of proximal development and the internalisation of externally provided guidance. According to this perspective early learning is mediated by interactions with competent others (Rogoff, 1990; Vygotsky, 1978) resulting in ongoing stimulation and motivation for learning, as well as support of a more metacognitive (or self-regulatory) nature. A key function of self-regulatory behaviour is its role in helping children become active participants and controllers of their own learning (Zimmerman, 1990).

From this perspective the value of using a computer to generate worthwhile learning experiences lies in the prestructured content and in-built cognitive supports of its software (for example, feedback, memory aids, sequencing, and the predictable and predetermined flow of a computer-based activity), as well as its highly motivating and engaging nature. When these contextual features are supported by co-participants (teachers and peers) the result should be increased cognitive activity. Complementary peer interaction, as well as the individual's own structuring of the activity, results in cognitive and metacognitive dialogues that serve to support and mirror knowledge and thinking.

METHODOLOGY

Our latest research concerns three groups of randomly selected 'at-risk' preschool children, two groups working in pairs at a computer with adult assistance and a control group. Each group was given a pretest, and four weeks later a posttest, using the TEMA 2 instrument. Children in the experimental groups received about fifteen twenty-minute computer sessions over a four week period, the control group children had no computer sessions, and all children took part in the usual day to day activities of their preschools.

An essential element of this research is the role of the adult. For one experimental group the adult took on the role used in the previous research: that of a caring and supportive guide. The adult encouraged children to remain on task, and provided reinforcement. For the second experimental group the adult adopted what we refer to as 'a metacognitive approach' to teaching. In this approach the adult went beyond the caring and supporting role so that as the child was asked a question through the computer and answered it, the adult extended the activity by giving explanations and additional instructions, and by asking the child a range of questions. The adult gave direct guidance, provided cues and questions, and modelled and demonstrated aspects of the material to be learned. These approaches are described more fully in Elliott (1991). The interactions between adult and children were intended as the link between existing and new knowledge. The computer software focusses the child on new learning, the adult's interventions have learners link this new knowledge with their present knowledge. The adults intervention is essential in ensuring there is a 'metacognitive' aspect to the computer sessions. The adult's directions and questions typically took the form of 'show me three fingers', 'clap three times', 'count my (three) fingers', 'let's count these together', 'draw a three in the air with your finger', 'what song did we sing yesterday that had three in it?', 'tell me about the story we read earlier today where there were three kittens', and 'write a three for me'. Data were gathered during 1991, with some follow up data in 1992. These data are still being analysed, with some results outlined below, and a report is in preparation (Elliott and Hall, 1992).

RESULTS

Data from this research are still being analysed, but early indications are that both teaching approaches provide greater mathematical learning than the mathematical content of the preschool's existing program. There is support too to show that a metacognitive teaching approach will result in greater mathematical learning than a typical caring, concerned teaching approach. That is, there is a specific teaching approach that encourages learning of mathematical concepts and skills, and it is a better approach than would typically be used by teachers in preschools. Some results are displayed in Tables 4 and 5. Other indications are that children work effectively in pairs, that the computer provides strong motivation for most children, and that the adult and the computer create an environment that keeps children's learning focussed.

Table 4: Metacognitive, caring and control groups: pretest versus posttest scores (one tailed t-test)

Group	Raw scores	Percentile scores
Metacognitive (n = 18)	.0001	.0001
Caring, supportive (n = 17)	.0001	.0001
Control (n = 19)	n.s.*	n.s.*

* not significant

Table 5: Metacognitive versus caring groups: pretest and posttest scores (one tailed t-test)

Pretest scores		Posttest scores	
Raw	Percentile	Raw	Percentile
n.s.	n.s.	.0527	.0456*

* significant at .05 level

DISCUSSION

Further work in this area will be concerned with the role of metacognitive processes in early mathematics learning, the teaching of mathematics in preschools, the role of the computer in this process, and the professional development of early childhood teachers. For example, one of the authors is presently working on a curriculum development activity involving 'at-risk' preschoolers learning mathematics through computers, and we intend to develop a teacher's kit to be used in professional development activities. Some data and resources are presently available for this, but extra funding will be sought shortly.

In research terms, further work needs to be completed:

(a) in refining the metacognitive model we are using. In particular which contextual supports seem most effective in maximising metacognitive activity and academic competence?

(b) in contrasting learning through an intensive program such as our 1991 work, with regular computer sessions throughout the year;

(c) in examining the impact of this experience on these children's first years of school;

(d) on the professional development of trained and untrained teachers;

(e) on the appropriate content of preschool mathematics programs for these children; and

(f) on the specifications of computer software for these children.

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TEACHERS' ATTITUDES TOWARDS THE STUDENT USE OF CALCULATORS IN K-6 MATHEMATICS CLASSES

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This report examined the findings of a study into primary teachers' present attitudes toward the student use of calculators in primary (Kindergarten-Year 6) mathematics classes. Data were collected from a questionnaire administered during 1990 to a sample of 147 teachers undertaking their fourth year of study for a Bachelor of Education (Primary) at three university campuses in New South Wales, Australia.

The results of the study showed 32% of primary teachers had their students using calculators, though 85% supported the student use of calculators in primary mathematics classes. Teachers indicated that calculators were not being used because either none were available, there was no school policy on their use and/or they were not considered relevant for the class being taught. Those teachers who supported the primary student use of calculators believed that calculators are a technological tool for use in mathematics, that they increase childrens' confidence, they take the focus off computation in doing problems and that children use them outside the classroom. It was found that of these teachers, 55% supported the introduction of calculators before the end of Year 2. A total of 4% did not support their use at all in primary mathematics classes.

This study concluded that there was not overall support from primary teachers for the introduction of the calculator into Kindergarten as recommended in the *National Statement on the use of Calculators for Mathematics in Australian Schools* (1987).

A LITERATURE OVERVIEW

There has been little in the last 25 years related to mathematics education that has caused more debate and concern amongst educators and the community than the introduction of the calculator into primary mathematics classrooms. Over the last 20 years calculators have become readily available to the community and their use in schools has been shown to be increasing (Moursund, 1981; ICME, 1984; FASTS, 1988;). During this time its great promise to many has been "that it enables a student to concentrate on the problem at hand, without it being obscured with what, to some students, seems to be an impenetrable barrier of calculation" (Willis, 1979, p. 3). Yet there has continued to be a controversy within the community concerning the use of calculators in the mathematics classroom.

During the 1970's and 1980's there had been a gradual international move towards the recommended student use of calculators from Kindergarten. The 1970's had seen research focus on the effect of the calculator on students' computational skills. This research and the ensuing discussion led to the publication of the NCTM's *Agenda for Action* (1980) including Recommendation 3 on the use of calculators in the mathematics class :

That mathematics programs take full advantage of the power of calculators and computers at all grade levels. (NCTM, 1980, p.8)

There then followed the release of the Cockcroft Report (1982), in England, which further discussed the place of calculators in the mathematics classroom. With the publication of Hembree and Dessart's meta-analysis in 1986 the NCTM (1987) reaffirmed its policy on the student use of calculators. This build up over 10 years resulted in a definite statement being issued within the *Curriculum and Evaluation Standards for School Mathematics* (1989) which recommended calculators be made available to all students in all grades. This has set the focus for the student use of calculators in the USA for the 1990's.

In Australia, over this time Blane and Willis (1986) had recommended that a National Statement on the use of calculators in classrooms be developed. This national policy statement was released (CDC and AAMT, 1987) with all state educational authorities supporting the view that calculators be available to students from Kindergarten. In 1991 there was the release of a *National Statement of Principles on Mathematics in Australian Schools* which supported the use of calculators from K -12.

Of most importance 1990 saw the report of work carried out in the United Kingdom on the *Calculator - Aware Number* curriculum. The work of the PrIME project and specifically the CAN curriculum work (1989), which focussed on the early primary years, found that when calculators were used the children's knowledge of number was beyond that expected from them at that age.

The children certainly show greater competence in mathematics at an earlier age than would normally be assumed. This is particularly true for large and small numbers, negative numbers, the ability to recognise patterns and awareness of their significance, and the understanding of place value. The growth in the children's development of concepts such as fractions, decimals, square roots, has also become more evident during this second year of CAN development. (PrIME, 1989, p.12)

There have been clear directions given by research (Hembree and Dessart, 1986), reports (NCTM, 1980; Cockcroft, 1982), national statements (CDC and AAMT, 1987) and state education systems (NSW Department of School Education, 1989) as to the place of calculators in mathematics classes.

Their role in the classroom was perceived to be varied. The literature suggested that calculators could be used for games (Koop, 1977), in problem solving (Lambert, 1985), for computation work (Reys, 1989), number exploration (Wilson, 1986), estimating and checking (Discipline Review Report, 1989) and in developing mathematical language (Del Campo, 1986). There was also the acceptance that they were part of our everyday world (Koop, 1979) and a part of the technological change that is taking place in our society.

There was consistent research evidence that the use of calculators did not reduce computational skill. Indeed they may even have a positive effect on the performance of students (Bell et al, 1983; Hembree and Dessart, 1986; Willis and Kissane, 1989). In fact research findings indicated that the introduction of calculators into the mathematics class has a positive effect on children's motivation and enthusiasm in doing mathematics (Foster, 1979; ICME, 1984; Comstock and Demana, 1987; Suydam, 1987, 1989, p.12)

Presently there exists an increasing curriculum policy direction for the student use of calculators by all students in all classes, while at the same time there continues to be a community controversy and an apparent divergence of opinion amongst teachers (Bitter, 1980; Shult, 1981; Cooper and English, 1985; Del Campo, 1986; Perry, 1989; McIntosh, 1990) over their use.

METHODOLOGY

The major issues that emerged from the literature review were related to:

- the ways in which calculators are used;
- the grade from which calculators should be used;
- the continuing community controversy over the use of calculators;
- the support teachers required for the effective introduction of the calculator into primary mathematics classes.

These and other issues formed the basis of the research questions used within a questionnaire administered to an available sample of 147 teachers during May/August 1990. The final format comprised 20 questions categorised into two sections. Section 1 identified the respondent's demographic informationon while Section 2 was directed at issues involving the student use of calculators in primary mathematics classes.

The sample of teachers chosen were those undertaking a course of study towards their fourth year qualification, Bachelor of Education (Primary), at three Sydney based university campuses. It was assumed that this sample would contain currently practising primary teachers and that their attitudes would be representative of teachers as a whole. A limitation of the study was that all teachers in the sample share the characteristic of undertaking their fourth year of study and thus the findings cannot be generalised to the population of all the primary teachers in New South Wales. It could also be assumed that this sample population may differ from the remaining teachers in the state in that they may have a heightened awareness of curriculum policies and advanced knowledge of calculator usage through the course work that they were completing.

ANALYSIS OF DATA

The collected data were analysed for response rates for each variable from each campus and then the overall response rate for each variable from all completed questionnaires. There were two procedures used in the analysis of the data:

1. A frequency count of responses for each variable from each campus was calculated.
2. A cumulative frequency count of responses to each variable from all collected data was calculated.

Chi square was used to test for significant differences across the variables of age, gender, years of teaching experience and class presently taught. This resulted in identifying Kindergarten teachers as having a significant difference of opinion as to the relevance of the calculator as compared to other teachers.

GENERAL DISCUSSION

In comparing the results of this study to that of Blane and Willis (1986) there was evidence of a growth in support over the last five to six years for the student use of calculators in primary mathematics classes. There was evidence of a growth in teacher support for the earlier introduction of calculators when compared to the report given at ICME 5 which indicated that "a study in 1983 in New South Wales, Australia, 40% of grade 5 students used calculators in the classroom" (Mohyla, 1984, p.69). It was also different to the findings of the UNESCO/CDC (1986) study which found that "the modal age at which it was considered most appropriate to use calculators in mathematics classes was '14', that is in Year 9 in schooling in Australia" (Blane and Willis, 1986, p.12).

Teachers strongly supported (85%) the introduction of calculators into the primary years though only 32% responded that they were currently using calculators. They acknowledged that calculators were mathematical tools available to and used by children outside the classroom and that there were a number of

uses for them in mathematics classes. Teachers (13%) who did not support the student use of calculators expressed concern that they reduced the need for children to learn their basic facts and that children would not learn pencil and paper computational skills.

YEAR FOR INTRODUCTION

However, there was an obvious divergence of opinion amongst teachers concerning the Year of primary school that calculators should be introduced. The teacher responses ranged from Kindergarten to not at all, though the vast majority of teachers believed that they should be introduced by the end of Year 5. This did not support the national calculator policy (AAMT/CDC, 1987) that recommended their introduction from Kindergarten. Perhaps the majority of teachers did not support the student use of calculators from Kindergarten because they were unsure of their use in the early primary years. This may add support to the need for further investigation of the student use and relevance of calculators from Kindergarten.

The divergence of opinions amongst teachers reinforced the need for clearer directions from educational authorities on the student use of calculators in primary schools, the need for further research into the appropriate use of calculators in the early years and the need for the ongoing professional development of teachers to disseminate research findings and to generate discussion regarding this issue.

REASONS FOR LACK OF USE

The vast majority of teachers supported the student use of calculators in primary mathematics classes but they were currently only being used in a minority of mathematics classes (32%). This should be a concern for two reasons. Firstly, the NSW K-6 Mathematics Syllabus (1989) recommended the student use of calculators from Kindergarten and it is apparent that this was not occurring. Secondly, the teachers who participated in this study were completing their fourth year of study towards a Bachelor of Education

(Primary) and it could be assumed that these teachers would be aware of current curriculum policies and yet only a third had their students using calculators.

The reasons given for not using calculators were varied but centred on calculators not being available, no school policy on their use and not being seen as relevant for the class being taught. These reasons need comment.

1. That there were no calculators available seems to be an oversight in supporting the N.S.W. K-6 Mathematics Syllabus which recommended their use. It also raised the question of why calculators were not available. It may be that the school personnel responsible for making decisions about resources may:

    - not be aware of the syllabus' recommendation;
    - not support that recommendation;
    - not know how to implement the use of calculators in the mathematics classroom;
    - be concerned about parent reaction to the use of calculators.

    Perhaps schools believed that it was the responsibility of the educational authorities to supply schools with an adequate number of calculators for teachers to introduce them into their mathematics classes as recommended in the N.S.W. K-6 Mathematics Syllabus (1989). The lack of calculators was identified by teachers as a major reason why students were not using them in primary mathematics classes.

2. Schools need to develop policies that support and reflect the relevant syllabus documents. It appeared that a number of schools have not yet developed a calculator policy and this was identified by teachers as a reason why students were not using calculators.

    The student use of calculators has generated such community discussion over time that schools need to discuss the issue and develop a school policy on their use.

3. Many teachers did not see calculators as being relevant for their class. There were 26 teachers who indicated this and when identified 12 of these teachers were Kindergarten teachers. In the total study there were 16 teachers who said that they were currently teaching Kindergarten. Of these, 12 said that calculators were not relevant for their students. This needs further investigation for if this was the case amongst all Kindergarten teachers it would be difficult for educational authorities to implement a state and national policy on calculators that recommended their introduction from Kindergarten. It may also be that since many schools have not developed a policy on the student use of calculators, teachers have not discussed their place in Kindergarten, leading teachers to believe that they were not relevant for the first year of schooling. There needs to be further study undertaken to investigate the use of calculators in Kindergarten in order to gauge their relevance for this grade. It may well be that the introduction of calculators into Kindergarten is not supported by teachers or the community.

These three issues must be addressed by educational authorities, researchers and schools if teachers are to use calculators effectively and efficiently in their mathematics classes.

The sample for this study comprised primary teachers who were completing their Bachelor of Education (Primary) which limited the generalisation of findings to all primary teachers. A broader study needs to be carried out to investigate the support amongst teachers of varying backgrounds towards the student use of calculators. Such teachers could be those in executive positions within the school, teachers in various classes and teachers with varying years of experience.

The majority of teachers involved in this study were in their first five years of teaching. Perhaps the lack of student use of calculators could be due to the teachers just beginning their careers and they may be more concerned with surviving the day to day rigours of teaching. Perhaps they will begin to have students use calculators more as they gain experience and confidence in their teaching.

It could also be argued that these teachers were among the most recent graduates from teacher training courses and that the mathematics units within these tertiary courses should have prepared them to use calculators in their mathematics lessons. These courses need to be evaluated in the light of recent research and syllabus documents concerning the issue of calculator use to better prepare initial teachers for the role of calculators in the primary mathematics curriculum.

SCHOOL POLICY STATEMENTS

Teachers indicated that a number of schools did not have policies on calculator use mainly because they were not aware of official policy, were waiting for directions or it was not seen as a priority. Concern should be expressed that schools were "unaware of official policy" or "waiting for directions for their use in schools" for they have been given an official syllabus and directions in the form of the NSW K-6 Mathematics Syllabus (1989). These directions may not have been effectively communicated to teachers through the syllabus and clearer statements may need to be made.

In NSW the K-6 Mathematics Syllabus (1989) and nationally the calculator policy (AAMT/CDC, 1987) have recommended the use of calculators from Kindergarten. One could have assumed that school policies would reflect these statements. This was not always the case. School calculator policies varied regarding the years in which students were allowed to use them. Some had them available from Kindergarten others from Year 4. A clear policy statement should be developed by schools on how and when calculators can be used by students. Such a policy needs to be made available to teachers, students and parents so that a positive direction regarding their use is given to all within the school community.

The issue of the student use of calculators has been a community concern for twenty years. With the increasing student access to calculators and the directions given to its place in the mathematics syllabus the development of a school calculator policy should be seen as a priority and discussion of the issue needs to be entered into by school communities.

SCHOOL-BASED PROFESSIONAL DEVELOPMENT COURSES

Teachers emphasised the importance of school based professional development courses and their willingness to attend if they were organised during school time. Schools need to respond to this stated need and organise such courses through their staff development plan. To assist teachers and schools in their discussion of this issue educational authorities could disseminate research findings to schools. Schools and teachers need a research base for these discussions. Hembree and Dessart's (1986) meta-analysis considered only 16 studies relating to grades Kindergarten to Year 3 (Saxon, 1987). The work carried out and reported through the CAN study (1989) focussed on the early primary years. The findings from this study need to be disseminated to teachers and schools.

Professional groups either nationally such as the Australian Association of Mathematics Teachers (AAMT) or state based such as the Mathematical Association of New South Wales (MANSW) could investigate ways in which they could assist schools with structured workshop material and dissemination of research findings.

PERCEIVED PARENT VIEWS

It was apparent that teachers were aware of the parent concern over the student use of calculators as many teachers believed that parents would not support the student use of calculators. Such a view was found in Blane and Willis' study (1986) which reported that at the upper primary level "parents were perceived to be strongly against" (p.8) the use of calculators. There was also evidence that teachers were uncertain at the present time of how parents viewed the student use of calculators in the primary years. Such a degree of uncertainty is another area in need of investigation perhaps focussed on 'Parent views on the student use of calculators in primary mathematics classes'.

Calculators are here to stay and there was strong support from teachers in this study for their use by students in the primary years. Though presently, a minority of teachers had their students using calculators. The reasons teachers gave for not having their students using calculators need to be addressed by educational authorities if they are to be introduced effectively into the primary mathematics curriculum.

CONCLUSIONS

This study investigated teacher attitudes towards the student use of calculators in primary mathematics classes. The findings from this study have been generalised to the following conclusions.

1. Calculators were not in general use in the primary grades.

2. There was not the same support for calculators being used in computational work as there was for it as a tool for problem solving and to reinforce estimation skills.

3. Teachers were not using calculators primarily because there were none available and/or there was no school policy on their use.

4. There were a number of teachers who did not see calculators as being relevant for their class.

5. There was strong support amongst teachers for the use of calculators in primary mathematics classes.

6. Teachers did not see that the use of calculators would reduce individual differences.

7. Teachers were concerned that calculators reduced the need for children to learn the basic facts and that they would not learn pencil and paper computational skills.

8. There were many schools that did not have a stated calculator policy.

9. There was a difference in stated school policies as to the student use of calculators. Some stated all students Kindergarten to Year 6 should have access while others stated only students from Year 4.

10. Educational authorities had not effectively communicated to schools their policies on the student use of calculators in primary mathematics classes.

11. There was not general support among teachers that calculators should be used from Kindergarten.

12. The main influence on teachers in forming their attitudes was their own opinion from teaching students.

13. School based professional development courses on the use of calculators in their primary mathematics classes had not been offered to teachers.

14. The vast majority of teachers were willing to attend school based professional development courses. Of these teachers, most would attend courses only within school time.

15. Most teachers were either unsure of parents' attitudes towards the student use of calculators or felt that parents would not support their use.

RECOMMENDATIONS

Based on the findings that have emerged from this study and the conclusions that have been drawn, the following recommendations are made.

A. Recommendations for Curriculum Change

1. Educational authorities must communicate more effectively syllabus directions for student use of calculators in primary mathematics classes. There should be a statement issued to schools by the relevant educational authorities clarifying the role of calculators in primary mathematics classes.

2. Educational authorities need to take on the responsibility or liaise with professional bodies to update schools and teachers with emerging research findings. This could be done in the form of a research findings bulletin published twice a year outlining the possible implications for the primary mathematics syllabus. This could be a joint publication from relevant educational authorities and professional bodies such as the NSW Board of Studies, the Mathematical Association of New South Wales and the Mathematics Education Research Group of Australasia.

3. Calculators should be made available for student use in primary mathematics classes. Schools could be supplied with calculators, schools could purchase calculators or students could purchase calculators as part of their school requirements.

4. Schools need to develop a policy on the student use of calculators. This policy needs to be developed through discussion between teachers and parents.

B. Recommendations for Professional Development Courses

Teachers could be offered structured school based professional development courses on the use of calculators in primary mathematics classes. These courses would need to be offered within school time for all teachers to attend. Such courses would be part of a school's staff development plan.

These could be structured courses involving discussion sessions, source material and classroom research tasks for teachers and students. The teacher/student observations from the research tasks could form part of each discussion session. Outcomes from such a course could be disseminated to other schools within a region across the state or nationally.

C. Recommendations for Future Research

1. A research study be set up across Australia or within a state to investigate the student use of calculators in primary mathematics classes, initially, over a six to twelve month period. This study to comprise a range of schools and classes from Kindergarten to Year 6. The study to focus on calculators being available to all students as part of the mathematics class with the teacher/student use of calculators being recorded and the findings reported.

   The research needs to be school based with calculators being available to students at all times. The study would be directed at gauging the usefulness and relevance of calculators to various grades. Student access to the calculators would be complemented with specific calculator lessons as the students or teachers identify a need. Teachers and students would be asked to record their work with the calculators, the results of which will be discussed amongst other teachers in school based workshops with research material or commentaries on the issue being available to supplement the teachers' classroom experiences.

2. The findings of such a study to be incorporated into the development of state and national syllabus statements leading to a calculator integrated mathematics curriculum (K-12).

3. This study suggested that there was uncertainty among teachers as to parent attitudes towards calculator use. In order to gauge community reaction a study be set up to investigate community attitude towards the student use of calculators in primary mathematics classes. Such a study would be school based and be set up with discussion workshop sessions being given for parents. These sessions could include semi-structured interviews being given before and after the workshops to gauge parent attitudes towards the use of calculators and to gauge any change to parent attitudes that may come about as a result of their participation in the workshops.

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TEACHERS MAKING MEANING IN MATHS OR: WHAT DOES IT MEAN TO LEARN MATHS?

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The 15 year old boy next door is "no good" at maths. He failed. He got 6% in the exam after sitting there, for an hour a day, five days a week, 40 weeks in the year, for 4 years. He failed maths. Or, maths failed him. We know about those failures. We even expect them. But I also failed maths. After many years of apparent success and increasing abstraction, I admitted defeat. I didn't know how to ask questions, and how can you do a PhD without questions? Or did maths fail me too? What kind of discipline is it that produces, amongst its best, incurious students?

So, like many other teachers of mathematics, I am concerned about the large numbers of people who fail/do not achieve at / do not like mathematics. (Behind this question is another important question: does it matter? if so, how, and to whom?) As a teacher of teachers of maths, I have become more and more interested in what baggage intending teachers bring with them: what are their assumptions about maths, maths learning and maths teaching? In teaching intending teachers over the last four years, in two different institutions, I made an examination of these assumptions a focal point of the courses, by getting the students to keep a journal and to carry out a case-study. It is the case study that I shall discuss here, although the journal also provided valuable complementary insights.

For the case-study I have drawn on work done by Deborah Ball (1988) in the Department of Teacher Education at Michigan State University. She argues that if, as a constructivist perspective claims, learning is a product of the interaction between what is taught and what is brought to the learning situation, then this view could enrich our understanding not only of children's learning but of the learning of teachers themselves. She describes a project that she developed - for an introductory elementary teacher education course which was "designed to surface and challenge what entering elementary teacher candidates know about mathematics and how it is taught and learned." I see the case-study working on two levels: as a small but powerful intervention for the intending teachers, and as a source of valuable insights for those of us involved in teaching teachers. The more general insights - about maths, about learning - are not particularly new. It is more the consistency of misconceptions which underlie them and the struggles to change that intrigue me.

THE CASE-STUDY

Ball chose permutations as the subject matter of her project, on the basis that many of her students would not have studied it formally, and of those who had, most would be likely to have only a procedural understanding: they would know how to use the formulae. Unlike Ball's project, this case-study used subject-matter that all students had at some time studied formally: a problem about area and perimeter. The material used evolved out of work done by Vicky Webber and myself, with a variety of adult mathematics classes over a number of years. We have found the solving of this particular problem to be counter-intuitive for most students and particularly powerful in provoking heated arguments and bringing to the surface assumptions about mathematics and how it is learnt. By emphasising reflection and distancing the need for (at least mathematical) success, the case study allowed the students to re-examine both the actual mathematics and the way we learn and teach it. Following Ball's basic framework, but with some variations, the case-study was constructed in three phases: learning maths; watching learning and teaching; and teaching. For the report the students, like the students in Ball's study, were not given any references about the mathematical content of the exercise, and were encouraged instead to read, if anything, more broadly in the area of learning. In particular they were given four references to look at: Eleanor Duckworth's *Teaching as research* (1987), David Hawkins' *I, Thou and It* (1984), John Holt's *How children fail* (1964), and Vivian Paley's *Wally's Stories* (1981). The students' notes said:

CASE STUDY OF LEARNING AND TEACHING MATHS

> In tutorials, you will participate in a case study in which you yourself will first **learn** about aspects of area. A relevant video will be shown to give you a chance of **observing** children being taught related concepts. We then ask you to go home and **teach** these concepts to someone else - child or adult- and finally to **write** a report on the whole case study.
>
> In the report we would like you to include your experiences both as a learner and a teacher, and draw some tentative conclusions about mathematics, about the teaching and learning of mathematics and about learning to teach mathematics. The purpose of the study is **not** for you to be either a model learner or a model teacher at this point, but to reflect on how you and others learn maths, how you feel about it and why, and what implications this has for teaching maths.

phase one - learning: The actual mathematical focus of the study was:

> A question: if you have a fixed length of fencing with which to make a holding pen for some sheep, will the sheep have the same total amount of grass inside the pen, no matter what shape you make it?

The students were given a moment or two, by themselves, to think about this question and to write down a quick spontaneous answer. This was followed by a series of related statements (eg if I increase the grass area of the pen, then I'll need more fencing as well; two pens with the same grassy area will need the same amount of fencing.....) which they were asked to read, noting down as they went an immediate true or false response to each statement. They were then asked to organise themselves into groups of three or four to discuss the statements and to try to come to a group consensus about the answers. If some members of the group did not understand, it was the responsibility of the others to convince them, using whatever means they found mutually acceptable. At least half an hour was spent in these small discussion groups, before the tutor asked the groups to share any findings or questions with the larger group. The role of the tutor was generally to extend or refocus questions rather than to answer them.

phase two - watching learning and teaching: The students were shown an Open University (UK) video, from the series Thinking Mathematically, in which primary children first come to grips with concepts of area at school. An experienced teacher is shown with several different age-groups of children, involved in activities from covering irregular pieces of paper with potato prints to finding a shortcut -a formula- for finding the area of rectangles.

phase three - teaching: Students tried to teach a person of their choice, from a wide range of ages, achievements and interests, what they themselves had learnt about area and perimeter.

LOOKING AT THE REACTIONS

The initial response

> I was quite shocked to find out I was wrong when we began to experiment...When I saw the proof in the numbers I could see I was definitely wrong, but I couldn't see why....

> The first reaction was an immediate "yes". After a few seconds of consideration doubt crept in....Subsequently a high degree of doubt formed.

In the first group, out of 200 people, about 80% - students and their students, old or young, men or women, studying university maths or not - got the answer wrong intuitively, ie almost everybody said, yes, if you've got a fixed amount of fencing, then you'll have a fixed amount of grass inside. Most of the few who said no, couldn't explain why, and only a few of the ones who could explain were able to do so without using a formula. Fewer than ten could really *see* why not. (Amongst those few were the three 5 and 6 year olds that students taught - are we teaching incompetence? creating blindness?) After discussion, the writing of the report gave the students a chance to examine their own reactions in some depth, and to reflect on their own underlying assumptions about maths and how it is learnt.

Similar results were obtained with students in other groups, in following years, from a different institution. The first group were the 108 students taking the course *Learning and teaching mathematics in schools* at Macquarie University in 1989, and included prospective secondary teachers, with a majority of intending primary teachers. The other three groups -approximately 50 people each year- were students enrolled in the Graduate Diploma in Adult Basic Education at the University of Technology, many of whom were already skilled teachers, predominantly in the literacy rather than numeracy area.

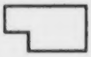
Why was 'intuition' so overwhelmingly inadequate, or even wrong? What can we learn from the intriguing consistency of the responses? What did the students learn from the whole experience?

IDEAS ABOUT AREA

Trying to come to grips with why their "intuitive" responses had been so sure and so wrong, three students teased out their ideas of area:

> [One of my assumptions] was that area is length x breadth, or whichever formula was appropriate for the shape.....The concept of area has been impressed on me in terms of the equations that give it value. My understanding of area wasn't developed from within me, I didn't own it. Rather I had adopted the formulas by rote.

> Area was a formula which was worked out by multiplying the length by the width. Shape appeared to have little to do with it. I believed if you adjusted the perimeter you must adjust the area. This seemed perfectly logical to me.
>
> The thought never occurred to me that 'area' meant space inside a region- when we were finding area I never thought of it in terms of finding how much space was inside a certain region
>
> for example: finding that! ----->



> Instead I perceived that the concept of area involved a certain formula which was applied to a certain figure and ascertained that shape's 'area'...I remember thinking when watching the film -"why are they using potato shapes to display area to children?" It was not until our next tutorial that the point was made in discussion that the potato shapes were used to show children that the concept of 'area' involved space inside a certain region!

All three students point to the divorce they experienced between area as a concept with meaning, and area as a formula to be applied. For one of them, area *is* the formula; the fourth student was surprised when she found that the concept in fact could have meaning. Her experience of the divorce is reflected in the language she used: her first sentence, pointing to the realisation of meaning, flows naturally; the second, talking about rules, is awkward and formal. These students were by no means atypical, and were not confined to people who had dropped maths early, as this comment from a student studying university maths shows:

> At school we had done a lot of maxima and minima problems involving area and perimeter, and if you were building rectangles the maximum area always turned out to be a square. I knew that the same perimeter did not always give the same area but I had never thought about why. I had never really worried about why the maximum area turned out to be a square - I had just thought it was a coincidence. With this activity however, I had to think about what area and perimeter actually were; my brain began to work overtime on what I had previously considered to be very simple. .....I had forgotten, if I had ever realized at all, that area was the amount of space that a shape covered and that perimeter was the distance around the shape.

Confronted time and time again with comments like these, I am astonished, not that so many people fail maths, but that so many pass it. With an understanding of area so fragile, so purely dependent on rule, so unanchored in the everyday world, the achievement of passing exams, of remembering which rule to apply to which set of words, is truly impressive. One of the very few students who gave a confident " no" answer to the main question firmly located his understanding within a practical context:

> At about the same time my father asked me to design a new chicken pen. On presenting my elaborate rectangular design, he suggested a more square-shaped design, explaining that the area would be no different, but that the materials and man-hours would be less as the perimeter would be smaller.....Coming from the country I always conjured in my head a picture of a grassed area surrounded by a fence... Thus I had no trouble in reaching the conclusion that the longer and thinner a paddock was, the less area it enclosed....

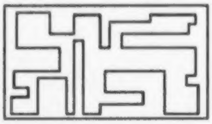
SOME MISLEADING ASSUMPTIONS

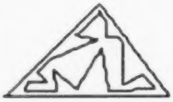
If we can work out why the responses of most of the intending teachers were so very uniform, and so very wrong, then we can begin to work what it is about their experience of maths and maths learning and teaching that has failed them. One common interpretation of the situation was to see the sheep pens only as rectangles; almost invariably this was done unconsciously rather than as an organising strategy:

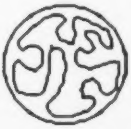
> ...I had made an interesting assumption: I thought about the problem in terms of rectangles. This was also characteristic of the person I taught, and so raises some interesting questions. In my experience, most questions dealing with fenced areas involved rectangular or square fields...

Why did we as a group restrict ourselves to rectangles? Is it because the learned formulae A=L x B and P = 2(L + B) are so deeply entrenched that it is hard to think of area in any other way? ....

I am [now] confident you can always find a larger perimeter enclosing a smaller area if you are not bound to maintain the shape of the figure eg



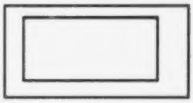




Another appeal was to something the students saw as "logic", which seemed to be based on an assumption of a direct, linear relationship between the two measures, perhaps deriving from their common dependence on 'L' and 'B', and the fact that they are usually taught together:

> I had previously believed that area and perimeter were in some [direct] way interrelated, after all we had learnt about them simultaneously and I can never remember having been told that they were not related...
>
> One of the first things I learnt... is the importance of not only teaching concepts but of relating them to each other... For instance R and I both made assumptions of the relationship between area and perimeter which show that we both knew about the two separate concepts but never really needed to relate the two (even after doing the HSC!)
>
> I found that it was easy to get the wrong impression at first glance. It seemed logical that if the perimeter of the pen was increased then the area would increase as well and if we just think of rectangles then this is true.

Rectangles here are assumed to be all (mathematically) similar; this perception of rectangles being " the same shape" was common. Something like this diagram seems to be how the relationship is visualised, but never illustrated:



So, area *was* the formula; it was, somehow, to do with rectangles; perimeter and area were learnt side by side, but not together; and logic had little to do with meaning in the everyday world.

IDEAS ABOUT MATHS MORE GENERALLY

The perceived chasm between the formal language of maths and more everyday meanings was not peculiar to the concept of area but was how students saw "doing maths" more generally:

> When I first heard that I was required to learn some aspects of area, I felt reasonably confident and relaxed. From my experience, area problems were always solved by applying the relevant formula. As I remembered such formulae, I felt prepared to receive what I expected to be a work-sheet covered in shapes and measurements. When I received the actual sheet containing not a single number or shape, my feelings changed to uncertainty and confusion. Whilst clearly mathematical in nature, the questions asked could not be solved by using one of my answer-producing "recipes".... On reflecting on my reaction I have become aware of the underlying assumption I held, that maths was only numbers, rules and formulae.

There were variations, but the theme was inescapable:

> The strangest thing happened when I first told S that we would be learning about area. Straight away he started muttering under his breath all the relevant formulas he could remember: "area of a rectangle equals length times breadth, area of a triangle equals half base times perpendicular height" etc. He finished by saying a confident "right" as he had proved to himself that he knew it ...more importantly, this showed quite clearly that mathematics to S was merely a collection of formulas and rules.

> People are used to following set rules in maths, to the point where when they are given freedom to experiment they become suspicious and insecure. ..The formula had been taken away and considering I have come to rely solely on it to explain area, I felt marooned without it.

The image of maths as "a subject that is based on correctness", as another student put it - also came through strongly, though again not unexpectedly, in other reports:

> For a subject that I had always felt contained only RIGHTS/WRONGS and reliable formulae, mathematics was proving very ...um...interesting.... challenging.

It had obviously not been part of M's maths experience to feel safe with guessing an answer before it was worked out. It seemed high on his priorities not to get an answer wrong.

> I was too anxious to find the correct answer, to the point of being totally unconvinced about any of my own attempts.

One student described her attitude to maths as a belief in its "givenness", its formulas, its procedures, its expected answers, - woven together, for her, with some pleasure:

> Maths has been a belief in a set of formulae. When you apply the formulae correctly to the question given you arrive at the "right" answer. This may sound completely unappealing, but in fact I enjoyed the game, its twists and turns and arriving at the correct answer. Maths was a static process. It was all written down in the book. You learnt it, you did it. I don't remember ever applying it to any part of my life....It also required another belief - belief in the authority of the teacher to be right.

Nowhere did students express any feeling that maths, like any other branch of knowledge, had been constructed by humans in response to human concerns and questions, that it had been something they themselves could make up and contribute to. For most, bridging the chasm between maths and their everyday lives was inconceivable.

PERCEPTIONS AND MEMORIES OF LEARNING MATHS

Ideas about maths itself and ideas about how it was learnt, were tightly woven together: maths was formulas, learning maths was learning formulas, in an environment separate from "life":

The way it always was - teacher out the front, demonstrating some process, working a couple of examples on the board then having children work through examples in their books ...

... maths was learnt in set steps with set outcomes and creative thinking wasn't a concern..

> No wonder I couldn't remember any maths. It had always consisted of learning formulae, using the correct process and coming up with "right" answers. It had never been relevant to my life. I'd never applied maths to everyday experiences. Not that the "mind games" of doing maths hadn't been enjoyable. They had. I had a good memory and I did well. But if I missed the sequence of the maths formulae, or, as in deductive geometry, you needed to rely on creative problem solving, I was lost.

Talking, discussion, was not part of doing this maths, either for the teacher....

> "Get it clear in your mind, and have proof to show others, before you open your mouth," a teacher of mine used to say.

or for the pupils:

M. felt that often he knew how to do the maths but not how to use it, that a problem could be right before him that he knew the maths to solve if only the question wasn't disguised in words. If maths was taught the other way round, from the "I have a problem to solve" angle first, with the theory resulting from the investigation, maths might seem more relevant to students.

Disguise... the words seemed designed to catch you out....

> My mind said, " Well, there has to be a catch - it'll be what I don't expect to happen, otherwise we wouldn't be doing it.... so ...."
>
> I carried out this case study with my mother....she told me the first two statements were true, but changed her mind after drawing some diagrams. Then she said something which amazed me. ...She said, "Oh well, the rest of the questions seem true, but they must be false as well."

Learning and "doing" maths meant learning and applying formulas, working alone, looking out for tricks, with no expectation of meaning or connections with the everyday world.

EFFECTS OF THE CASE-STUDY ON THESE VIEWS... OF MATHS, AND LEARNING

What was particularly impressive, and more hopeful, was the eagerness with which students worked, once they started to find meaning in what they were doing. Almost all the participants in the case-study became seriously engaged both with the mathematical question, and, through their attempted explanations of that question, with questions and insights about mathematics and their own learning. Some students had difficulty in owning a self in relation to mathematics, and in asking their own questions. This was reflected in an initial reluctance of some to use the word "I" in their reports and in their realization that they had hardly ever talked about their mathematical ideas. As the validity of their pictures of mathematics and learning were challenged by the experiences of the case-study, students recorded changing feelings and an understanding - even excitement - about the potential for change. Working in small groups, with discussion and concrete materials, raised the possibility of different, more interactive ways of learning, in which the learners could play some part in constructing mathematical meaning:

> Having studied 3-unit mathematics at school and now doing maths at University, I expected to find the questions in this case study easy. Instead, I found them challenging. This is because investigating properties of area is something I had never been asked to do before.
>
> I made an amazing discovery about maths during this case study ...I was finding that the subject matter was actually becoming relevant to me. I saw that maths could be effectively transformed to the level of the ordinary person in the classroom and was no longer a sacred subject which only those with a degree in this area could discover or alter.

One aspect of the relevance was a growing perception of the way in which maths grows out of, and feeds into, the everyday world:

> [My student] to all intents and purpose would be considered a failure at maths. He was unsuccessful at maths in the HSC... [but] he had no problem with these questions. He answered them both quickly and correctly. The question that had caused me so much confusion was not at all confusing to him because he has spent time on farms and was able to give me an example as to where it would actually occur.... and I was supposed to be the "clever" one at maths!

Some reports differentiated between merely doing maths, and understanding it:

> If I am serious about becoming a maths teacher I am starting to realize that I am going to need to make sure that I understand what I am doing rather than simply doing it, which I feel I have been doing for years. I would like to break the cycle of students feeling that mathematics is just something you do -not something you understand. However first I am going to have to make sure that it is something that I understand.
>
> I could see with my own eyes that indeed the area did change with equal perimeters but differently shaped areas. The experience for me was like the proverbial light bulb flashing on, but also one of frustration and anger. Why hadn't I been shown WHY before?

Students became aware that they became engaged in the mathematical question not so much when they made their initial response but when they tried to make sense of that response, either for themselves, or for others, particularly when responses in the group differed:

> This may sound very strange but I think it was very fortunate that the answers obtained by members of our group were not the same. This meant we had to work through problems, over and over until we were all satisfied with the derived conclusion. It was not enough just to state your conclusion was right: I found I had to listen, query, agree and most importantly I had to attempt to explain my own understanding - not an easy task.

As we formed groups and discussed our intuitive answers to the questions, our cluster of women realized with some consternation that what had seemed previously reliable in our heads, clashed with the understanding of others and would therefore require 'real' working out, or 'proving'.

A central key in the development of meaning was, many students discovered, interaction, either with materials, or with other people. Talk began to be seen, not as cheating, but as a powerful way of constructing meaning:

> My main AMAZING discovery in this tutorial clearly demonstrated the worth of ...the recommendation that students verbalise their thoughts and strategies. As I endeavoured to explain to my group (using fairly inept hand movements) how I imagined the fencing to be, using my own prior experiences with a guinea-pig holding, suddenly like a light-bulb, I was 'turned on' -I UNDERSTOOD!
>
> As a student majoring in mathematics I feel that I am learning maths every day in my lectures, where I am told that some mathematician found a theorem, this is what it is, and this is how you use it. When however I sat down in my [case study] tutorial where we discussed our ideas and used examples to verify our results, you can imagine my confusion. I felt that I was and had been deceived in all my school and university maths education. I began to cast my mind back over all my other schooling experiences such as science classes where we did experiments and English where we discussed the themes of novels. My entire schooling appeared to occur in a sharing, interactive way except for maths. In these classes learning seemed to be a

> passive activity where you had to just absorb what you had been told. I was amazed when I found out in tutorial that this didn't have to be so. Learning can be defined as "the modification of behaviour through interaction with the environment" (Macquarie Dictionary). Mathematics, I realized from the case study, doesn't have to be the exception to this definition.

CONCLUSION

By presenting the students with a qualitative, relational question, involving them in working in small, relatively unthreatening groups, to convince each other of the reasonableness of their responses, and then requiring them to reflect on their reactions throughout, the case-study seems to have given the students an opportunity to re-view their ideas about maths and learning maths. The strong overall direction of change could be seen as a clear shift from the first to the second of Frankenstein's (1989) three ideologies of learning mathematics (adapted from Giroux's categorisation of ideologies underlying various approaches to native-language literacy [1982]). The first, instrumental ideology, sees knowledge as external, objective and neutral with instruction focusing on mastery of atomised content. The second, interactional ideology, sees knowledge as a human construction, with teaching focusing on meaning, being student paced, and emphasising problem-solving, process, and cognitive dissonance.

The separated formula-learning of the instrumental paradigm, so consistently described by the students, is sometimes referred to as abstraction, but we need to distinguish between the free-floating products of abstraction, and the process of abstracting. We abstract *from* some material reality, and in the impoverished vision of maths that we as a generation of teachers have given these students, there is no material reality to begin with: students simply expect, and are expected, to work with the already abstracted products. The more difficult, and powerful, achievement is that of actually abstracting. In the shift to interactional learning, this process element is beginning to emerge.

And surely the real power of abstraction itself lies in the possibility of its reconnection to the material world. It seems that the more abstract the subject the more status it has in our education system. Latin, physics, maths - study these for the HSC, and your marks soar. The separation of logic from meaning, however, encourages alienation from the everyday world and a terrible irresponsibility. Do the many mathematicians working in jobs funded by the military make any connection? The intending teachers in my study, a good proportion just out of high school, still saw maths in this passive, dangerous way. Could we not see the weaving of the abstract with the concrete as an even higher thinking? As Hegel (1975) says:

> [the analytical approach] labours under a delusion if it supposes that while analysing the objects, it leaves them as they were: it really transforms the concrete into an abstract. And as a consequence of this change the living thing is killed: life can only exist in the concrete and one. Not that we can do without this division, if it is our intention to comprehend....The error lies in forgetting that this is only one half of the process, and that the main point is the reunion of what has been parted.

In an article comparing the maths that has evolved in a hunter-gatherer society with that of our western, industrial society Denny (1986) argues, that "the isolation and control of key variables is needed for the high degree of alteration of the environment achieved by industrial society"; and, according to the mathematician Hyman Levy, "Pure mathematics is the method of isolation raised to a fine art." Perhaps what we mathematics educators should be valuing, however, is not only this skill of isolation, but also the skill involved in relating that abstract knowledge back to concrete experience, the skill of extracting from and putting back into context. Denny argues that, for the hunter, "inclusive knowledge of the whole pattern of natural processes will be imperative." We too, in our industrialised societies, with their proliferating global problems, increasingly complex and inter-related, may find that, as well as the fine art of isolation, the inclusive knowledge of inter-relating patterns will be essential to survival.

Thus, for me, the challenge is to work towards Frankenstein's third, critical ideology of mathematics learning, which includes a recognition of political conflict and power differences. It argues that learners actively generate their own knowledge from actual situations and a minimum of givens, and emphasises the ways in which maths can empower students to understand and act upon the world. This case-study has hardly begun to address the challenge of this critical framework for learning. Though quite powerful, the study is a very small intervention. However, without such beginnings, it is not likely that our future teachers will be any more concerned with connecting the abstract to the material world, with the accessibility of mathematics, with its potential for engendering critical awareness - with making meaning in mathematics - than our past ones. I would like to give the last word in this paper to one of those future teachers:

> Immediately the question was placed in front of me my heart started to beat faster, my eyes raced across the words, and my mind immediately went searching for formulas, rules and mathematical jargon .. I came up against a brick wall, I could not think, nothing was registering, no immediate answer came to mind. ...I re-read the question...I tried to visualise...My thoughts were slowly moving from viewing area as length x breadth to thinking of area in terms of space inside a shape...Alas! alas! I find that I am what John Holt calls an answer-centred child "who sees a problem as a kind of announcement that far off in mysterious Answerland there is an answer that they are supposed to go and find." .....I found that discussion, taking time, using concrete materials and getting familiar with the problem rather than rushing into it made the learning process much easier and less frustrating....you need time to think and experiment. .... Although I truly love maths personally, I was scared by the thought that I would have to teach it . I was frightened that a pupil would say, "But Miss, I don't understand why", and I would have to say something like, "it just is - that's the rule", because I didn't fully understand the concept myself - I just knew the operations to follow and the rules to follow. I now realise that maths can be understood.....and I see maths as an integral part of everyday life not just some executive subject belonging to the geniuses of society.

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KNOWING AND MEANING IN MATHEMATICS CLASSROOMS: PERSPECTIVES DRAWN FROM WITTGENSTEIN'S PHILOSOPHY OF LANGUAGE

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In discussing Wittgenstein's philosophy of language it is usual to distinguish between his early and late work. His early work, the *Tractatus Logico-Philosphicus* (1921), is famous for its *picture theory* of language. Propositions, he argued, offer us pictures of states of affairs which maintain in reality. They function ostensively, that is by pointing out the facts to which they refer. Later, however, he repudiated this representational view of language, stating

> In the *Tractatus* I was confused about logical analysis and ostensive definition. I thought at that time that there is a 'connection between language and reality'. (Wittgenstein, 1967b)

Wittgenstein's later work, in contrast, focussed on the way language functions as an activity or *language-game*, in which meaning, intention, and purpose cannot be separated out "by a simple formula" from the question of meaning.

This shift in Wittgenstein's view of language from representation to activity, from "purest crystal"(1953/1991, §97) to "blurred edges"(§71) has important lessons, I think, for research in mathematics education

> We have got on to slippery ice where there is no friction and so in a certain sense the conditions are ideal, but also, just because of that, we are unable to walk. We want to walk: so we need *friction*. Back to the rough ground! (§107)

If the objects of theory become too idealised, then ultimately the theory will cease to facilitate practice. Thus, in later Wittgenstein writings great stress is placed on laying bare how language functions, in its "roughness" and diversity.

This paper attempts to apply ideas such as these to mathematics education (*cf* Bloor, 1976, 1983; Confrey, 1981; Hamlyn, 1989; Kanes, 1991a, 1991b; Watson, 1988, 1989). In doing so, the discussion introduces a model (set out immediately below) which attempts to trace out the key relationships between critical elements in any theory of mathematics education: in the first instance these are taken to be variables in the domains of practice (mathematics, learning and teaching) and theory (epistemology, learning theory, pedagogy). Following an analysis of these elements and their interrelationships a third domain of *interior practices*, based on concepts drawn from Wittgenstein's philosophy of language, will be identified and discussed. Note that availability of space has lead to a substantial condensation of the exposition and analysis of the ideas presented here.

SCHOOL MATHEMATICS AND EPISTEMOLOGY

One source of interest in epistemological questions can be related to the collapse of the *process-product* model for conceptualising student/teacher interactions (Putnam, Lampert & Peterson, 1990). In this now discredited model, pedagogy was viewed in purely functional terms: process variables such as teacher proximity, classroom climate *etc* were thought to directly control product variables as measured by student performance in prescribed tasks, affective orientation and so on. In practice, this approach has not proved effective in leading to high quality teacher/learner interactions. Moreover, in criticising this model authors (Romberg and Carpenter, 1986) have questioned whether mathematical knowledge can be validly fragmented in the way commonly practised in research design, instructional design, assessment of the learner's conceptual development and other analytic procedures consistent with this approach. The suggestion is that such epistemological practices are inappropriate for the needs of education in mathematics.

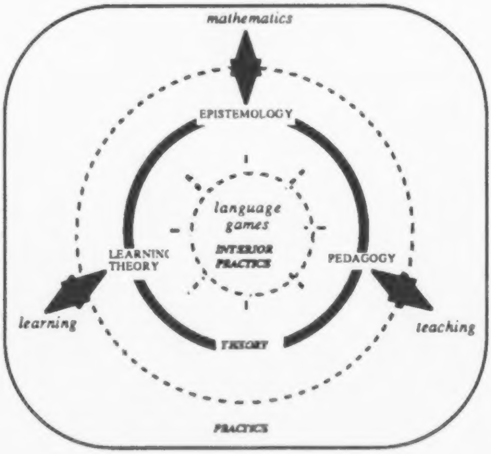

Figure 1: Wittgensteinian model for learning, teaching and epistemology in mathematics education practices

Another source of interest in epistemological questions has been a growing scepticism towards essentialist notions of mathematical knowledge. Relativist views, in which mathematical knowledge is held to be thoroughly social in character (Bloor, 1976, 1983) and dependent on social (Walkerdine, 1988) and cultural (d'Ambrosio, 1985; Bishop, 1988) contexts, are receiving more favoured treatment than in former times. Mathematical knowledge, so it is claimed, is neither culturally neutral nor independent of history. In North America these views have been developed by the so-called radical constructivists (von Glasersfeld, 1984): those who believe the world is a mental construction residing in the knower's mind. Recently, radical constructivist authors such as Paul Cobb (1992) have argued that the neo-pragmatist philosophy of Hilary Putnam and Richard Rorty provides theoretical support for placing mathematics within the sphere of social knowledge.

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THEORY AND PRACTICE IN TEACHING AND LEARNING MATHEMATICS

In the literature, there are a number of general approaches to this relationship. An outline of these, including a brief discussion of their relationship to elements of the model, follows.

Reflective practice (Schon, 1983). On this view expert practitioners are skilled in the task of incorporating technical-rationality (theoretical knowledge) within knowing-in-action (expert practice). Schon agrees that the development of the later is enhanced by the process he calls reflection-in-action. This form of knowledge is closely allied to knowledge *of* (see above).

Action research (Carr and Kemmis, 1986). As part of the action research paradigm the teacher is conceptualised as practitioner (both creator of the (lesson)plan and executor of the plan) and researcher (observer and analyser), in a rolling sequence of these roles. Constructs derived from cognitive theory are consistent with this conception. For instance, in Glaser's view (1984) the teacher constructs a pedagogic theory (planning phase) which is similar to, but tellingly different from the content based theory structures held by the students (Carpenter and Peterson, 1988). Action follows (teaching), and this finds students testing, evaluating, and modifying their current theories thereby resolving conflict between these and the teacher's. Observations based on social interactions, including instruments of assessment and evaluation, lead into the fourth phase: teacher analysis and construction of new pedagogic theory depending on teaching goals and the students' conceptual development. The action research approach differs from others in this review in that it is part of the programme to openly treat theoretical constructs explicitly in their propositional form.

Cognitive apprenticeship (Collins, Brown and Newman, 1990). This view is complementary to reflective practice, but contrasts with other approaches in the close proximity established between the teacher and the learner. Emphasis is placed on procedural knowledge. The teacher acts as an exemplary practitioner in the content domain, the learner's knowledge is encoded as site or context specific. Learning is situated in sites of domain practice, thus theory tends to remain implicit, non-propositional in form.

Teacher cognitions and the wisdom of practice. In this research a sharp distinction is made between teacher cognition and teacher actions. Cognitions, including beliefs (Leinhardt, 1990) have been studied using various taxonomies for teacher knowledge (see for example, Shulman, 1987) and include teachers' thoughts during lesson planning and teaching (Peterson, Fennema, Carpenter and Loef, 1989), and beliefs about students, classrooms, and learning. One result to emerge from this research is that, for practitioners, theory derived constructs sit side by side with what Shulman (1986) has called the "wisdom of practice". These distinctions recall the contrasts between Schon's term "technical-rationality" and "knowing-in-action".

Another result is that cognitions and actions are not related in straight forward ways (Clark and Peterson, 1986). This indicates that theoretical constructs are often not, even for the expert practitioner, related "ecologically" (Kagan, 1990) to actual teaching action. Support for Schon's notion of reflective practice in which, for the expert, technical-rationality is related to knowing-in-action, is not therefore confirmed by this body of research.

PARADOXES IN LEARNING THEORY AND PEDAGOGY IN MATHEMATICS

Paradoxes within learning theory and pedagogy and in the relationship between these two, provide a further source for investigating the elements of the model in Figure 1 and their interrelationship. For instance, constructivist learning theory (in which the learner is conceived as being actively engaged in the processes of modifying, transforming, and extending current knowledge structures) although widely accepted (Resnick, 1983; Bereiter, 1985) suffers from what Bereiter has referred to as the learner's paradox. The paradox is that

> if one tries to account for learning by means of mental actions carried out by the learner, then it is necessary to attribute to the learner a prior cognitive structure that is as advanced or complex as the one to be acquired. (p. 202)

Recall that Plato mentions a similar paradox in the Meno (80e): If the student had the knowledge, then there would be no need to seek it, but if the student lacked knowledge, then how would the student even know what to look for? Socrates concludes from this that students can *not* learn what they do not already know. The standard rebuttal of the paradox points to an apparent confusion about the meaning of words, for example, "having knowledge". In Kanes (1991b, 1992) the writer argues that, as with its classical echo, the learning paradox arises from all too narrow views about what constitutes mathematical knowledge. If this is correct, then the significance of the paradox is that it points once again to important weaknesses in the epistemological assumptions upon which cognitive theories of learning and teaching are usually based.

Cobb (1988, 1992) and Kanes (1991b) discuss a related problem, this time concerning the attempt to construct a pedagogy consistent with constructivist learning principles. For instance, talk of "instructional representations" to be used by the teacher in order to prompt the learner's construction of an "internal representation" consistent with the instructional target, is inadequate for constructivist purposes because it leaves unexplained how learners are expected to identify instructional targets within their own internal representations. Once again the underlying issue is epistemological, and the key questions relate to the limits for pedagogical purposes of representational knowledge.

Paradox has also been noted in pedagogy based on more general theoretical foundations. Brousseau (1988) for instance argued that teacher and student enter a *didactic contract* in which the teacher must ensure that the student has an effective means of acquiring knowledge, and in which the student must accept responsibility for learning, even though not being able to see or judge, beforehand, the implication of the choices offered by the teacher. Brousseau argued that the contract is driven into crisis and ultimately fails, for

> all that he [the teacher] undertakes in order to get the pupil to produce the expected patterns of behaviour tends to deprive the pupil of the conditions necessary to comprehend and learn the target notion: if the teacher says what he wants he cannot obtain it. (p.120)

Similarly, Steinbring (1989) observes that the teaching process of making all meanings *explicit*

> leads to the effect that by the total reduction of the new knowledge which is to be learned to knowledge already known, nothing really new can be learned. (p. 25)

Brown, Collins, and Duguid (1989) also emphasise this same point

> Whatever the domain, explication often lifts implicit and possibly even non-conceptual constraints out of the embedding world and tries to make them explicit or conceptual. These now take a place in our ontology and become something more to learn about rather than simply something useful in learning.

Brown *et al* argue that the new construct of situated cognition (sketched below) can afford frameworks for understanding the relationship between explicit knowledge and implicit understanding. Such a theory may provide a better articulation of the epistemological grounding of cognition.

WITTGENSTEIN'S PHILOSOPHY OF LANGUAGE AND PEDAGOGIC INTERACTIONS

For Wittgenstein, human activity is purposeful and thoroughly social in character. Underscoring this is his analysis of language in which the use of words and sentences always form part of an activity. Images of "tools" - language as a tool - and games are therefore recurrent in his later works. Wittgenstein is interested in recovering or demonstrating the multiplicity of meanings carried by ordinary words and sentences: words, on this view, as for poststructuralists, do not have a unitary meaning. Frequently, however, we fall in to the trap of universalising our language-game, failing to notice our grammatical entanglements. According to Wittgenstein,

> We remain unconscious of the prodigious diversity of all the everyday language-games because the clothing of our language makes everything alike. (p.224).

Thus practices of language lead to "conceptual confusion" in ordinary life, in the sciences (*eg* psychology and mathematics, p232), and in philosophy. Conundrums often arise when investigators become overhasty to see the similar in the diverse, and when in an effort to establish this element of similarity they destroy the local character or specificity of particular interactions. As an antidote to this, Wittgenstein suggests it is the task of philosophical analysis simply to make visible the troublesome language-games, those hidden from view by super-familiarity (*cf* §97) and habit. Thus

> Philosophy simply puts everything before us, and neither explains or deduces anything. -Since everything lies open to view there is nothing to explain. For what is hidden, for example, is of no interest to us.(§126)

Consequently, the dictum "Look and see" (§66) applies to Wittgenstein's philosophical method. "One cannot guess how a word functions. One has to *look at* its use." (§340, italics in the original).

Wittgenstein believes it is only in an understanding of the particular purpose in each human action that we are brought closer to understanding language, and through this conceptual knowledge. This contrasts, however, with the view that understanding is generated by working upwards through a hierarchy of ever increasing conceptual generalisations. Wittgenstein assumes we must study activity itself - not propositions or concepts, as in his earlier philosophy - and not activity in general, but the minutiae of specific activities.

Another assumption made is that the study of language and the formulation of language is important because knowledge *eg* mathematical knowledge, is determined by language. This view of mathematics, for instance, implies that it is rooted in social practice. Nevertheless, Wittgenstein maintains a distinction between *facts* which are objects of experience and *concepts* which are social constructs and are therefore the products of language (Wittgenstein, 1967a, p.66e; Wittgenstein, 1953/1991, §364). Metaphysical realism ("Platonism") is denied (as this valorises a notion of 'concept' anterior to activity), as is, for example, Hilbert's formalist program for the foundations of mathematics.

These assumptions combine to explain the significance in his work of the *language-game* construct.

> Here the term "language-game" is meant to bring into prominence the fact that the speaking of a language is part of an activity, or form of life.(§23)

In this same paragraph he provides a list of examples of language-games, or ways in which "symbols", "words", "sentences" are used, these exhibit the scope intended for this concept.

In order to illustrate these ideas, the pedagogic interaction between students and a teacher on the application of 'dummy variables' as indices in expressions involving complex algebraic manipulations will be analysed. The transcript analysed was taken from a video recording of the episode. The normal classroom teacher, Ms X, was not present during the interaction reported. Students are from a Year 11 class in a state run secondary school, located in the southern suburbs of Brisbane.

The transcript set out below is organised against a 'narrative' which, provisionally, divides the interaction into four sub-episodes or 'chapters'. Separate language-games are supposed to correspond with three of these. in the manner shown below. It may be helpful to indicate the purpose of each of these language-games:

- *Language-game of Reference*: Utterances replace (substitute or represent) former symbolic expressions with alternative expressions;
- *Language-game of Action*: Utterances deploy symbols, make them available for action, mathematical procedures;
- *Language-game of Form*: Utterances organise the juxtaposition of symbols and the linkage between terms.

TEACHER/STUDENT(S)	DESCRIPTIVE NARRATIVE
T;. Now the very first step here, where you've got $\arg(z_1/z_2)$, Ms X wants you to focus on z_1/z_2. Now the first thing that she did was to write that out in a trigonometric form, or a polar form. And she wrote on the top line, what did she write? Sarah: r_1 ... (inaudible) T: Outside of? Alice: Inside the brackets, I think it's $\cos q_1 + i \sin q_1$ T: Why did she say r_1 and q_1? Ss:(Several students at once) Because that's the modulus and argument for z_1!	**Chapter 1: Reference** Students and teacher, starting with the language-game of reference, substitute and represent one formulation of symbols with another. This activity is further emphasised by the students' tightly closed representational response to the question "Why did she say r_1 and q_1?"
T: Sorry, just explain? Sorry, who's talking? Alan: Because, well we've got subscript '1', for z_1, we sort of use (*sic*) the same subscript, probably.	**Chapter 2: Action** The teacher abruptly breaks into a new language-game of action. Actions (explaining, talking) are immediately emphasised. A student responds by spontaneously focussing on how symbols are deployed ("use"), and what facilitates this deployment *ie* the subscript.
T: Would it have mattered what subscript? If (*sic*) she'd written '2', would that have been wrong? (*sic*) If she had written r_2 would that have been wrong?	**Chapter 3: Form I** The teacher acknowledges student contribution, and henceforth focusses on subscripts. However, emphasis now shifts to an analysis of logical structure.("would that have been wrong?"). Formal structure, linkage between terms (*eg* uses of syllogism) are emphasised. Implications for the overall structure or form of certain symbol deployments is suggested for consideration.
Alice: Only if (*sic*) she had have, it would have been confusing, because you've got z_1 and z_2, and then (*sic*) you've got, it would be easier to have r_1 and q_1 then (*sic*) they've got, it makes a link (*sic*) there, so you have, you say that it's with the same, the same problem (*sic*). Sarah: And also, it linked up (*sic*) to what we did yesterday, because we used z_1 and q_1 and r_1 when we were doing the multiplication as well, so it just tied in (*sic*) with what we did yesterday.	**Chapter 4: Form II** Students respond by using the language of logical structure (notably syllogism), linkage ("link", "linked"), juxtaposition ("tied in") and identity ("the same, the same problem").

Figure 2: Transcripts

'INTERIOR PRACTICES' WITHIN THE MODEL FOR LEARNING THEORY, PEDAGOGY AND EPISTEMOLOGY IN MATHEMATICS EDUCATION

The writer now turns to consider what implications this analysis has for the model of research and teaching set out earlier in this paper. In this model learning theories, pedagogy, and epistemology are interrelated theoretical domains. Exterior to these, but also in mutual dependence, are the variables of learning/teaching, and school mathematics within the domain of practice. Suggested here is an extension of this model which places language-games as an area of practice embedded within, and therefore interior to theoretical practice. Language-games are therefore thought of as *interior practices*. In summary then, this model consists of three domains distinguished in the following way:

Exterior practices. The domain of practice. What we understand about what we are doing when we teach, learn or perform in a mathematical way.

Theory. In contrast, this domain is about what we understand about our practice *as* teaching, learning or mathematics.

Interior practice. This central domain focusses on what we are *actually* doing when engaged in the activities of exterior practice or theory.

Language games have a generative function. In their activity and practice, larger meanings and purposes, *forms of life*, are generated and these, I think it can be shown, constitute in various ways the theoretical elements of learning, teaching, and knowledge. In what follows two attempts will be made to illustrate the plausibility of these claims.

LANGUAGE-GAMES AND EPISTEMOLOGY

The links between the language-games reference, action, and form (see transcript) and the theoretical domain of epistemology illustrate the point that theories can be built up by the complex interplay of relatively simple language-games. In this case, there is an analogy between Ryle's knowledge *that* (1949) and the language-game of reference signalled here. Propositions match and/or represent actual situations, and thus operate as a representational system. But matching involves referring. We conclude that the linguistic substratum of propositional knowledge belongs to something like the (invented) language-game of reference illustrated in the text above.

Likewise, knowledge *how* is analogous to the language-game of action. Both view words, symbols, gestures, diagrams *etc* as tools (§§23,53) and involve the deployment of these in order to perform activities or functions. (Language-games of reference and form, in contrast, merely displace or juxtapose words and instruments.)

The third category into which the logical objects of knowledge fall is knowledge of persons, places and things. As already indicated, this kind of knowledge, knowledge *of*, is akin to re-cognition (Polyani, 1967; Pring, 1976; Hamlyn, 1970, pp103-111). Since language-games of form involve the articulation of structure and the recognition of juxtaposed elements, there is a strong family resemblance (§§67,77) between the language-games of recognition and those of form.

A further example in learning theory is situated cognition. Authors (Brown *et al*, 1989; Greeno, 1991; Lave, 1988) propose that the traditional forms of education be shifted from a primary concern for conceptual representation; situated or distributed cognition conceptualises such a shift. According to this theory formal representations of knowledge are indexed by references to context. Thus much of the context is built into or hidden within encoding structures; only the context specific experience of the learner enables knowledge encoded in this way to be utilised. Interior practices provided for here, supply the micro-foundations for these conceptual developments. For example, in the transcript, participating in the language-game of action, Alan constructs a sentence in which he evidently has little faith (hence the terminal "probably") yet nevertheless it is taken as a move in the new language-game of action. How did he construct this utterance? Employment of the word "use" (L16) provides the key. The word is used as an index for the deployment of symbols and words for the purpose of performing a meaningful action. As such, there is a relationship between "use" and similar utterances *eg* "the first thing that she did was to write" (L3), "what did she (ie Ms X) write?" (L6), "explain" (L14), and "talking" (L14) - even where, as in the first two instances, these belonged to different language-games. As part of the original language-game the utterance "the first thing that she did was to write " functioned to highlight the Teacher's reference to the past. This contrasts with the indexical relationship which later becomes established - for here actions are emphasised, and the students deployment of the word "use" repositions the utterance "the first thing that she did was to write" as part of the new language-game of action. In L15-17 Alan has reworded L7-10, however, in doing so he has emphasised the action of setting out the symbols '1', 'z_1', 'r_1', 'q_1' (language-game of action) and down playėd the activity of (merely) referring to them (language-game of reference). Activity implied by the word "use" is situated in other contexts (another language-game) although it is registered in present actions. This example sketched how language-games become entangled, although remaining self-signifying. The two situations mentioned - situated cognition and the function of indexical representations, and the language-game practices associated with indexical entanglements - are both strikingly similar in correspondence of form and function.

CONCLUSION

The writer has in broad outline shown the connection between interior practices of theory and practice in mathematics education. What implications does this model have for practice? Is the model adequate to provide direction and guidance? The writers answer is, no. Wittgenstein himself did not see philosophical practices as changing the world - on the contrary he claimed that when analysis was correct it was so precisely because it left every thing "just as before". And so it is for the model presented here.

It would be unsatisfactory, however, to leave the matter there. In this it is worth recalling Marx's comment, with his sights set on Hegel's metaphysics, that hitherto philosophy had set out to explain the world, however, the task is to change it. It may be true that in "setting the world before us" Wittgenstein's work threatens to become mere description. If so, then parallels between Wittgenstein and Hegel may be found. In (almost all) other respects, however, these two philosophers take contrary positions for, as Wittgenstein himself pointedly remarks, Hegel works to recover unity in the diversity of phenomena, whereas the theory of language-games pulls the other way - it is interested in showing that

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often what we are inclined to think of as being one (mathematical knowledge, for instance) is really a multiplicity of self referring practices or activities. Wittgenstein argues that in the function of language, we cannot rely on the crystal clarity of unambiguous distinctions. The edges, he maintains, are blurred and *must* be blurred. Unity gives way to diversity.

Another, but related difficulty with Wittgenstein is that, again like Hegel and unlike Marx, his work proceeds without reference to concepts of power, in which political and economic issues arise. For instance, the relative status of teacher and students, participants in the construction of interior parameters (language-games), and the relationship of these to traditional practices, the material conditions of work, executive authority, policy, and the state, are not considered.

It is clear that in order to join the diversity spoken of in Wittgenstein's analysis, with an analysis of the constraints just referred to, considerable broadening of the conceptual structures available for theory building would need to be facilitated. Foucault's notion of *discourse* as a practice that systematically forms the objects of which it speaks, may be the construct needed here. For Foucault, it is the extra-representational function of a discourse which renders it significant for any theory of knowledge and society.

> discourses are composed of signs; but what they do is more than use these signs to designate things. It is this 'more' that renders them irreducible to language (*langue*) and to speech. It is this 'more' that we must reveal and describe. (Foucault, 1972, p. 49)

The description of this 'more' may be no more urgent in any subject domain, than in school mathematics.

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A COMPARISON OF PATTERN-BASED AND EQUATION-SOLVING APPROACHES TO ALGEBRA

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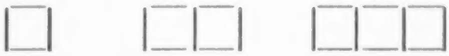
The National Statement on Mathematics for Australian Schools advises that algebra learning begins with the study of sequences and patterns leading to their description as algebraic rules relating dependent and independent variables. This study assessed the success of 512 students in seven schools at year levels 7 to 10 in recognizing and describing algebraic rules relating two variables. When given a relationship described by a table of values for two variables, the majority of these students were unable to write an algebraic rule of the form $y = ax + b$. Comparison of the test results from schools using different approaches to algebra suggests that the pattern-based approach (as implemented at the schools taking part in the study) was no more helpful than traditional approaches.

In *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990), it is suggested that algebraic thinking should be based on initial experiences with patterns and sequences. It is expected that in the primary years children will investigate patterns and sequences and make generalizations about them in everyday language. For example, a pattern of matches:



could be described by children as "You start with four for the first square and then add on three for each extra square" (*National Statement*, p. 191). In the secondary school students learn how to express such generalizations mathematically using algebraic symbolism. Some of the first algebraic skills to be developed, according to the *Statement*, are (a) describing in words relationships and rules for generating elements in a pattern or sequence, (b) expressing one of the rules algebraically, and (c) generating further elements of a pattern from a given verbal or algebraic rule.

Introducing algebra in this way as a language for expressing relationships between two variables, which we will call the "pattern-based approach", represents a clear break with tradition. It seems likely to facilitate the later study of formulas and functions. It is aesthetically pleasing to teachers, and students can make good use of concrete materials for building designs like the two-dimensional pattern above. As well as being endorsed in the *National Statement*, the pattern-based approach appears in the *RIME Lesson Pack* (Lowe and Lovitt, 1984), in the New South Wales mathematics syllabus (NSW Board of Secondary Education, 1989) and in some currently-used secondary school textbooks (e.g., Blane & Booth, 1991; Tomlinson, Ardley, Mottershead, Thompson and Wrightson, 1987).

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Other textbooks in common use (e.g., McLeod, Ganderton, Creeley and Tanti, 1988; Schnabl, Schnabl and Wagstaff, 1985) continue to introduce algebra as a technique for finding an unknown number, in which the earliest activities for students are manipulating expressions and solving equations.

The two approaches are based on two different meanings of algebraic letters. In the pattern-based approach, letters represent variables with a range of values, and in the more traditional approach, letters stand for specific but unknown numbers. In the United States, the NAEP study (Herscovics, 1989) showed that although most students with one or two years of algebra experience could recognize a simple pattern linking two variables (e.g. "add 7") from a table of values, about one-quarter of them were unable to write the corresponding equation (e.g. $y = x + 7$). The systematic research of the British CSMS study (Kuchemann, 1981) described four levels of understanding of algebra based on the interpretation of letters demanded by an item. For items at the lowest levels, where most students were successful, algebraic letters could be ignored, immediately evaluated or interpreted as objects. Items at level 3 required the use of letters as unknown numbers in very simple expressions, but the majority of 13, 14 and 15 year olds could not deal with these items. Items at level 4, in which students were required to interpret a letter as a variable, were dealt with successfully by less than 10% of 15 year olds. Textbooks used by British students were criticized at that time as tending to present algebra "as an afterthought to other work, for example on number patterns where from the child's viewpoint it serves no useful purpose" (Kuchemann, 1981, p. 117).

In view of the results from Britain and USA, it may seem foolhardy of Australian curriculum planners to advocate the introduction of algebra through an approach based on the concept of a variable. However the attractiveness of the approach has been outlined by Pegg and Redden (1990) who point to the opportunity it provides for students to see algebraic notation arising as a natural and useful consequence of expressing generality, the focus on letters as standing for numbers rather than names, and the opportunity it offers for students to see different correct expressions of generalization for the same pattern which can lead to opportunities to establish informally the rules of algebraic manipulation. To illustrate using the example above: students who see the pattern as "four matches for the first square and then three for each extra square" should write the algebraic rule $N = 4 + 3 \times (s - 1)$ where N is the number of matches required for s squares. On the other hand, a student who saw the pattern as "start with one match and add on three for each square" would want to write $N = 1 + 3s$. Comparison of these two correct expressions establishes some aspects of the distributive property.

As frequently happens in curriculum innovation, the recommended change in the method of introducing algebra has not been grounded in any empirical research. No study comparing the effectiveness of the two approaches has been published. The evidence which is presented in this paper suggests that the pattern-based approach may be no better than other approaches for the majority of students.

This paper presents a measure of students' success in recognizing a linear relationship between two variables when given a table of corresponding values of the variables, and in writing it algebraically. In particular, we wanted to find out what percentage of students at various year levels can:

1. interpret a table to read off corresponding values of two variables x and y;
2. fill in a missing entry in the table;

3. use a rule, rather than counting up, for calculating y;
4. write a rule relating the two variables algebraically.

Some students involved in the study had been taught with a pattern-based approach, whereas others had been taught with a traditional approach based on manipulating expressions and solving equations.

PROCEDURE

Two test items were prepared, and included in a classroom test. The items, called A and B in this paper, are shown in Figure 1. The items could be expected to favour the schools where the pattern-based approach was being used. The test was given in seven schools to 512 students in 26 classes from Year 7 to Year 10. Information about textbooks and other materials that had been used for these classes was obtained from the teachers concerned. Students had sufficient time to complete all the questions reported in this paper. All test papers were marked by the researchers. Results and comments were sent to the teachers.

A. Look at the numbers in this table and answer the questions.

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	..
8	..
..	..

(i) When x is 2, what is y?
(ii) When x is 6, what is y?
(iii) When x is 20, what is y?
(iv) Use algebra symbols to write the rule connecting x and y.

B. Look for the pattern in this table.

First number	0	1	2	3	4	5
Second number	-1	9	19	29	39	..

(i) Work out the missing number.
(ii) Explain how you worked it out.
(iii) Use algebra symbols to write a rule for working out the second number. In your rule, use F to represent the first number and S to represent the second number.

Figure 1: Test Items

RESULTS AND DISCUSSION

Table 1 shows the percentages correct for each part of Item A, and for the missing value and the rule in Item B. Almost all students could read and continue the number patterns in Items A and B, and therefore were correct for A(i) and (ii) and B (i). Working out the value of y when x is 20 (item A(iii)) was more difficult. As may be seen in the table, for both the items, and especially for Item B, many students were unable to write a correct algebraic rule.

Table 1: *Percentage of correct responses to items A and B*

		Item A				Item B	
Year	n	(i) $x=2$, $y=6$	(ii) $x=6$, $y=10$	(iii) $x=20$, $y=24$	(iv)Rule $y=x+4$	$F=5$, $S=49$	Rule $y=10x-1$
7	179	95%	92%	69%	33%	98%	7%
8	211	99%	97%	89%	74%	98%	47%
9	66	94%	91%	74%	44%	100%	14%
10	56	100%	98%	82%	46%	96%	11%

Results for item A (iii) and (iv)

As shown in Table 1, calculating y when $x = 20$ caused many errors, especially at Year 7 level where 31% were not successful. The most frequent wrong answers over all year levels were 16 (13 students, probably subtracting rather than adding 4) and 23 or 25 (21 students, probably not using any rule but losing track when counting on). Ten students had probably used a direct proportion rule which Stacey (1989) found that many students invent when counting becomes too hard. For example, for $x = 20$, seven students wrote $y = 28$, twice the value of y when $x = 10$. On the basis of Stacey's findings, we believe that more students would have used direct proportion if the value of x in question (iii) had been larger, so that counting was impractical, and if it had been a simple multiple of the value of x in question (ii).

For A (iv), in which students were required to write the rule $y = x + 4$ or equivalent, results overall were disappointing. As shown in Table 1, only one-third of the Year 7 sample wrote a correct rule, and the Year 9 and 10 students were not much better at around 45% correct. Common incorrect attempts to write the rule were xy, $x + y$ and the reversed equation $x = y + 4$. Eleven students (2% of the sample) wrote $x + 4$. At each year level there were two or three students who wrote rules involving 5, (e.g. $y = x + 5$ and $x = 5y$) which we assume to be derived from the first row of the table. Errors in the year 10 sample, for example, include the following:

$$y = 5 - x, \quad xy, \quad x = 1 + 4y, \quad 1x = 5y, \quad x = y, \quad x = y + 4.$$

At all year levels, the forms of many responses reveal lack of understanding of the meaning and use of algebraic notation.

Results for item B

Completing the missing entry in the table for Item B was a simple task for almost all students. However, as may be seen in Table 1, most students in the sample, at all levels, were unable to write an algebraic rule relating the two variables.

As expected, most descriptions of how students had worked out the missing number were attempts to say "I added ten". Many of the incorrect rules are obviously intuitive attempts to write a next-value rule such as "Add ten to the previous number" or "If you increase the first number by 1, then you increase the second number by 10". They include $F = S + 10$, $S = S + 10$ (this would be correct in BASIC programming) and $F + 10 = S$ (written by 64 students), as well as variety of expressions containing either $F + 1$ or $F = 1$ together with $S + 10$ or $S = 10$ (written by 21 students). The incorrect rule $F = S + 10$ was the most common error for the Year 10 classes, accounting for almost half of all attempts to write a rule. It may represent students' naive attempts to translate "Add 10 to the second number" into algebraic symbols, or it may be related to the "reversal error" (see MacGregor, 1991, for a summary of the related literature) which leads students to associate the numeral with the larger of two variables when attempting to write an equation. A few students had focussed on the "9 times" relation which they saw in the second column of the table, and tried to incorporate it into their rules as, for example, $S = F$ ¥ $9 + 4$ (which produces the correct missing value 49).

The results provide documentation of an important difficulty obstructing students' construction of formulas from tables. This is the tendency for students to focus on differences between successive values of the dependent variable, rather than on the relationship connecting dependent and independent variables. For example, many primary and junior secondary students will predict the next number in the sequence 1,4,9,16,25, ... by noticing that the differences between the terms are increasing odd numbers, not by recognising that the numbers are squares. Arzarello, (1991), working with 16-year-olds, described this phenomenon by saying that students' thinking was locked into arithmetic concepts which caused them to search for ways to predict the next number in a table from the value of its predecessor. A similar observation has been made by Clement, Narode and Rosnick (1981) and MacGregor (1991) who also report students' attempts to write such a "next-value rule" algebraically. Pegg et al. (1990) described this as a "completing-the pattern" response and commented that it is a typical early response.

Performance in items A and B related to approach to algebra

Performance varied considerably between schools and classes. For example, on Item A(iv), at year 9 level 73% were successful in one school, whereas in another only 18% were successful. On Item B, in several classes at all levels not more than three students were correct. On the other hand, there were four Year 8 classes, all at the same school, with more than 60% correct. This variation in performance is illustrated in Figure 2 which shows the percentage of students from each class who wrote correct algebraic rules, averaged over Items A and B. Figure 2 also indicates the classes which had been taught using a pattern-based approach. The performance of these classes suggests that they were no more successful than the classes who had learned algebra in other ways. It would be unwise to draw firm conclusions about the effectiveness of the pattern-based approach from this data because of the number of schools (seven) in the sample, the method of selection (teachers' interest in using the test for their classes) and uncontrolled factors. As well as possible differences in teaching style, amount of time spent on algebra, recent revision or practice before the test, and the learning environment, there is the distinct possibility that the choice of text made by a school is related to the general level of academic skills of the students. These inadequacies in the selection of schools for testing and in the control of interfering factors must be taken into account when drawing conclusions from the data. However there is no evidence in the results to indicate that learning algebra through a pattern-based approach equips students better to identify relationships between variables and express them algebraically than does a traditional approach.

Figure 2: Scattergram showing facility averaged over two items for classes taught with different approaches to algebra.

Very few of the verbal explanations for Item B, at all year levels, were written in clear and well-formed English. It may be concluded that describing a mathematical procedure or an algebraic rule verbally was a very difficult task for the majority of these students. Some typical examples of their English descriptions and algebraic rules are shown in Figure 3.

Category	Explanation	Rule
Pattern involving 9	It goes up by 9's. [Yr 7]	S+F
	A pattern of 9 being the last number. [Yr 8]	F+S+10
	Every number that has 9 in it. [Yr 9]	F+10S
	Every 10th number has 9 added on. [Yr 10]	S=10+9F
Add ten	The second the number is counting by ten. [Yr 7] -	
	Bottom line is going up by tens. [Yr 8]	Fx10-1=S
	Add 10. [Yr 9]	F+1=S+10
	10 is between every set of numbers. [Yr 10]	F=S+10
Describe two sequences	I added 10 to each number by the increasing 1 number. [Yr 7	F+10=S
	First number progressed by 1, second number progressed by 10. [Yr 8]	F=-1+F10
	When the first number increases by 1, the second number increases by 10. [Yr 9]	no attempt
	Each first number is added on ten in the second. [Yr 10]	no attempt
Relate two variables	Subtract 1 from the first number multiplied by ten. [Yr 7]	Fx10-1=S
	10 times the top column and take 1. [Yr 8]	S=10F-1
	Multiply the first by 10, subtract 1. [Yr.9]	Fx10-1=S
	Subtracting one from the first number where it is multiplied by 10. [Yr 10]	no attempt

Figure 3: Students' explanations and rules for finding the missing number in item B

CONCLUSIONS

The results highlight the extreme difficulty experienced by many students in trying to describe a numerical relationship in words as well as their difficulty in using algebraic symbols to write a rule. For example, a Year 7 student wrote for Item B, "You cent to you get to 9 and then you cent how much figers are up", and a Year 10 student wrote, "We are follow the first and second number with the distance by 10". These two students, and many others in the sample, appear not to have yet developed the level of English language proficiency necessary for talking and writing about mathematical relations. Even the very simple statement required in Item A - for example, "I would add 4" or "It is 4 more" or "You plus it by 4" - was not produced successfully by half the sample of students. It is hard to believe that the English language competence of students in Australian secondary schools at the present time is insufficient for a majority of students to produce a simple statement such as these. Clearly other factors are operating, but in any case it seems likely that language competence is affecting algebra learning.

The difficulty of the items raises questions about the degree of success that can be hoped for when algebra is introduced via the verbal description of patterns. It was apparent that many students' descriptions of perceived patterns, relationships or rules, whether clearly expressed or not, could not possibly be transformed to an algebraic form without extensive paraphrasing. For example, a Year 8 child wrote, "I saw a progression of 10 per unit of *F*", and another wrote, "1=10, so you just minus one". Both these students wrote the algebraic rule correctly, but clearly had not used their written statements as a basis for translation. It is still an open question whether being required to describe a complex relation in English helps in the construction of its algebraic form (MacGregor, 1990).

It is worth noting that the approach to beginning algebra that had been used for the most successful students was based on the view of algebraic letters as specific unknown numbers rather than variables. In the material which the teachers reported having used in years 7 and 8 (based on units developed by Quinlan, Low, Sawyer and White, 1989), algebraic expressions and equations represent arrangements of concrete objects, some of which are hidden and therefore "unknown". Very little emphasis is given to presenting algebra as a way of describing relationships between two sets of variables. The least successful students (i.e., four classes - two at year 7, one at year 9 and one at year 10 level - in which no student was able to write a correct equation for item B) had used a pattern-based approach.

The success achieved by some classes suggests that students in lower secondary school are capable of learning how to formulate algebraic rules from number patterns, while the great discrepancy between results for different classes and schools suggests that different approaches, teaching materials, teaching styles or the learning environment have a powerful effect.

Two interpretations of the pattern-based approach

A close analysis of the way in which texts such as *Mathematics Today* (Tomlinson et al, 1987) and *Moving Through Maths* (Blane et al, 1991) implement the pattern-based approach, as well as the suggestions in the *National Statement*, reveal different routes from observing a pattern to deriving an equation. These differences are possibly of crucial importance for learners. One common textbook approach takes a geometric design, immediately derives from it a table of values and then seeks an algebraic formula which will produce the numbers in the table. Students may work out the formula in a rote fashion by using the constant difference as the coefficient of x (the independent variable) and then adjusting the values by adding or subtracting a constant. Teachers may not ask students to express their generalizations in words, which might help the relationship make sense. In contrast, the version of the pattern-based approach advocated by the *National Statement* and trialled by Pegg et al. (1990) does not derive the algebraic rule from the table of values. Instead, the geometric features of a pattern or design are directly translated into a statement about the relationship, first in an English sentence and then in algebra. If we use the pattern given at the beginning of this paper as an example, the first version of the pattern based approach immediately constructs the number pattern 1,4,7,10, . . . and a formula starting with $y = 3 \times x + ?$ is sought to generate this pattern of numbers. In the other version, a student may notice that "there are the same number of matches along the top as along the bottom but the number of vertical matches is one more than this." With guidance from the teacher, this observation is linked with the expression $x + x + (x + 1)$, and finally with the formula $y = 3x + 1$. Problems in the transition from a verbal expression to an algebraic rule that have not been resolved are (a) how to help students with poor English skills construct a coherent verbal description, and (b) how to proceed when the verbal description cannot be translated directly to algebra.

There appear to be no other published studies of the effectiveness of introducing algebra via a study of patterns and rules relating two variables and their verbal description. The results reported in this paper suggest that it is a difficult way to approach algebra for the majority of students and that it offers no guarantee of success.

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LEARNING ENVIRONMENTS IN MATHEMATICS: EXPLORING PUPIL PERCEPTIONS

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INTRODUCTION

The development of an effective learning environment has for some time been the concern of the mathematics education community as demonstrated over the past two decades by teachers introducing strategies such as grouping for instruction and cooperative learning methods into their classrooms. In major Australian and international documents emphasis is given to the creation of a suitable learning environment in the classroom (e.g. The Mathematics Framework P-10, 1989; MCTP Curriculum Package, 1988; *The National Mathematics Statement*, 1990; and Professional Standards for Teaching Mathematics, 1991). Physical, social, organisational and academic factors are perceived to contribute to the creation of a learning environment where classrooms are seen as mathematical communities fostering the development of students' mathematical power. The 1991 teaching standards document (NCTM) makes reference to physical space, materials, goal-setting, exploration of problems, and listening to, responding to and questioning one another as some of the factors which may be present in a classroom which supports the teaching and learning of mathematics.

Studies related to the primary classroom environment include those of Fraser (see for example, 1984; 1986). His 1984 publication details five instruments for gathering student perceptions of the classroom environment, one of which, the My Class Inventory (MCI), has been developed for use with primary level pupils of eight to twelve years. This yes-no format questionnaire gathers data on pupil perceptions of the following aspects of the class environment: Cohesiveness, Friction, Difficulty, Competitiveness and Satisfaction. Actual and preferred versions have been developed. While, according to Fraser, the actual classroom environment version has been used more frequently, both versions have been utilised in studies, for example in Fraser's (1986) investigation of discrepancies between grade three children's preferred and actual classroom environments.

Muscella (1986) developed a perception of classroom environment instrument which was suitable for use with younger children. To explore interpretations of the class environment, the researcher asked kindergarten pupils to select favorite learning situations from representations given in adult-drawn sketches of the actual class situations. Muscella's instrument facilitated expression of ideas by children younger than those who are able to respond to written questionnaires such as the My Class Inventory.

It is apparent that data collection from younger children in the primary school, in particular, continues to pose challenges for the researcher, and that very little learning environment research has gathered data on the perceptions of primary school children. Few studies have actually asked children to indicate what they believe is for them an effective learning environment, particularly for the curriculum area of mathematics. In studies such as those of Fraser, analysis has focussed on factors such as competitiveness, difficulty, and satisfaction. However, as demonstrated in the literature (for example

Victorian Ministry of Education, 1988; NCTM, 1991), other more concrete factors such as grouping structures, manipulatives, and tools for discourse also contribute to the mathematics classroom environment.

In the current study a drawing and description instrument has been developed for use in the primary school which, most importantly, can expand the research data base for claims regarding effective learning environments, particularly for the learning of mathematics. It is considered that the instrument can have the potential to be used on a small scale by the primary level class teacher as a diagnostic tool providing insights into the preferences and needs of the individuals in his/her class, with its larger scale use in the present study providing data for comparison by the individual teacher or other researchers. The instrument provides insights into pupil preferences for learning environment factors different from those which have been the focus of many previous studies.

METHODOLOGY

Subjects

Data were gathered from five multi-age schools and four conventional, straight grade schools, resulting in the administration of the instrument to 26 multi-age classes and 25 conventional, straight grades. The gender groups were represented in similar numbers with 645 girls and 600 boys in the population of the study.

Sampling and Administration

Schools were selected through opportune sampling with the objective of a study population of approximately 1200 students, with similar numbers of students from each year level 1 to 6, with similar numbers of students from multi-age classes and from conventional straight grades, and with a gender balance.

Participant class teachers administered the instrument to their students. The instruction "Think of a situation in which you are learning maths well. Draw it. Show the people, the things around you and what you are doing." was written on the blackboard and read through with the class. Children proceeded to draw a picture of their preferred situation. Upon completion of the picture a six item questionnaire was distributed. Children used this to describe their drawings. Some of the younger children's questionnaire responses were transcribed by the teacher, with the children describing their completed drawings in their own words. Teachers provided information on the depicted tasks by completing a Teacher Perception Sheet for each child.

Once data were collected they were divided into sub-groups according to school type (multi-age and straight), class level (1, 2 , 3, 4, 5 and 6) and then gender. Following this, a one-third random sample of each sub-group was selected. A table of random numbers was used. The year 1 level multi-age data was treated differently; as only 28 responses were gathered, sampling did not occur. However, as the multi-age level 1 responses came from three classes, this was considered justifiable.

Analysis

Drawing, Description and Teacher Perception data were combined and considered one data source for each child. They were examined in terms of:

The sample population: n=427;
Class type: multi-age=213, like-age=214; and
Gender groups: males=206, females=221;
Year level: 1 to 6.

An analytic structure had been developed in stage one and two of the study. The analytic categories were: Location; Instruction Type; Adult Role; Adult Presence; Tools; Task Type; and Most Helpful Factors. They each consisted of elements considered in the analysis which employed frequencies, means and chi square contingency tables.

FINDINGS

Location

While it had not been specified that the learning environment be a school one, the most frequently depicted location among the sample population of 427 primary school pupils, for a personally effective mathematics learning environment, was the Classroom (70%). The Home was the second most frequently depicted location (15%).

Within the whole sample Class Type was a source of significant variance in student preferred location (n=427, Chi square=22.33 DF=5; p=.0005). Table 1 shows the location element results which most markedly differed from the expected values. The expected values are shown in brackets. It appears that the greatest source of variance was Home data. This location was seen as an effective learning environment in mathematics by 22.43% of the straight grade children and by 7.98% of the multi-age class children. Another major contribution to the variance came from Classroom location, chosen by 62.15% of the straight grade children and by 78.87% of the multi-age class children.

Further analysis of the results from the whole sample showed that grade level, but not gender, was a source of significant variance (n=427, Chi Square=48.73, DF=25, p=.0039). The trends observed were that frequency of Classroom depiction was higher for grades 4, 5 and 6 than for the sample as a whole with the greatest differences being for years 5 and 6, and the frequency of Classroom depiction was lower than expected for grades 1, 2 and 3, with the greatest difference being in grade 2. This suggests a basic trend for children to increasingly associate learning with school environments as they progress from infant to upper primary classes.

Table 1: Student Preferred Locations, Multi-age and Straight Grades - Observed and Expected Frequencies

	Class Type	
Location	Multi-Age	Straight
Classroom	168 (150.15)	133 (150.85)
Playground	4 (6.48)	9 (6.52)
Home	17 (32.42)	48 (32.58)

The Multi-age sample Location results did not show any significant variance within grade level or gender group results. Within the Straight grade sample there were no significant differences in Location preference among the gender groups. However, within the Straight grade sample, grade level was a source of variance in choice of Location for the effective learning of mathematics (n=214, Chi Square=49.98, DF=25, p=.0021). This suggests that the significant differences observed in the overall sample data largely stemmed from the straight grade data. The results from the straight grade sample are given in Table 2. It should be noted that the Expected Frequencies are presented in parentheses in Tables 2 to 5, as was the case in Table 1.

Table 2: Student Preferred Locations, Straight Grades - Observed and Expected Frequencies

	Grade					
Location	1	2	3	4	5	6
Classroom	20 (25)	16 (21.9)	16 (20.6)	25 (22.5)	28 (20)	28 (23.1)
Playground	1 (1.7)	0 (1.5)	4 (1.4)	1 (1.5)	0 (1.4)	3 (1.6)
Home	15 (9)	14 (7.9)	10 (7.4)	5 (8.1)	3 (7.2)	1 (8.3)

While a Home situation was represented by only 15% of the whole sample it is clear from the above results that more of the younger children considered this environment instrumental in their effective learning of mathematics than did the older children. This result suggests that a number of the younger children were able to identify a mathematics learning situation which did not carry the label of a 'mathematics lesson' as such. This adds weight to the belief that the instrument is suitable for the younger children and that they are able to give a considered response to an instruction which asks them to identify a situation in which they feel they are learning maths well.

Instruction Type

Over the whole sample the most frequently depicted Instruction Type was Individual Non-Instruction (33.25%), meaning that the child was working by himself or herself with no direct assistance from a teacher or other adult. Group Non-Instruction was the second most frequently depicted Instruction Type (26.23%). Group Non-Instruction situations did not include direct contact with a teacher or other adult. Class Instruction and Class Non-Instruction were the least frequently depicted elements.

By combining the elements into the three sets of Class, Individual and Group, the preferences for grouping structure alone can be examined. This reveals that Individual was preferred by 51.99% of the respondents, Group was preferred by 38.88%, and Class preferred by 9.13%.

Class Type (multi-age or straight) was a significant source of variance in preferred Instruction Type for the effective learning of mathematics (n=427, Chi Square=39.91, DF=5, p=.0001). As seen in Table 3, Individual Non-Instruction was the element of least variance. This element was depicted by 31.92% of the multi-age sample and by 34.58% of the straight grade sample.

The multi-age classes showed a higher frequency of preference for Group Non-Instruction, Group Instruction and Class Non-Instruction than was the case for straight grade classes. Class Instruction and Individual Instruction were depicted more frequently by those in the straight grades than by those in the multi-age classes. The whole sample class type Instruction Type results are detailed in Table 3.

Further analysis of the Instruction Type data in terms of gender groups and grade levels, across the whole sample and according to each of the class types, revealed no other sources of significant variance. Of the variables addressed in this study, Class Type was the only source of significant difference in preferences for effective mathematics learning environment Instruction Types.

Table 3: Instruction Type, Multi-age and Straight - Observed and Expected Frequencies

	Multi-age	Straight	Totals
Class Instruction	6 (12.47)	19 (12.53)	25
Class Non-Instruction	9 (6.98)	5 (7.02)	14
Individual Instruction	22 (39.91)	58 (40.09)	80
Individual Non-Instruction	68 (70.83)	74 (71.17)	142
Group Instruction	33 (26.94)	21 (27.06)	54
Group Non-Instruction	75 (55.87)	37 (56.13)	112
Totals	213	214	427

Adult Presence

46.84% of the sample group chose an effective learning situation in mathematics which was characterised by the absence of a teacher or other adult. The majority of drawings did include an adult in one role or another (see Adult Role, below). Analysis for the whole sample showed that neither Class Type nor Gender nor Grade Level were sources of significant variance.

Within the multi-age sample Grade Level was a source of variance in depiction of Adult Presence (n=213, Chi Square=17.299, DF=5, p=.004). In grades 3 and 5 preference for adult presence was higher than for the sample as a whole, while in grades 1 and 2 it was lower. The full results are shown in Table 4.

Table 4: Adult presence, Multi-age - Observed and Expected Frequencies

Grade	1	2	3	4	5	6
Adult Present	7 (13.93)	11 (15.43)	27 (18.91)	21 (21.4)	23 (19.91)	17 (16.42)

Adult Role

Of the 7 elements within the Adult Role category, six of which involved some sort of adult presence and one of which did not, the No Adult element was the most frequently depicted (46.84%). Adult and Child situations were the second most frequently (21.55%).

Analysis of the sample data revealed that Class Type was a source of significant variance in the preferences for Adult Role in the preferred mathematics learning environment situations (n=427, Chi Square=17.891, DF=6, p=.0065). As shown in Table 5 which details the element data to which the variance is likely to be attributed, the children from multi-age classes showed stronger preferences for Adult and Child (Group) and for No Adult while the data from straight grade respondents revealed higher than expected preference for Adult and Child, and Adult and Class.

Table 5: Adult Role, Multi-age and Straight - Observed and Expected Frequencies

	Multi-age	Straight	Totals
Adult and Child	32 (45.89)	60 (45.11)	92
Adult and Child (Group)	41 (31.43)	22 (31.57)	63
Adult and Class	10 (13.97)	18 (14.03)	28
No Adult	107 (99.77)	93 (100.23)	200

Grade level and gender group data did not reveal any statistically significant differences in adult role depicted in the preferred learning environments. Sources of variance were only found in the Class Type data.

Tools

Only frequencies were calculated as the data category elements were not mutually exclusive. Figure 1 shows the results from the overall sample for the elements: Computer, Calculator, Other Materials, Text, pencil and Paper/Worksheet etc., Nothing, Unclear.

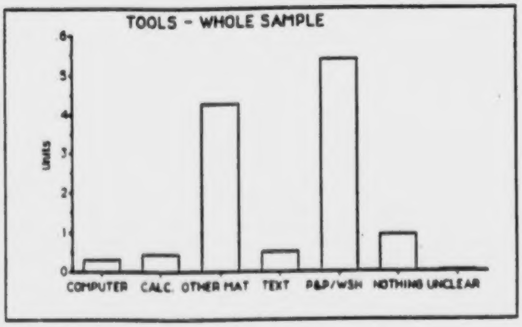

Figure 1: Tools, Whole Sample, Frequency of Depiction

It is clear that items of the Pencil and Paper/Worksheet category were the most frequently depicted (229). Materials were depicted 212 times in total (combination of Computer-12, Calculator-18, and Other Materials-182).

Closer examination of the class type, gender group and grade level results indicated some interesting differences. These are presented in Tables 6 and 7:

Table 6: Selected Tools, Multi-Age and Straight Grades

	Multi-Age	Straight
Calculator	4	14
Text	19	2

Table 7: Selected Tools, Male and Female

	Male	Female
Computer	8	4
Other Materials	77	105
Text	7	14

The Grade Level results indicated preferences which might have been expected. These included more preference for Other Materials in the lower grades than in the upper grades and less preference for Pencil and Paper/Worksheet in the lower grades than in the upper grades. There were more drawings by children in the upper grades in which no tools were depicted than in the lower grades.

Tasks

To gather information on depicted tasks teachers provided their perception of the child's task in terms of five criteria. The results presented in Table 7 indicate that for the whole sample, the tasks were predominately judged to be largely of a closed, other-directed nature. Possible values were 1 to 4, with a lower value indicating an open, self-directed, creative task.

Table 8: Tasks, Whole Sample, Mean values

Task Type	Mean	Standard Deviation
Inventing/Practising [Method]	3.314	.932
Discovering/Recalling [Information]	3.014	1.09
Exploring/Using [Tools]	2.972	1.072
Self-directed/Other-directed	3.007	1.118
Open/Closed	3.227	1.043

Analysis indicated that Class Type was a source of variance for Discovering/Recalling (n=422, Chi Square=9.424, DF=3, p=.0242). The multi-age sample scored higher than students in conventional classes, indicating more preference for recall tasks.

Gender was not a source of variance for any of the task descriptors. In contrast, grade level was a source of variance for all five items:

i) Inventing/Practising (n=421, Chi Square=38.979, DF= 15, p=.0006) The most notable results were at the grade 1, 2, and 4 levels which showed higher levels of Inventing than was true for the sample as a whole. This trend was particularly evident at the grade 2 level.

ii) Discovering/Recalling (n=422, Chi Square=42.262, DF=15, p=.0002) The most obvious variation was in the grade 2 group which was perceived to be Discovering twice as often as students in the sample as a whole. The general trend was preference for more Recall tasks in the upper school than in the lower school..

iii) Exploring/Using (n=394, Chi Square=29.135, DF=15, p=.0155) Grade two showed the most variation from the total sample, being nearly twice as likely to have depicted Exploring tasks (that is, having scored 1).

iv) Self-directed/Other-directed (n=421, Chi Square=28.213, DF=15, p=.0203) Grades 1 and 3 scored more ones (self-directed) than other grade levels. However, when the frequency of occurrences for both the 1 and 2 self-directed scores were combined, it was found that grades 1 and 2 scored higher on the self-directed end of the scale.

v) Open/Closed (n=422, Chi Square=43.944, DF=15, p=.0001). The grade 2 sample was most distinctive, clearly scoring a greater proportion of ones than other groups.

It is not entirely surprising that the trends all point in the one direction as the items are obviously related and as the scores given on each of the five items related to the one situation depicted by the child. However, it is interesting to note that it is at the grade 2 level that the children in this study showed more preference than at other levels for self-directed, open tasks.

Most Helpful Factors

The final data source was the responses to questionnaire item six, an item of an open nature which asked the respondents to identify the aspect(s) in the picture that was most helping them to learn maths well. This item encouraged focus upon, and identification of, the particular factor or factors which were perceived by the child to be instrumental in his/her effective learning of mathematics. Factors classified by the researcher under the Tools category (for example, manipulatives, pencil and paper) were the most frequently mentioned, Interaction category items (for example, teacher encouragement, peer assistance) were the second most frequent and Activity category items (for example, measuring, written procedures, mental computations) were the third most frequent . For each category there were similar results from the two class type sample groups. However, a break-down of the Class Type Interaction data produced some interesting results shown in Table 8.

Table 9: Selected data, Most Helpful Factor(s), Multi-age and Straight grades

	Multi-age	Straight
Adult Interaction	41	56
Peer Interaction	17	0

Closer analysis showed that the multi-age sample's preference for peer interaction came only from grade 4, 5 and 6 multi-age students.

SUMMARY

In summary, the following points can be made.

- Preference was shown for a Classroom location by the sample as a whole. However, a group of children in the lower grades, particularly from the straight grade sample, identified Home as an effective learning environment for mathematics. This suggests a trend of increasing association of learning with school as students move from Infant to Upper Primary grades.

- The most frequently depicted Instruction Type situations were Individual Non-Instruction and Group Non-Instruction, each of which involved no direct contact with an adult. The multi-age sample showed significantly high levels of preference for Group Non-Instruction, Group Instruction and Class Non-Instruction and, as a consequence the straight grade sample showed correspondingly higher preferences for Individual Instruction and Class Instruction.

- In the whole sample more children portrayed an adult in their picture than did not. However, the differences were minimal. In the multi-age sample, grade level was a source of variance, with more grade 3 and 5 children and less grade 1 and 2 children portraying an adult.

- Twenty-one percent of children showed an adult having direct interaction with themselves, and fifteen percent showed an adult having direct contact with their group. Class Type was a source of variance with the straight grade sample showing a higher preference for "Adult and Child" and "Adult and Class", and with the multi-age sample showing a greater preference for "Adult and Child (Group)", and "No Adult".

- Combined materials were represented 212 times, Pencil and Paper/Worksheet 229 times and text 21 times. The multi-age sample showed a small, but clearly greater preference for a text than did the straight grade sample. Females showed more preference for materials than did males.

- Grade 2 children were perceived to prefer more tasks of an open or exploratory nature as compared to the other grade levels.

Whether the differences which have become apparent in the preferences of children in multi-age and straight grade classes are a result of their present learning experience is not known. The fact that some children are known to have chosen situations which they experienced in previous years indicates that this is not necessarily the case. However, it has become apparent that information on when the experience occurred and on the stability of a child's perception of an effective learning situation in mathematics would be of value. These are possible avenues for further research.

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BRIDGING MATHEMATICS STUDENTS: ATTITUDES, AUTONOMOUS LEARNING BEHAVIOURS, AND PROBLEM SOLVING

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This paper describes a pilot study for a research project investigating aspects of autonomous learning behaviours, attitudes towards mathematics, and problem solving performance of students enrolled in a year-long bridging mathematics course. Six attitudinal variables were selected for investigation. A problem solving exercise was used as the context for investigating three autonomous learning behaviours. Further directions for research are discussed.

INTRODUCTION

The Study Context

Subjects. In recent years, students whose mathematics achievement level is not high enough to enter university science courses, have been given increased opportunities to attain the mathematics level required, by completing a bridging course in mathematics. These courses range from intensive short 4-6 week courses, to "Year 12 equivalent" studies taking a whole academic year. Such courses not only provide "catch-up" opportunities for recent school leavers, but also provide opportunities for mature-age people to return to study.

The students in this project were enrolled in a year-long Bridging Mathematics course, designed to meet requirements for entry into first year mathematics courses. Most of the students were older students returning to study.

Background to the study. It has been proposed (Fennema and Peterson,1985) that autonomous learning behaviours (such as independent learning, perseverance, taking risks, and willingness to choose a challenging task) are necessary in higher levels of mathematics; that these behaviours develop over time, and are related to the student's internal belief system, as well as external/societal factors. The student's belief system includes attitudes towards mathematics. Attitudes that Fennema and Peterson (1985) found to be important were confidence in learning mathematics; causal attributions and perceived usefulness of mathematics.

In secondary school mathematics students' confidence has been found to be a good predictor of future mathematics achievement and participation (Eccles and Jacobs,1986; Rowe,1988). Causal attribution patterns in secondary school mathematics students have also been proposed as important (Wolleat, Pedro, Becker and Fennema,1980). Elliott (1990) found that these attitudes, as well as the student's perceived usefulness of mathematics, were also important in the mathematics achievement of tertiary students. Hence these three attitudes were chosen for investigation in this study.

Many Bridging Mathematics students, who have enrolled in the course in order to achieve entry into the tertiary course of their choice, appear to be highly motivated. Many also show signs of anxiety, especially at the start of the course. Hence the attitudes of intrinsic motivation, maths anxiety, and attitude towards success in mathematics were also selected for investigation.

In recent years there has been an increasing emphasis on problem solving in mathematics courses. Problem solving, rather than only straightforward applications of algorithms, has become an important objective in the learning of mathematics. The student's belief system and its influence on problem solving performance, was cited by Silver (1985) as an area of research which could make a valuable contribution to the understanding of how mathematics is learnt. Problem solving performance was incorporated into this study as a measure of the level of mathematical thinking of the students, and because of the influence of affective variables in problem solving (McLeod,1989).

Research questions

This project was a pilot study investigating aspects of autonomous learning behaviours, attitudes towards mathematics, and problem solving performance of Bridging Mathematics students. The research questions were:

1. What attitudes towards mathematics are held by students entering the Bridging Mathematics course? How do these attitudes change during the year?

   Specific attitudes:
      - confidence in learning and doing mathematics
      - motivation
      - perceived usefulness of mathematics
      - mathematics anxiety
      - attitude towards success in mathematics
      - causal attributions in mathematics

2. What evidence of autonomous learning behaviours is shown by Bridging Mathematics students?

   Specific autonomous learning behaviours:
      - independent learning
      - perserverance
      - choosing to work on a challenging task

3. How does problem solving performance develop during the year?

METHOD

Attitudes

The testing instrument used was an Attitudes Questionnaire, consisting of two parts.

Part A was based on the Fennema-Sherman Mathematics Attitude Scales (1976) and consisted of forty (40) statements, using the five-point Likert scale. It contained five (5) subscales, which were:

confidence in learning mathematics
perceived usefulness of mathematics
motivation
mathematics anxiety
attitude towards success in mathematics

Part B investigated causal attributions: whether success/failure in mathematics was attributed to ability level; amount of effort; ease or difficulty of task; or environmental factors such as luck or teacher input. This part of the questionnaire was based on the Mathematics Attribution Scale (MAS) (Fennema, Wolleat and Pedro,1979).

The questionnaire was administered three times. It was administered to eighteen (18) Bridging Mathematics students at the beginning of Semester 1. It was administered to fourteen (14) of these students at the beginning of Semester 2, and at the end of Semester 2.

Of the initial eighteen (18) subjects, three (3) were female and fifteen (15) were male. Five (5) were less than 21 years old, and thirteen (13) were over 21 years of age. Of the fourteen (14) students who completed the year, two (2) were female and twelve (12) were male. Three (3) were less than 21 years old, and eleven (11) were over 21 years of age.

Autonomous learning behaviours

The main investigation of autonomous learning behaviours was a Problem Solving Exercise completed by five (5) Bridging Mathematics students towards the end of Semester 2. The subjects were asked to rank a set of eight (8) problems from easiest (1) to hardest (8). Each subject then selected three (3) problems to attempt. The problems chosen by the subject indicated the subject's willingness to work on tasks perceived to be challenging.

For each problem attempted, the subject was asked to show all working on the solution; keep a record of the time taken, and answer a questionnaire relating to the problem after it had been attempted. The time spent on a difficult problem indicated level of perserverance, and the extent to which other help was sought suggested level of independence.

Problem solving

An initial set of six (6) problems was given to the Bridging Mathematics students at the beginning of Semester 1. The problems were multiple-choice, but students were asked to record all working out of the solutions.

The Problem Solving Exercise (which involved autonomous learning behaviours, as described above) was administered towards the end of Semester 2, to ascertain the problem solving development of the five (5) students tested.

RESULTS

Attitudes

Questionnaire: Part A. The mean scores for motivation, attitude to success, and perceived usefulness of mathematics were high for Bridging Mathematics students at the start of Semester 1. Anxiety was also high, and confidence ranged from high to very low. When students were separated by gender, it was found that females had a much lower confidence level than did males (Mann-Whitney U 15,3 = 4.5, $p<.025$); a much higher anxiety level (U 15,3 = 2.5, $p<.005$), and a stronger attitude towards success (U 15,3 = 2.5, $p<.005$)

When students were separated according to whether they had left the course by the end of Semester 1, or had completed the year, it was found that those who left had less confidence (Mann-Whitney U 14,4 = 13, $p<.1$), less motivation (U 14,4 = 7.5, $p<.025$), and did not see mathematics as useful as those who stayed (U 14,4 = 8, $p<.025$)

An analysis of the questionnaire for the fourteen (14) students who completed the course, showed that the only attitude which changed significantly over the year was mathematics anxiety. (ANOVA, F = 6.227, $p<.01$). A post hoc test (Tukey's HSD = 3.34, $p<.05$ for $u1 = u2$) supported the conclusion that anxiety decreased significantly during Semester 1, and then did not change in level.

Questionnaire: Part B. The results of the attribution scores of the fourteen (14) students tested three (3) times over the year, showed that the class, on the whole attributed success in mathematics to effort and environmental factors. Failure in mathematics was attributed largely to lack of effort. There was no significant change in the scores for these attributions over the year.

Overall, only two attributional variables changed significantly during the year. Attributing failure to difficulty of task decreased over Semester 1, but not over the whole year (ANOVA, F = 2.667, $p<.01$; post hoc test Tukey's HSD = .265, $p<.05$ for $u1 = u2$). Attributing failure to environmental factors decreased over Semester 1, and then showed no further change (ANOVA, F = 5.271, $p<.01$; post hoc test Tukey's HSD = .486, $p<.05$ for $u1 = u2$).

Autonomous learning behaviours

The Problem Solving Exercise given to five (5) students during Semester 2, provided much qualitative data concerning the three (3) autonomous learning behaviour variables selected. Table 1 contains a summary of these results. The students showed evidence of autonomous learning behaviours, especially independence and perseverance.

Table 1: Autonomous Learning Behaviours

Subject	Independent learning	Choosing a challenge	Perserverance
1	worked independently	Ranks:1,2,3 Problems chosen for ease of task.1 problem chosen for interest	Completed 2 problems quickly. Spent 35 mins. on the third; did not complete.
2	worked independently; referred to notes.	Ranks:1,2,3 2 problems chosen for ease of task; 1 chosen for interest,challenge.	Completed 2 problems in short time. Persisted to completion the third problem (56 min)
3	worked independently	Ranks: 1,2,6 Chose 1 problem for ease of task; 1 for interest,1 for challenge	Completed 1 problem quickly. Persisted to completion 2 problems returning to each several times (24 min , 100 min)
4	worked independently	Ranks:2,4,5 Chose 1 problem for ease of task, 1 for interest,1 for challenge	Completed 2 problems quickly. Persisted with third problem for several hours (not completed)
5	worked independently; looked up formula	Ranks: 6,7,8 All problems chosen for interest , challenge	Persisted with all 3 problems; 2 completed (21 min , 30 min) ; 1 incomplete (150 min in 7 sessions)

Problem solving performance

Initial problem solving exercise. The mean number of correct answers to the set of six problems given to the Bridging Mathematics students at the start of Semester 1 was 4.47 (s.d. = 1.46)

The written solutions were analysed using a schema constructed by Charles and Lester (1982). A score of 0,1 or 2 was given to each of three aspects of each solution, depending on the completeness or correctness of the aspect. The three aspects were:

1. Understanding the problem (mean = 1.76; s.d. = 0.243)
2. Solving the problem (mean = 1.63; s.d. = 0.303)
3. Answering the problem (mean = 1.52; s.d. = 0.458)

The students mostly understood the problems, and gave at least partial solutions and answers, but the range of problem solving performance was greater for the latter two aspects.

Later problem solving exercise. The solutions to the problems attempted by the five students who did the Problem Solving Exercise in Semester 2 were analysed using the schema described above. The results of that analysis, and the results of the analysis of the first set of problems done by the same students, are found in Table 2.

Table 2: Mean Scores for Initial and Later Problem Solving Exercises

Subject	Exercise	No. Problems Correct	Understand -ing (0 to 2)	Solution (0 to 2)	Answer (0 to 2)
1	Initial	6	2	2	2
	Later	1	1.67	1.33	1.33
2	Initial	3	1.5	1.33	1.17
	Later	1	2	1.33	1.33
3	Initial	6	1.83	1.83	2
	Later	2	2	1.33	1
4	Initial	6	1.83	1.83	2
	Later	2	2	1.33	1
5	Initial	6	2	2	2
	Later	1	2	1.76	1.33

The number of problems in the first exercise was six (6), and for the later exercise was three (3).

Table 2 shows that students got proportionally less correct solutions for the later exercise, while still showing a high level of understanding of the problems set. It should be noted that the first set of problems were multiple-choice, so that students could have chosen the answer by elimination,or by guessing. The written solutions for the later exercise in fact showed that the students had considerable problem solving skills.

DISCUSSION

Attitudes

Results from the three Attitude Questionnaire tests were largely as expected from the literature and observation. Students were highly motivated, perceived mathematics to be very useful, and had a strong attitude towards success. These attitudes remained high for the whole year. Confidence levels were widely spread, with a fairly high mean, and remained so for the whole course. Mathematics anxiety, which also varied widely amongst the students, decreased during Semester 1, but did not change further.

During the Bridging Mathematics course, students consistently attributed success in mathematics to their own efforts, and to environmental factors, rather than to their own ability, or ease of task set. Failure in mathematics was attributed to lack of effort. During Semester 1, students became less likely to attribute failure to difficulty of task, or environmental factors. Overall, the students attributed success or failure to their own efforts, but did not seem to recognize their mathematics ability.

Autonomous learning behaviours

The students who participated in the Problem Solving Exercise showed evidence of autonomous learning behaviours. All worked independently, and most showed a high level of perserverance. Four chose at least one problem as a challenge.

All the participants were in the upper half of the class for mathematics achievement. The analysis would have been more valuable if less capable students had also participated in the exercise. A further direction for research would be to set the exercise earlier in the year, as well, so as to investigate the development of autonomous behaviours.

Problem solving

The solutions produced by the students in the Problem Solving Exercise in Semester 2, were rich in qualitative data, which was not adequately analysed by the schema used. A deeper analysis of the solutions presented could give more information about the problem solving performance of the students.

Since the initial problem solving exercise in Semester 1 contained multiple-choice questions, the resulting worked solutions could not be properly compared with the later set of problems done in Semester 2. In further research, the initial set of problems should not be multiple-choice, thus allowing for a clearer study of the development of problem solving performance.

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WHY CHILDREN DO NOT DRAW PARALLELS

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After successfully learning to identify pairs of parallel lines, samples of Year 1 and Year 4 students looked for parallels in geometrical figures and copied figures containing parallels. Oblique parallels were usually, but not always, more difficult to find than horizontal and vertical parallels. 62% of the children copied at least one parallel with an error of more than 5 degrees, but most of them judged they had drawn parallels. Asked to check, most realised their error and were able to draw an accurate parallel. The results suggest that young children attempt to preserve parallels in their 2D drawings but are hindered by the complexity of the figure from noticing inaccuracies. Failure to check more carefully might also be a general trend resulting from current teaching methods.

In my keynote address to MERGA-14 in Perth (Mitchelmore, 1991), I reviewed research showing that children often find it difficult to draw parallels, both when making drawings representing 3D figures such as cuboids and when copying straightforward 2D diagrams. The difficulty is particularly pronounced when the parallels are incident to an oblique line; errors are greater for smaller angles of incidence and decrease with the age of the child but do not vanish altogether even among adults.

I discussed some hypotheses which had been put forward to explain this effect:

- children cannot recognise parallels in 2D figures;
- children cannot recognise parallels on 3D objects;
- children cannot draw simple pairs of parallel lines accurately;
- children do not notice parallels on 2D or 3D figures.

All these hypotheses could however be rejected on the basis of research (summarised in Mitchelmore, 1985) showing that children in their first year of primary school can easily recognise and draw simple parallel lines, and that drawing their attention to the parallel lines in a figure does not significantly increase their drawing accuracy; parallel results were obtained for Year 4 students in respect of 3D figures. These findings suggested three guidelines for further research.

1. 2D and 3D drawings must be considered separately: the nature of the drawing task and the distribution of errors are quite different in the two cases.

2. There seems to be a complexity effect. Young children often fail to recognise parallels in figures which contain extra lines, and the drawing error also jumps significantly.

3. Children's errors may be caused by a different concept of what constitutes an "exact copy". Piaget showed that young children often conserve topological rather than Euclidean properties (Piaget & Inhelder, 1956) and Beilin (1984) found that testing of congruence by superposition first arises at about 8 years of age. Perhaps increasing accuracy in drawing parallels is an indicator of a maturing concept of congruence.

The study reported in this paper was designed to investigate whether children's concepts of an "exact copy" of a 2D figure included the preservation of parallels, and how judgements of parallelism were effected by the complexity of the figures.

METHOD

Two groups of 16 children were selected for individual interview, one group from Year 1 (average age 7.1 yr) and one from Year 4 (average age 9.9 yr). Eight children (4 males and 4 females) were selected at random from one class in each year in each of two Catholic schools located in predominantly middle class areas.

As in previous research, parallel lines were called "friends" to avoid possible linguistic problems and interference with verbal knowledge. Using transparent plastic strips, friendly lines were defined as lines which "go along side by side without bumping into each other" however far they go. After four demonstration items, children were asked to identify the friends in 16 pairs of lines at various orientations; they were encouraged to use the plastic strips if they were in any doubt. Errors on the first 8 items were corrected immediately, but no feedback was given on the second 8 items.

Children were then asked if they could see any friendly lines in 12 more complex figures (see Figure 1). Children were prompted ("Are there any more friends?"), if necessary, to ensure that they analysed each figure fully.

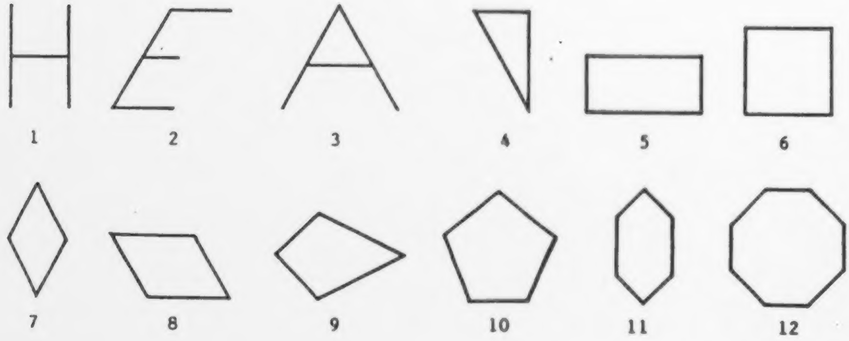

Figure 1: Shapes used for identification of parallels

Children were then given nine drawings to complete. For each item, a diagram containing one red line and one or more black lines was presented on a card. The child had a booklet on which each page showed only the black line(s) and completed the figure by drawing a red line to make it look "exactly like" the given diagram. In three cases children copied parallels, in three cases perpendiculars and in three cases neither; the three drawings in which the target line was parallel to a given line are shown in Figure 2. (The perpendiculars were included as part of a concurrent study of the students' concepts of perpendiculars; see Mitchelmore (1992) for details). Children had already done these copying tasks as a warm-up activity, and now it was suggested to them that looking for friends might make such drawings easier. Before making each drawing, children therefore had to identify any friends in the given diagram; and afterwards, they were asked if there were any friends in their drawing.

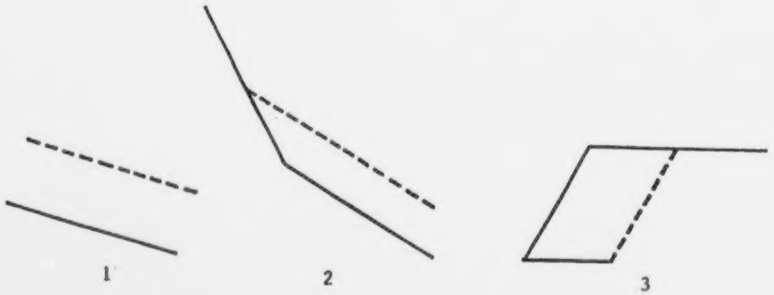

Figure 2: Parallel drawing tasks. The broken lines show the target lines (given in red and copied by the subject).

In a final de-briefing, children were asked if and how identifying friends had helped them do the drawings, whether they had ever thought about friendly lines, whether they could see any friendly lines around them, and whether they had ever studied anything similar in school.

RESULTS

The results confirmed that young school children can easily learn to recognise simple parallels. The Year 1 children made 15 errors on the first 8 figures (a 12% error rate) but only three on the next eight figures (2%); only one Year 4 child made any errors and then only on the first eight figures. In distinction to older students' verbally expressed knowledge of parallels (Mansfield & Happs, 1989), the fact that some pairs of parallel lines were of unequal length did not seem to cause children any difficulties. But occasional comments such as "They would be friendly if they were walking in one direction but not if they were walking in the other direction" (referring to a pair of non-parallel lines) suggest how difficult it might be for children to verbalise the concept of parallelism even when they can recognise parallels without difficulty.

The success rate on the more complex figures (see Figure 1) varied from 94% on Item 1 to 29% on Item 7. In general, the oblique parallels tended to be more difficult to recognise than the horizontal and vertical parallels; in Item 12, for example, the respective success rates were 46% and 85%. Recognising oblique parallels included realising that not all oblique lines are parallel; for example, several children stated that the two lines at the top of Item 11 were parallel.

Table 1: Item difficulties in parallel recognition task (percentages)

Item Group	Item No.	Inclination	Student Group			
			1	2	3	All
A	1	V	100	100	100	100
	6	HV	100	100	100	100
	11	V	90	100	100	96
	5	HV	82	100	100	93
B	8	H	73	100	100	90
	3	O^-	70	100	100	89
	4	O^-	75	94	100	88
	12	HV	60	100	100	86
	8	O	55	100	100	83
C	2	H	40	73	100	64
	10	O^-	22	75	100	61
	9	O^-	38	63	67	56
	12	O	30	38	100	41
	7	O	27	27	67	31
	11	O	0	38	100	31

* H = horizontal, V = vertical, O = oblique parallel, O^- = oblique non-parallel.

An attempt was made to clarify the sources of difficulty by grouping the various parallels and non-parallels in a manner which would discriminate consistently between children. As a first step, three children (including one from Year 4) who stated that there were no parallels in almost all items were eliminated. Horizontal and vertical parallels were then separated from the obliques, and all item difficulties recalculated. The data were then rearranged in order of item difficulty and student total score, and defensible item and student groupings sought. The result of this rather subjective process was three item groups and three student groups, as shown in Table 1.

Student group 1 consisted of seven Year 1 and four Year 4 students; they were usually successful in Item group A, often wrong in Group B, and usually wrong in Group C. The students in Group 2 (seven from Year 1 and eight from Year 4) were almost always correct in Item groups A and B, but still had difficulties in Group C. The three students in Group 3 (all from Year 4) were successful on almost all items. We note that not all horizontal or vertical parallel items were easier than the oblique items; the presence of oblique lines seemed to make horizontal and vertical parallels more difficult to recognise. Also, differences between Student groups 1 and 2 extended across all orientations.

The parallel in Item 1 (see Figure 2) was generally drawn accurately; only one drawing (made by a Year 1 student during the warm-up) showed an error of more than 5°. On the other hand, the obliques in Items 2 and 3 caused the usual difficulties. In several drawings, the target line started out almost parallel to one line and then curved round to be almost perpendicular to the incident line; in such cases, an "average" straight line was inferred. The numbers of inaccurate drawings of Items 2 and 3 are shown in Table 2. The improvement in accuracy from Year 1 to Year 4 was to be expected, as was the slight difference in difficulty between the items; and the complete absence of any difference in accuracy between the two drawing sessions confirmed that drawing error is not due to lack of attention to parallels in the stimulus figure.

Table 2: Percentage of parallel drawings with an error of more than 5°

	Item 2		Item 3	
Drawing session	Year 1	Year 4	Year 1	Year 4
Warm up	38	12	50	31
After identifying parallels	31	19	50	38

Most interesting in view of the purpose of this study were students' reactions on being asked, in the second session, whether the lines they had drawn were "friends". The great majority, including those whose lines were decidedly inaccurate, claimed their lines were friends (see Table 3). Of the seven accurate drawings not recognised as friends, two were made by a student who did not see any parallels in any of her drawings; in four cases, the parallels were not recognised in the stimulus; and one line was judged as too curved by a student who said "I tried to draw a friend, but it's difficult". Of the five inaccurate drawings judged not to contain friends, two were made by another student who could not see friends in any of his drawings; in two cases, students immediately and spontaneously redrew their lines to be friends; and one student thought the line should be a friend, saw that it was not, but did nothing about it. All in all, the data are strong evidence that the children were expecting to draw friends whenever they saw friends in the given drawing.

Table 3: Percentage of second session drawings claimed to show parallels

	Item 2		Item 3	
Drawing session	Year 1	Year 4	Year 1	Year 4
Error up to 5°	100	80	75	67
Error more than 5°	80	0	63	100

In discussions after all drawings were complete, 30% of the children spontaneously stated that they had copied friends by drawing friends; and all the rest agreed, when prompted, that this is what they had been doing. The 20 children (11 in Year 1 and nine in Year 4) who had made large errors on one or more of Items 2 and 3 in the second session were then asked if their lines really were friends; 75% stated immediately that they were not friends and a further 20% agreed they were not friends after checking with the plastic strips. Of the 19 who now judged that their lines were not friends, 89% indicated that their drawing was not a good one and proceeded to improve it; and of these 17 students, all but one were able (after up to four attempts) to draw an accurate parallel.

Several comments illuminate students' difficulties. They clearly were expecting friends: "It's a friend, but a bit too up", "They look like friends, but slightly off"; and they could see what was wrong: "The line should have been closer here ", "I need to draw it wider". To explain their errors, some students assigned an independent existence to their line: "It wanted to be a friend", "The line turned off a bit"; others blamed themselves: "I didn't know they weren't friends", "You don't really realise it's a bit bent [inaccurate] until you look closer". The task of drawing parallels could seem extraordinarily difficult: "It happened again!" said one student after the third unsuccessful attempt; but most students eventually succeeded: "I stopped now and again and made sure it was straight [parallel]".

No student admitted to having thought about friendly lines before, but all could pick out parallel lines in the interview room. None of the Year 1 students knew the word "parallel" (several had difficulty even saying it) but only one Year 4 student did not know its meaning and most remembered having studied parallels in mathematics lessons.

DISCUSSION

The results of this small investigation confirm many previous findings: parallels can be easily recognised, but they are difficult to draw, and not because children do not pay attention to them. The results also strongly suggest that young children do indeed try to copy parallels and that their idea of an "exact copy" does therefore include parallelism. Many children in this study made this explicit in the interviews, and most of those who did not, acted as if it was this principle that was guiding them; only three out of 32 students showed by their action or comments that they did not believe parallels mattered. Moreover, in contrast to accuracy of recognition and drawing of parallels, and children's experience of parallels in school, there was no discernible difference between Year 1 and Year 4 in terms of the apparent strategies used to copy figures containing parallels. We must reject the hypothesis that children's difficulties in drawing parallels are due to an immature concept of congruence.

A most striking aspect of the interviews was the frequency with which a simple prompt from the interviewer brought about an improvement in children's responses, both in terms of recognition and drawing accuracy. For example, although 20 of the 32 children were apparently content with an inaccurate copy of parallels, after being asked to check their drawings, this number fell to four. (Three of these were from Year 1). However, some children did spontaneously correct unsatisfactory drawings without prompting; there were only two cases of this in Year 1 but eight in Year 4. It seems likely that the real reason why young children often do not draw parallels accurately is that they do not check their drawings carefully enough; and there is some suggestion that the supposed tendency to check one's drawings increases with age.

But why don't children check their drawings? And why should they check their drawings more carefully as they grow older? We can suggest two factors.

Firstly, there were several indications that children had difficulty isolating the various parts of a figure and considering the relation between two lines while ignoring all the other lines. For example, one student had difficulty saying how many lines there were in Items 1 and 2 in Figure 1. Other comments are equally revealing:

- "The two side lines [in Item 1] would be friends if that [middle] line was gone".
- "These lines [the horizontal lines in Item 2] are friends except for that line [on the left]".
- "The lines [in Items 5 and 6] get in each others' way".
- "If they [the opposite sides in Item 5] were apart, they'd be friends".

Also, several students covered up distracting lines when judging friendliness. Such difficulties could explain both why students often did not recognise parallels in complex figures and why they did not check their own drawings carefully enough. The skill of disembedding relevant visual information has a long history of research based on the assumption that it indicates a personality trait known as field independence (Witkin, 1981); it has also been reported as a factor in geometrical performance (Bright, 1975). One thing is clear from this research tradition: disembedding skill improves with age during childhood. One explanation of young children's failure to check their drawings is therefore that checking requires disembedding skills which are still developing. (Development of disembedding skills may also be interpreted within the van Hiele model as one aspect of the transition from the first, global, stage to the second, more analytical but still concrete, stage.)

A second possible explanation is based on social conventions. In primary mathematics classrooms, children may be programmed into believing that the correctness of their work can only be judged by their teachers. The habit of checking one's answers, at least in a mathematical context, may therefore simply not be learnt. (Certainly, secondary school mathematics teachers complain often enough about how difficult it is to get their students to check their answers.) But even if a student checks an answer and sees that it may be incorrect, it requires a degree of courage to admit that to an adult. A sign from an authority figure (e.g. the interviewer) that it is acceptable - or even desirable - to criticise one's own products may bring a barrage of confessions.

The fact that dramatic improvements in drawing accuracy occurred when students were encouraged to check their answers suggests that many students did possess the necessary analytical capability but simply did not bring it into play. The tasks were within their "zone of proximal development" (Vygotsky, 1978). We may surmise that, through the process of making the rationale for their global visual judgements explicit, those students who showed that they could copy parallels more accurately when challenged would in future draw parallels more accurately at the first attempt. Unfortunately, no follow-up drawing tasks were included in the interview protocol to test this conjecture.

IMPLICATIONS FOR TEACHING

Two implications for teaching arise from this study: one narrow and one general. Actually, they are more like reminders than implications.

In narrow terms, the results emphasise that the adult tendency to see geometrical figures as consisting of sides and vertices which can be treated in isolation is something which has to be learnt. Early activities such as making models of common shapes using drinking straws, colouring components (such as opposite sides) of drawn shapes, and finding the number of triangles in a pattern of intersecting lines might enhance the development of such disembedding skills.

More generally, the results remind us of the tremendous gains that can be achieved if students can be taught to check their "answers" (be they the result of a numerical calculation or a geometric construction, or whatever). This cannot be achieved by the teacher repeatedly castigating the students for not checking their answers. Instead, the students must come to feel that there is always a rational basis for deciding whether an answer is correct and that they themselves can make this judgement. That means in practice that, instead of teachers routinely telling students whether their answers are correct, they should expect students to convince themselves that they are correct and occasionally ask them to explain why. In the process of convincing themselves and others, students would learn that mistakes are inevitable and acceptable. (The only non-acceptable mistake is not to realise that one has made a mistake.) This would bring substantial improvements not only in the narrow area of geometrical drawing, but, I believe, in every aspect of mathematics.

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CONSTRUCTIVISM: EPISTOMOLOGY TO PRACTICE

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CONSTRUCTIVISM AND SCHOOLING

Theory building in classrooms

> The mind not only "holds" beliefs, it perceives, construes and interprets information about the world, then hypothesizes, conjectures and reasons about this information. This results, at times, in decisions, beliefs, knowledge and at other times in confusion, wonder and understandings ... the mind is an intermediary that interprets and directs all perceptions and action ...
>
> (Welman, 1992, p.88)

This paper explores the notion of conceptual change in mathematics classes being a product of theory construction (Carey,1986; Chandler and Boyse, 1982; Gopnik,1984; Welman, 1992) by both teachers and students. Its main thesis is that while there might be a growing recognition of constructivism as epistomology, the institutionalisation of practices which reflect such philosophies is inhibited by our 'knowledge' of what mathematics schooling is.

The way that teachers in classrooms act out their roles, and hence the way they constrain the potential realities of others, is based on theories they have developed about what it is to be a teacher or a learner, about what it is to do mathematics and even about the nature of mathematics itself. These theories have been socially constructed: they have been built up from prior experience, itself tempered by active interpretation and linguistic interaction. If we are to accept the principle that knowledge is not passively received, but actively interpreted by the cognising subject (Lerman, 1989; Wheatley, 1991), we need to research interpretation of personal rôles in classrooms. If we acknowledge that subjects construct viable explanations of experiences (von Glasersfeld, 1987), we should study how teachers make sense of and theorise about their own impacts on the meso-world of the classroom.

Developing understandings of what it is to educate

The assumption of transmission-based pedagogies that major conceptual changes in students can be brought about through expressive action by teachers (paired with relatively inactive reception by learners) has been brought into question by the constructivist movement[1]. This cluster of perspectives is based around the alternative notion that learners actively establish learning goals, pursue these by selectively interacting with communications from others, then create for themselves appropriate schemas of understanding[2]. Meaning will be tempered by current understandings, individual ways of organising knowledge and personal experience of both the concepts and social situation being interpreted. Thus received ideas are sifted from the broader communication media, fitted with prior experiential understandings, reconstrued and used as a basis for further action (including subsequent building of new ideas) by individuals.

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[1] There is good reason for the irony that Ackerman (in press) notes - that for mathematics education the meanings of the labels "constructivism" and "constructivist" can never been entirely clear. If we believe that knowledge is constructed by individuals, we cannot make a claim that there is one form of constructivism, for this would be self-contradictory.

Central to this view of the development of personal constructs (Kelly, 1955) is the notion of knowledge being a product of choice (Gergen, 1985), a concept closely related to the notion of ownership of what is learned (Labinowicz, 1985; Kamii, 1985; Steffe, 1987, 1990). Choices, however, are not consciously made and not free of constraints: the shifting set of theories on which constructivism is based share the notion that constructive action within the mind of an individual is a social artifact (see, for instance; Gergen, 1985; Sullivan,1984; Watzlawick, 1984; Wexler,1983). Learning, for the constructivist, is representational theory-building involving re-organisation and re-building of communications to fit with developing schema (von Glasersfeld, 1989).

Creating space for construction of concepts

As Pateman and Johnson (1990) assert, constructivist ideas have influenced pedagogical practice. They note that the movement is impacting on the planning, enaction and evaluation of learning activities. If we believe that students need to construct mathematical concepts, then we need to create the temporal, discursive, and cognitive space for this to happen[3].

This paper aims to link epistomology with practice by identifying some moments where teachers or students create space for themselves and others. The use of the word 'moments' should not imply that these times are discrete or independent: the complexities of classroom interaction defy both iconic and linguistic simplicity. 'Moments' was chosen to reflect the immediate nature of split-second periods in which the potential of any lesson is re-shaped.
The following table lists those moments identified in the recent research project outlined below. It is not suggested that the moments are either linear or paired: they form a dynamic web of interaction. It should be noted that both teachers and students are practitioners in that both contribute to pedagogical practices and outcomes.

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[2] Even the radical constructivist (see, for instance Kamii, 1985, Sutching, 1992; von Glasersfeld, 1990) does not deny the role of teachers in offering knowledge verbally. Constructivism merely shifts the focus to examination of the ways that discursive interactions are handled through perceptive and active cognition.

[3] This is not as easily done as imagined: many writers about constructivist theories (see for instance Ackerman, in press; Duckworth, 1987) note the practical difficulties of putting such theories into action given the culture of classrooms.

Teachers		Students
Interpreting		
Planning		
Presenting	----------------------------->	Interpreting
Supervising		Planning
Evaluating		Performing
		Presenting
		Evaluating

In previous research (Mousley 1990a, 1990b, 1991b), I have explored constructivist classrooms where teachers allow mathematical activity to be planned, organised and controlled largely by children; but these classrooms are rare. And even in these classrooms, mathematical theories do not lie in the tasks, no matter who chooses them. Tasks are merely tools, with pedagogical theories-in-use shaping the ways in which they may be used to create different learning opportunities. At each of these moments, different teaching practices support theory construction to varying degrees.

The research project

In exploring the notion of constructivism in mathematics education, Nerida Ellerton, Ken Clements and I were interested to find out how a number of teachers and then groups of their students would interpret and develop a given activity. It seemed that recording the implementation of one activity in a number of mathematics classrooms would throw light on the way that teachers and students had theorised their rôles in the pedagogical process.

Eleven Australian classroom teachers (of a range of grades from Years 1-9) were involved in the project, as well as some teachers from South-east Asian countries. This paper, however, draws on data from only three Australian classrooms from the Year 5-6 range. The teachers demonstrated a variety of teaching styles and pedagogical philosophies: it would be wrong to imply that the observed teachers are constructivists. We were not so interested in teachers' espoused theories as in the ways that ideology is played out in pedagogy.

The project aimed to capture the moments listed above in order to provide data about how classroom actors construed and reconstrued the task. The teachers were given the activity a week before lessons were to be taught, and their reactions were recorded. Immediately before each lesson, teachers were interviewed about their planning strategies and intentions. Field notes were used to record discussions at these stages. Lessons were then observed, with teachers being videotaped as they introduced and concluded the lessons, and some groups of children video- and audio-taped (later transcribed) during the activity. Other groups were observed while working with field notes being taken. A few weeks after the lessons, teachers and students were interviewed and video snippets were used to assist recall of selected incidents. Discussion focused around participants' perceptions of their rôles at different stages of the lesson.

The text outlining the activity for teachers was a mere two sentences, written on an index card:

> From a given piece of cardboard, make a regular shape which holds one cup of birdseed. Make a similar shape which is twice as big.

One would assume that this activity would result in similar lessons in a number of classrooms. However, as von Glasersfeld (1983) points out, communication of concepts is not as straight-forward as we usually assume.

> The underlying process of linguistic communication, ... the process on which ... teaching relies, is usually simply taken for granted. There has been a naive confidence in language and its efficacy. (p. 43)

In fact, there were quite different interpretations by the teachers then various re-interpretations by their students.

Two moments of re-construction

This paper focuses the link between teachers' initial presentations of the activity and the students' re-interpretations of it. It was found that the former interaction impacted markedly on options pupils had for the latter.

Some teachers thought the given task too vague and realised that it might not lead to the discovery of the 'mathematical knowledge' they felt was bound within the task. One teacher (Teacher H, Year 6), for instance, shaped the lesson by writing on the chalkboard "Make a box which holds one cup of seed. Make another box which holds twice as much." At a later date, the researcher (R) interviewed this teacher (TH):

431 R I am interested ... to know why you thought the shape should be a box. Had you thought about the possibility of making other shapes?

432 TH Yes, but I wanted to build on this lesson to give them an understanding of volume.

433 R Good. So they will do that with box shapes?

434 TH Yes. They have to learn length by width by height and ... well, they couldn't do that with other shapes. ... Oh, I guess they could, but, like cones and other shapes - I didn't want shapes where they couldn't measure length and width and height.

In planning the activity around a particular learning objective - a formal rule - Teacher H presumably expected all students to take a relatively directed path of "discovery". His pupils later demonstrated an acceptance of this rôle context, clearly displaying characteristics of students waiting to be led.

463 Rob But it has to be a box. Not a box. He said a cube. That's the same all around. The same size - this way, this way, this way. Ask him how big. Darren, ask him how big to make it.

464 Darren How big would fit. You've got the cardboard. How big could we make it? It has to hold a cup.

465 Rob Just ask him, Daz. He knows.

466 Darren Okay. He knows. (Inaudible) Mr H...

Similarly, Teacher N (Year 5) thought that the activity could be used for children to discover what happens when all three dimensions of a cube are doubled. She first told the children to "Make a cube 5cm by 5cm by 5cm". When they had all finished, she asked them to "make one measuring 10cm by 10cm by 10cm" and to use the birdseed to compare the volumes of the shapes constructed.

Her students also displayed a reliance on the teacher to solve a mathematical problem they had:

134 Rhana It's too big. We were only meant to double it. You went wrong, Karen

135 Karen No. They are right. 5cm this one, 10cm that one. You try it. With the ruler.

136 Rhana Mmm, but there's too much seed. It should be two lots. Why is there seven and a half? It's mad. Two lots would be right. Don't write it down. I'm going to ask her if we did it right.

At this stage, following Rhana's question, the teacher called the class to attention. After a short discussion which led to the teacher expressing the generalisation that "the big one holds eight times as much", she proceeded to give a six-minute explanation of why this is so. No diagrams or concrete materials were employed, but terms such as *third dimension*, *multiply*, *multiples* and *comparative volume* were used. When asked later if she thought that all of the children would have understood, she claimed

231 TN Yes. That is why it is important to have the hands-on work first. Yes. They had seen it with their own eyes. That is why children need to do real things in mathematics - so they understand why the mathematics works.

In introducing and supervising the activity, both of these teachers created limitations to the way the task could be explored. In doing this, they also limited potential rôles students could play in the "discovery" process.

Other teachers, however, recognised that the task could be interpreted by students in different ways and saw this as opening up opportunities for people to take some ownership over the task. When Teacher M (Year 5/6) first saw the activity, for instance, she said

730 TM It will be interesting to see what they make of it. I'm not even going to explain what *regular* means. I wonder if they will ignore the word. ... Seeing the different shapes will be fun. And their reactions when they double different aspects.

Teacher M presented the task as given, by writing the instructions on the board. The groups in her class made various shapes, and some surprisingly difficult explorations were the result.

For instance, a group of three girls had built a square pyramid, left open at the apex so seed could be poured in using a small funnel. They had found that their shape held more than one cup of seed, so traced it onto another piece of card and trimmed the triangular parts of the net gradually until a pyramid of the right size was formed.

748 Silvia Don't take too much off. Remember … (inaudible) … you are not just taking that bit. That long bit. You are taking it four times. No, eight times. There and there and (etc.)

749 Binny And it's not just thinner. The shape. Look. It gets shorter so you are losing this bit. The top bit every time. The whole pyramid loses the top bit.

(Some trial and error followed)

783 Silvia Good. That's it. One cup. I'll tape it up. … Now. Two cups. Now for two cups.

784 Rachael Two cups. Yes. Or twice as big? Two cups isn't twice as big.

785 Binny (Inaudible, then laughed.)

786 Silvia Yes it is. But I know. But … I know what you think. Like … like two times the edges. Make the bottom twice as big. The sides too?

787 Binny Not the sides.

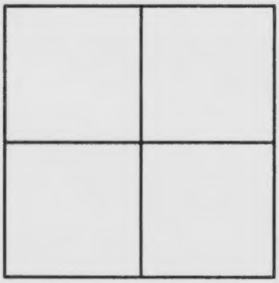
788 Silvia Why?

789 Binny We make the bottom twice as big. Right? Longer and wider. But that will be like this and this and this and this. Like four times as big. We need to decide if we want it really twice as big … or each side double - fourth as big. Anyway, then we could cut down the sides like we did … so it holds two cups.

790 Silvia Mmm. I get it. That way it's twice as big. Well, on the edges anyway.

791 Binny What edges. What are you on about?

792 Silvia These edges. The sides. The sides of the square. See. Look. You've got a square, right? (Silvia drew a square.) Then you double this one … and that one; and you don't have two. What do you have?



793 Binny Four. I know that.

794 Silvia Good. Four. So do we want twice or four times? Hey? I reckon only twice. Twice as big.

795 Binny Yeah, and …

796 Silvia And it holds two cups. So its twice as big like that.

797 Rachael But it needs to be twice as high.

798 Silvia Look, it doesn't say twice as high on the board. Or two cups. Look.

799 Rachael No.

800 Silvia We need to decide … to make up our minds. What are we going to call twice as big. Jees, I wish we'd started with a box.

801 Binny No, a box is no good. They are doing a box. Anyway, it's the same. We would be the same. Look, if you do this to the box … (Binny drew a sketch of a box then one twice as long) … you've got …

802 Rachael Two cups. But it's not twice as big.

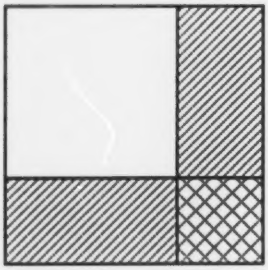
803 Silvia Yes. Yes it is.

804 Rachael But it's not twice as wide … or high … just long. Forget the box. I think we should not double the sides. Of the bottom. We should work out what would make twice as big really. Like. The square. The area of the square.

805 Silvia Yeah. Mmm.

806 Binny Times it … Times it by one and a half. Two is too big because you get four. No. One and a half …

807 Silvia That's too big. One point two five - or less? No, probably more that one and a quarter. It's not one and a half - it's less. Look. If you've got this square. Right? and it's half as long again … and half as long on to this side. (*Hatched areas, below*) Right? That's one and a half. But then you've got this piece *(double-hatched, below)*. That means one and a half is too much.



808 Rachael Can we work backwards? Jees. No. We don't know how many little squares. Get some graph paper so we can do it on little squares. Like, we count the squares. Then double it.

809 Silvia Yeah. Then work out how long the sides need to be.

810 Binny Would that make it hold two cups?

811 Silvia No. Rachael, where's the graph paper?

Learning, in this case could be classified as constructivist because it involved an unrestricted interplay between reality and possibility (Inhelder &Piaget, 1958).

Negotiation of meaning

Such incidents demonstrate the negotiation of meaning necessary for the completion of the task as well as the development of complex mathematical ideas. In reviewing transcripts of the groups at work in classrooms where children were free to take control of their own mathematical activity, it is easy to find examples of conjecture, reasoning, explaining, negotiation of both vocabulary and meaning, planning, generalising and responding. These established genres of social interaction are too often restricted by teachers taking strong control of the creation and supervision of tasks.

Each participant in the last discussion above contributed to its movement towards clarification of the task and the creation of possible solutions to the problems posed along the way. Individuals responded appropriately when others had not followed their reasoning and used alternative communication (sketches, hand movements, pointing to and handling the pyramid) to help others construe meanings which were shared enough to allow the discussion to continue. Choice of direction and ownership of the mathematics being explored rested firmly in the control of the learners. Such co-operation and freedom, however, were not enabled in the earlier examples because the teachers were perceived to be in control of instructional functions.

Students' notions of what it is to do mathematics in each of these classrooms would be likely to be quite different. Teacher H, while expecting students to make mathematical discoveries, planned the activity around these and his students accepted the implication that he had definite ideas about how tasks should be completed and the 'rules' which would evolve. Teacher N shaped the task carefully so that the activity would 'prove' the rule she planned to explain, and the students in her class provided no evidence of the notion that it is a student's role to discover mathematical ideas or to contribute to the learning of others. While there was knowledge development in these classrooms about the institutional and cultural contexts of schooling, there was little constitutive mathematical activity by students. In contrast, Teacher M's students looked for little direction and apparently thought it their task to negotiate and explore both the task and the resulting mathematical ideas. Mathematics, for them, was malleable: its meanings were to be construed.

CONCLUSION

The above snippets of classroom interaction are useful in examining the notion that knowledge originates from activity with objects. Many teachers have interpellated this claim by thinking that the provision of concrete materials or of hands-on activities is all that is required to relate mathematical concepts to the 'real world' of the child. However, we need to think of objects not as counters, blocks, cardboard, seed or other concrete materials. What students manipulate is developing mental constructs (Wheatley, 1991). Acting on the world may very well involve manipulating materials, but the objects of manipulation are also teachers' directions, written text, group discourses and perceptions of the roles of both teachers and learners in classrooms.

Also demonstrated above is how pedagogical moments can shape potential learning. Such moments, in combination, become processes of classroom interaction. Wheatley (1991) comments that with instructional processes which sanction natural instincts to construct meaning,

> Students come to realise they are capable of problem solving and do not have to wait for the teacher to show them the procedure or give them the official answer. Students come to believe that learning is a process of meaning-making rather than the sterile academic game of figuring out what the teacher wants. (p. 15)

The observations above, plus others recorded during the life of the project, support the claim of Candy (1989) that learners will not attain full and undisputed ownership over the learning situation while they construe the instructor as still exerting residual authority.

This claim has implications for our own practices as teacher educators. We criticize student teachers for not having clear learning objectives and well-sequenced lesson plans. We worry that teachers do not demonstrate learning sequences we believe are appropriate for different concepts. We model linear, transmissive curricula with carefully prepared tertiary programs. But it seems that presentation of loosely-defined tasks may allow for some exciting lessons about learning mathematics. This research suggests that we should lend more weight to Schratz' (in press) claim that

> The more interpretive space a teacher leaves in the learning process, the more likely the chances will be for the students to use their own knowledge of the world and everyday reasoning to tackle educational problems. However this process is also dependent on providing materials that can be subject to negotiation and multiple interpretations.

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CHILDREN'S SOLUTIONS TO PARTITION PROBLEMS

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Children's solutions to ten different multiplication and division word problem structures were analysed at four interview stages in a 2- year longitudinal study (Mulligan, 1992). The study followed 70 children from Year 2 into Year 3, from the time they had received no formal instruction in multiplication or division to the stage where they were being taught basic multiplication facts. A Teaching Experiment that encouraged children to represent a range of multiplication and division situations through language, modelling, drawing, symbolising and reflective writing, was conducted with 10 children in the later part of the Longitudinal study.

This paper reports the findings for the division Partition problems revealing that children very rarely used a sharing one-by-one (dealing) strategy at any stage in the Longitudinal Study or Teaching Experiment. Instead, a variety of counting and grouping strategies such as estimation and grouping, one-to-many correspondence and trial-and error grouping procedures was used. Knowledge of addition facts and skip (multiple) counting assisted children in forming equivalent groups.

INTRODUCTION

Over the past decade, researchers have analysed secondary students' solution processes to multiplication and division word problems based on differences in semantic structure, mathematical structure, size of quantities used, and student's intuitive models (Bell, Fischbein and Greer, 1984; Bell, Greer, Grimison and Mangan, 1989; Brown, 1981; De Corte, Verschaffel and Van Coillie, 1988; Fischbein et al., 1985; Nesher, 1988; Vergnaud, 1983, 1988). More recently, a growing number of studies on young children's solution strategies to multiplication and division problems has emerged (Anghileri, 1989; Boero, Ferrari and Ferrero, 1989; Brekke and Bell, 1992; Kouba, 1989; Mulligan, 1992; Olivier, Murray and Human, 1991; Steffe, 1988). These studies have provided complementary evidence that the semantic structure of the problem, an understanding of the problem context, and the development of counting, grouping and addition strategies may influence solution process.

Earlier studies by Gunderson (1953) on multiplication and division word problems, and by Zweng (1964) on division word problems, provided some evidence of Grade 2 children's responses to partition and quotition problem structures. Australian studies on division (Irons, 1977; Barr, 1980) and multiplication (English, 1982) word problems also focussed on differences in performance between problem structures but did not analyse the complexity of the solution strategies.

Davis and Pitkethly (1990) have investigated the cognitive aspects of pre-schooler's and Year 2 children's sharing and counting activities. The major features of the study suggest that dealing without counting is a widespread sharing strategy but that children needed to count after dealing to check for fair shares. Hunting and Davis(1991) conducted structured interviews with 75 pre-schoolers and sharing by dealing was demonstrated by 85% of the sample but children resorted to counting to check. However, Davis and Hunting (1990) found that with spontaneous sharing tasks in unstructured situations children did not use dealing at all. It seems that counting procedures were used instead of dealing.

The investigation of underlying intuitive models for division has shown that children employ three different models for partitive division: sharing by dealing, sharing by repeated taking away, and, sharing by repeated building-up (Kouba, 1989). It also appears that children may perceive partitive and quotitive division as more related than Fischbein et al.(1985) had suggested.

LONGITUDINAL STUDY

A purpose of this Longitudinal study was to analyse and classify the informal strategies children use in solving multiplication and division word problems (Appendix A). The results indicated that 75% of the children were able to solve the problems using a wide variety of strategies even though they had not received formal instruction in multiplication or division for most of the 2-year period. Performance level generally increased for each interview stage, but few differences were found between multiplication and division problems except for Cartesian and Factor problems. Most compelling was the evidence that the **semantic structure of the problem affected the choice of solution strategy**, more than it affected a change in performance. This was found to be consistent with the pattern of response identified in the Carpenter and Moser (1984) study on addition and subtraction word problems.

Solution strategies were classified for both multiplication and division problems at three levels:

(i) direct modelling with counting;
(ii) no direct modelling, with counting, additive or subtractive strategies; and
(iii) use of known or derived facts (addition, multiplication).

A wide range of counting strategies was classified as counting-all, skip counting and double counting. However the use of sharing one-by-one (dealing) was rarely found. Analysis of intuitive models revealed preference for a repeated addition model for multiplication, and a 'building-up' model for division rather than sharing or repeated subtraction.

PARTITION PROBLEMS

In order to determine whether children were influenced by different linguistic terms in the Partition problems, two different forms were used, with one problem type using the cue "**shared** equally": (a) There are 8 children and 2 tables in the classroom, how many children are seated at each table?; (b) 6 drinks were shared equally between 3 children. How many drinks did they have each? There were some differences found in performance between the two problem types with 75% of children gaining a correct solution for the (a) problem compared with 97% for the (b) problem at the final interview. When larger number combinations were used, again children performed much better on the (b) problem showing 83% correct compared with 55% for the (a) problem.

Table 1: Strategy Use for Partition (a)

Small and Large Number Problems : Interview Stages 1 to 4

STRATEGY TYPE	SMALL NO.				LARGE NO.			
	1	2	3	4	1	2	3	4
Direct Modelling								
Counting-all	6	6	5	2	4	10	6	8
Sharing			2			6	6	6
One-to-many	9				1			
Trial and error					13	13	6	17
Skip counting							3	3
Halving	3	1	2	2				3
No Direct Modelling								
Counting-all			2			1		
One-to-many	1	33	2					
Skip counting						1	2	3
Doubling	1		3					
Repeated subtraction	4	1	3	3				
Halving	34	6	26	38	3		3	6
Known Facts								
Addition	14	10	22	18				
Multiplication	3	3	6	5			2	6
Derived					1			2
Division			2	5				2
TOTAL % CORRECT	66	69	74	75	23	33	29	56

STRATEGY USE

A wide range of strategies was employed to solve the Partition problems and children generally relied on simple additive procedures or matching strategies. Children used one-to-many correspondence as a matching procedure for both Partition (a) and (b) problems. Table 1 compares the range of strategies used with halving shown as predominant at Stage 1, but one-to-many correspondence at Stage 2. Interview Stages 3 and 4 saw a return to the more efficient 'halving' process and the marked decrease in one-to-many correspondence. The use of trial and error grouping was identified for the large number combinations and was consistent with Kouba's (1989) findings for a similar partitive problem. On the other hand, one-by-one sharing was rarely found.

The Partition (b) problem (six drinks shared equally between 3 children. How many drinks each?) varied slightly in semantics from the Partition (a) problems, and a wide range of solution strategies was used. Table 2 shows one-to-many correspondence with and without modelling was preferred because children immediately represented 'six drinks and 3 children' as "two, two, two", "two for one, two for the next, two for the last." Because the number combination was easy to estimate the strategy of skip counting in 2's was also recognised. The large number combination (14 drinks shared between 7 children) was less easy to estimate, but doubling in 7's was used here. The sharing one-by-one was encouraged by the semantics in the problem "share" but the Quotition (b) problem , also using "shared" did not reveal any one-by-one sharing.

Table 2: Strategy Use for Partition (b)

Small and Large Number Problems : Interview Stages 1 to 4

STRATEGY TYPE	SMALL NO.				LARGE NO.			
	1	**2**	**3**	**4**	**1**	**2**	**3**	**4**
Direct Modelling								
Counting-all	1	1	2	3	4	4	8	6
Sharing	3	4		3	4	4	2	6
One-to-many	17	6	5	10	9	10	10	5
Trial and error		1				6		2
Skip counting	1	3	3		1	7	8	5
Repeated addition	1			2				
Doubling						1		
Halving								2
No Direct Modelling								
Counting-all						1		
Sharing		7	5	7	3	1	2	2
One-to-many	26	23	11	26	1	6	2	3
Skip counting	1	23	31	7	3	6	5	6
Repeated addition	4		3	7				
Doubling			3			3	5	
Repeated subtraction					1			
Halving		3	3	10		1	3	12
Known Facts								
Addition	3	3	9		4	7	14	17
Multiplication	3	3	6	15	1	3	6	15
Division			7					3
TOTAL % CORRECT	61	80	81	97	34	64	64	83

INTUITIVE MODELS

Following the analysis of solution strategies, underlying models of multiplication and division were investigated to complement the main findings of the study (Mulligan, 1991a). The underlying intuitive models for multiplication and division were found to be more complex and varied than previously found by Fischbein et al. (1985) with older pupils and were supported by evidence in the Kouba (1989) study on multiplication and division word problems with younger children.

Three underlying models were identified for division with small and large number combinations: sharing one-by-one, 'building-up' (additive), and 'building-down' (subtractive). Analysis of Partition, Quotition and Rate problems across the four interview stages showed a **widespread preference for the 'building-up' model.** This was based on an **underlying notion of addition or additive-type strategies.** The sharing one-by-one model was rarely observed.

Table 3 compares the percentage of correct responses for each intuitive model by problem structure. The three models were found across problem structures (except for sharing one-by-one), and across the four interview stages.

Table 3: Percentage of Correct Responses by Intuitive Model on Division Small No Problem: Interview Stages 1 to 4

PROBLEM STUCTURE		INTERVIEW STAGE 1				INTERVIEW STAGE 2				INTERVIEW STAGE 3				INTERVIEW STAGE 4			
		S	BD	BU	Tot	S	BD	BU	Tot	S	BD	BU	Tot	S	BD	BU	Tot
Part	(a)	0	27	39	66	0	32	37	69	2	31	41	74	0	51	24	75
	(b)	3	11	47	61	3	12	65	80	5	8	68	81	10	29	58	97
Rate	(a)	1	11	39	51	1	7	46	54	2	8	56	66	2	18	65	85
Quot	(a)	0	8	26	34	0	25	35	60	0	15	40	55	0	39	45	84
	(b)	1	9	37	47	0	23	41	64	0	16	53	69	0	26	67	93

S:	**Sharing one by one**
BD:	**Building down**
BU:	**Building up**
Tot:	**Total percentage of correct responses for each problem structure**

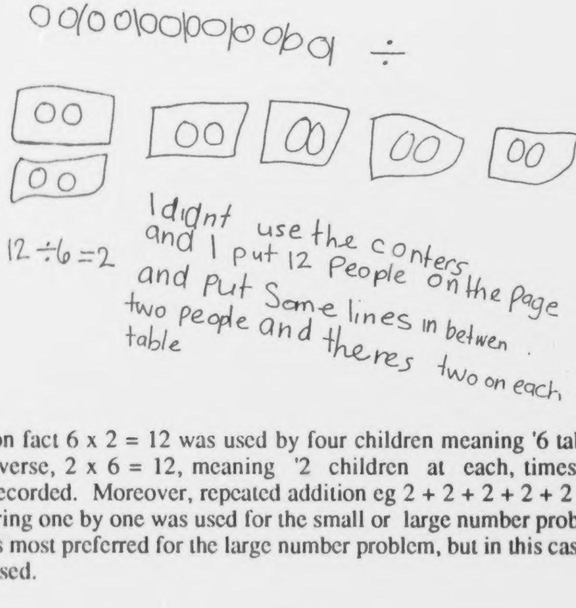
For the Partition (a) problem, 'building-down' was almost as prevalent as 'building-up' for Interview Stages 1 to 3. At Stage 4, though, the 'building-down' model was preferred; 51% compared with 24% ('building-up'), but this was an exception to the pattern found overall. The 'building-down' model found at Stage 4 could be explained by the use of the known division fact (8÷2 = 4). The reasonably high percentage of 'building-down' for Partition (a) in Interview Stages 1 to 3 could be attributed to this also. Similarly, the 'building-down' model was markedly increased at Stage 4 for the Partition (b) and both Quotition problems possibly for this reason. Comparison with the large number problems at Stage 4 showed that 'building-up' was preferred because children did not know the division fact (28+4 = 7).

THE TEACHING EXPERIMENT

The Teaching Experiment focussed on representing a range of multiplication and division situations through language, modelling, drawing, symbolising and writing (Mulligan, 1991b). Children related their informal strategies to more formal symbolic representations by linking counting and additive recordings to multiplication and division. As well, children were able to find patterns and relationships between problems and devise their own problems showing understanding for the operation involved. Some direct teaching strategies were employed and these were influential in assisting children represent and solve the problems; relating skip counting to the problem situation, and using a hundred square to represent patterns. Further evidence of children's underlying intuitive models for multiplication and division in pictorial and symbolic form were consistent with the results of the Longitudinal study.

The small number Partition problem presented a simple combination (12 children and 6 tables. How many children at each table?), and did not include the term 'share' as it was not found to be influential in the Partition (b) problem in the Longitudinal Study. There was a wide variety of symbolic number sentences and pictorial diagrams represented and this was consistent with the wide range of strategies shown in the Longitudinal Study.

Samantha used the most abstract representation (Figure 1) where the division symbol was correctly used showing a quotient of 2 even though formal division had not been introduced in the classroom. However, another child used 12 - 2 = 6 to represent the same picture.



12÷6=2

Ididnt use the conters and I put 12 People on the page and Put Same lines in between two People and theres two on each table

Figure 1

The multiplication fact 6 x 2 = 12 was used by four children meaning '6 tables with 2 at each', but the inverse, 2 x 6 = 12, meaning '2 children at each, times 6' was also verbalised and recorded. Moreover, repeated addition eg 2 + 2 + 2 + 2 + 2 + 2 = 12 was found but no sharing one by one was used for the small or large number problem. The 3 x 8 = 24 model was most preferred for the large number problem, but in this case the division symbolism was used.

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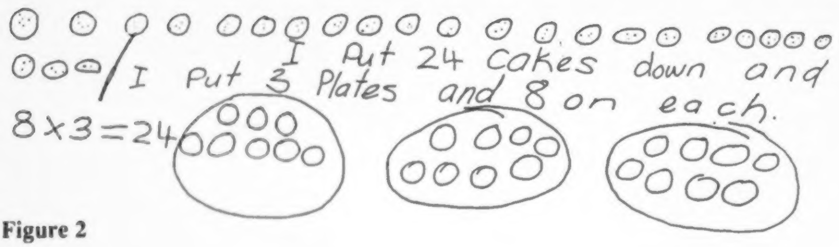
I Put 24 cakes down and
I Put 3 Plates and 8 on each.
8×3=24

Figure 2

Estimation, one-to-many correspondence and trial and error grouping were shown with a variety of symbolic recordings: 3 x 8 = 24, 8 x 3 = 24, 24 -8 = 3, 24 -3 = 8. Three children avoided symbolising and simply described their picture. 'Building-up' and 'building-down' models were observed with 'building-down' being used where children drew the 24 cakes first. These representations were less sophisticated than the 'building-up' models. Figure 3 shows how Michelle counted-all first, before grouping.

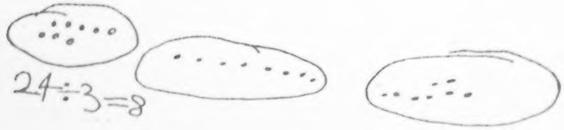

Figure 3

One explanation for the lack of a sharing one-by-one strategy may be attributed to the problem identifying the total dividend to be shared, whereby children did not need to count out the total to be shared. Their understanding of 'fair share' was demonstrated by the formation of equivalent groups and trial and error procedures to ensure that each group had the same amount. This constrasts with evidence from a number of studies with pre-school children showing that sharing by dealing was used widely (Davis and Hunting, 1990; Davis and Pilkethly; 1990; Hunting and Davis, 1991). However, some children who showed immature strategies based on counting-all and modelling also used sharing one-by-one, and by the final interview this strategy had changed to a more efficient one based on estimation of equivalent groups. It appears then, from the analysis of individual profiles that sharing one-by-one was used at early stages and that children progressed to more efficient division strategies with the development of skip (multiple)counting and addition skills.

The most compelling evidence in this analysis, that children used 'building-up' strategies based on addition and counting was not proposed by Fischbein et al. (1985) possibly because children in Grades 5 to 9 had been instructed in traditional models of sharing and repeated subtraction. The analysis of intuitive models has provided more insight into young children's developing notions of multiplication and division, and how these are inextricably linked to addition. Thus, these findings raise serious questions for teaching and learning methods that rely on the sharing and repeated subtraction models for division in the primary school. Furthermore, young children can be encouraged to develop efficient division strategies based on estimation, grouping and counting. The rigidity of traditional partitive and quotitive models needs to be redressed and the development of problem-solving strategies for multiplication and division contexts can be encouraged from the pre-school years.

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APPENDIX A

Multiplication and Division Word Problems

Multiplication	Division
i. Repeated Addition (a) There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (b) Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (c) I have three 5c pieces. How much money do I have?	**i. Partition (Sharing)** (a) There are 8 children and 2 tables in the classroom. How many children are seated at each table? (b) 6 drinks were shared equally between 3 children. How many drinks did they have each?
ii. Factor John has 3 books and Sue has 4 times as many. How many books does Sue have?	**ii. Factor** Simone has 9 books and this is 3 times as many as Lisa. How many books does Lisa have?
iii. Rate If you need 5c to buy one sticker, how much money do you need to buy 2 stickers?	**iii. Rate** Peter bought 4 lollies with 20c. If each lolly cost the same price, how much did one lolly cost? How much did 2 lollies cost?
iv. Cartesian Product You can buy chicken chips or plain chips in small, medium or large packets. How many different choices (of packets) can you make?	**iv. Quotition** (a) There are 16 children, and 2 children are seated at each table. How many tables are there? (b) 12 toys are shared equally between the children. If they each had 3 toys, how many children were there?
v. Array There are 4 lines of children with 3 children in each line. How many children are there altogether?	**v. Sub-division** I have 3 apples to be shared evenly between 6 people. How much apple will each person get?

FACTORS PERTINENT TO CHILDREN'S RESPONSIVENESS IN SPATIAL PROBLEM-SOLVING ACTIVITIES

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THE NEED FOR RESEARCH INTO CHILDREN'S SPATIAL THINKING PROCESSES

Despite the research in psychology, on spatial abilities and its development (see Clements, 1981; Eliot, 1988; Lohman, Pellegrino, Alderton, and Regian, 1987), there is little evidence that ordinary school-based lessons, particularly in the primary school, can improve students' spatial thinking processes (Eliot, 1988; Lean, 1984). The term "spatial thinking processes" is used in this paper to include (a) visual imagery skills such as recognition of shapes, transforming shapes, and disembedding parts and shapes, (b) spatial conceptualising, and (c) the interaction of visual imagery with the early conceptualising of shapes.

If a study is to consider school-based learning experiences, then it is necessary to consider relevant factors of the classroom context. Several studies have shown the effectiveness of factors such as the use of concrete materials (for example, Suydam, 1986), and the use of language especially as it is involved in cooperative groups (for example, Dalton, 1985; Davidson, 1990). Yet there is some evidence that a dominant member of the group can prevent a correct solution being obtained (Stacey, 1989). On the other hand, advocates of teacher-led discussion emphasise the importance of the teacher challenging, focussing, and questioning (Cosgrove & Osborne, 1985). No matter how the experiences are provided, the issue of accessing and assessing spatial thinking processes needs considering and an overview of this issue was given in an earlier paper (Owens, 1990).

THE STUDY

The study reported in this paper consisted of two interrelated sections. In the first part of the study, the aim was to consider whether a series of spatial problem-solving activities would have a significant effect on performance on spatio-mathematical tasks. In addition, consideration was given to the effect on performance of other factors, namely classroom organisation, gender, and Year at school. An experimental design was used to investigate these effects.

In the second part of the study, consideration was given to the nature of the thinking processes, together with the effects of student-student and student-teacher interactions associated with different classroom organisations.

EXPERIMENTAL STUDY

Design

The study employed an experimental matched-group design in which subjects from the same class and in the same cluster as determined by their ranking on total pretest scores, were randomly allocated to one of three groups. Group 1 students worked individually on two-dimensional spatial problems and participated in whole-class, teacher-led discussions. Group 2 students worked in small cooperative groups on the same spatial problems but Group 3 students, who formed the "control" group, worked in small cooperative groups on a series of number problems. The same teacher (the author) taught all the students. Groups 2 and 3 were together in the classroom as they were both organised into cooperative groups. Group 1 students were taken separately but to maintain the same student/teacher ratio for all students, this group had twice as many students as the others.

All students completed a spatial thinking pretest, an introductory activity, and then ten spatial (or "control") activities over a six week period. After the learning sessions, the students completed a parallel posttest, and eight weeks later (six weeks for one school because of annual holidays) completed a parallel retention test. The testing consisted of items involving two-dimensional space and items requiring transfer of thinking to three-dimensional tasks. Analyses of covariance with pretest scores as covariates (PRE2D for the 2D tests but PRE2D and PRE3D for the 3D tests) were used to determine the effects of gender, Year at school (2 and 4), and Group (1, 2 and 3) on posttest and retention test scores.

Subjects

There were 190 students involved from three schools in Sydney with a high proportion of students with non-English speaking backgrounds. Both a Year 4 and a Year 2 class were involved from each school.

Test Instrument

Two parallel forms of a test were developed for the purpose of measuring spatial thinking processes. There are six subtests:

Subtest 1 requires respondents to recognise congruent shapes in rotated or reflected positions. Some of the items in this subtest are more easily solved by analytic than by holistic procedures.

Subtest 2 is concerned with recognition of shapes made by tessellations of a smaller shape.

Subtest 3 has three types of items. Two types have diagrams of configurations made from matches and respondents are asked to complete shape outlines by drawing the matches that need to be added, and, in the second type of item, to mark the matches that need to be taken away. For the other type of items respondents are required to recognise and trace over embedded shapes in outlines of configurations.

Subtest 4 is about the construction of models made of parts from memory and/or under transformation.

Subtest 5 contains items concerned with nets of three-dimensional shapes; in particular with the mental transformation of two-dimensional shapes by folding.

Subtest 6 is primarily concerned with the recognition of congruent angles in shapes in different orientations or contexts.

Following a Rasch analysis, there were 46 items used for the 2D test and six items used for the 3D test (subtest 5) (Owens, 1991).

The Learning Experiences

The spatial learning experiences involved ten sessions over a six week period (plus an introductory session) and covered a number of activities using commonly available materials. These materials were:

1. the seven-piece tangram - students were to find the similarities and differences of the shapes, make shapes from other shapes, make outlines of the shapes using sticks, sketch shapes and configurations, and compare and order the angles of the shapes;
2. pattern blocks - students were to make the shapes larger, compare angles of the shapes, and make the shape outlines with sticks;
3. pentomino shapes - students were to make them from square breadclips, find symmetries, and tessellate the shapes;
4. matchsticks - students were to make designs of squares and see shapes within designs.

The activities were designed to encourage the students to solve problems through discussion and the invoking of visual imagery and pertinent spatial concepts. The use of open-ended and multifaceted activities catered for the needs of students with a range of prior experiences and existing concepts. The activities were used to provide a basis for challenging students to reflect on and, whenever necessary, to modify existing concepts, images, and skills. In particular, it was expected that teacher/student and student/student interactions would encourage the use of language and communication to facilitate the further development of concepts.

Results

With the two spatial groups combined, analyses of covariance of the scores on the posttests (POST2D and POST3D) indicated no significant effect on the variance due to the three main variables or their interactions. However, as shown in Table 1, significant effects were shown on the two-dimensional retention test (RET2D). The improvement for the groups learning from spatial problems was significantly better on two-dimensional spatial thinking than the "control" group learning from number problems.

A similar effect was not found with the three-dimensional retention test scores (see Table 1). This is consistent with the study by Rowe (1982) who found that students in Grade 7 who undertook a series of lessons on either 2D or 3D tasks improved in their scores on 2D items but not 3D items.

Table 1: Analysis of Covariance of Retention of Spatial Thinking Processes; Spatial versus Number Groups

	two-dimensions		three-dimensions	
	F	signif. of F	F	signif. of F
main effects				
group	5.072	.026 *	1.389	.240
gender	0.037	.848	4.146	.043 *
year	0.080	.778	0.550	.459
2-way interactions				
group gender	0.647	.423	0.228	.634
group year	1.576	.211	0.719	.398
gender year	2.248	.136	0.408	.524
3-way interactions	0.011	.916	0.588	.444

Note: * significant at the 0.05 level

Table 1 also indicates that the amount of variance in scores on the 3D retention test explained by gender was significant at the 0.05 level. Males improved more than females except for the Year 4 girls who were learning as cooperative groups but none of the comparisons for groups by year were significant at the 0.05 level as the intragroup variance is larger than the between group variance. The gender and gender by classroom organisation effects need further investigation.

There were no differences in improvement between the two groups using the different classroom organisations (an analysis of covariance was used for the analysis comparing just the two groups involved in spatial activities).

THE QUALITATIVE STUDY

Purpose

Although the experimental study showed that spatial learning experiences did affect children's spatial thinking processes, it did not explain **how** the learning experiences helped students to develop their spatial thinking processes or **how** they solved the spatial problems. In order to determine the relevant factors and their relationships, the qualitative study was undertaken. Earlier training studies did not provide this kind of data and did not explain how the children's spatial concepts, imagery, and other cognitive processes were used and developed.

Categories pertinent to spatial problem solving were developed from written records of tertiary students and observation of primary students working individually (Owens, 1990). These categories were the basis of the analysis using data from twelve children (four groups of three) working individually near others or as a cooperative group on all the problems and children in classrooms in Australia and Papua New Guinea.

Procedures Used in the Qualitative Study

In order to carry out this study, it was necessary to use mainly observational data, with the use of video-recordings, to decide on apparent cognitive processes. Some verbal support was given to these observations either by the comments students made while problem solving or in recalling their thinking on viewing the video playback. Each incident in which a significant development occurred was described and then categorised. The categories and subcategories used for aspects of interaction and cognitive processing are given in Appendix 1. The names of the major categories of cognitive processing, that is, imagery, concepts, heuristics, and affective processes, were influenced by Goldin's (1987) model of problem solving. The names of the visual imagery subcategories - concrete, dynamic, action, pattern, and procedural - were mostly influenced by Presmeg (1986) but modified to accommodate the approaches suggested by information processing theories, the comments of students, and the observational data. Other researchers, with experience in this field, agreed with the kind of cognitive processes indicated by certain behaviours and certain incidences as recorded on the video-tapes. The exploratory nature of this study into factors which influence spatial learning required a holistic approach and episodes were considered as a whole in deciding the best way to describe the learning.

The Importance of Interactions and Responsiveness

Based on the analysis of the video-data, the following description of students' participation in spatial tasks was developed. When a student was engaged in spatial problem-solving activities, there were inevitable interactions between the person, the influences of the context on the person and, in turn, the person's responsiveness. This is illustrated in Figure 1.

The cyclical interactions continued and changed throughout the period of engagement in the activity. Interaction normally encouraged further responsiveness as the learner became increasingly more involved in the problem-solving activities. There was a "snowballing" effect, not only on participation, but also on the extent and quality of imagery, concepts, understandings, and tactical approaches. Students would frequently monitor their own progress towards a solution.

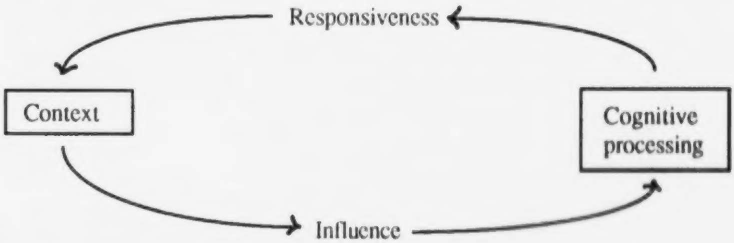


Figure 1: Interaction between context and cognitive processing

The term "responsiveness" is used to suggest that students responded to the task, the materials, and the other components of the environment by becoming actively involved in the problem-solving activity. Responsiveness implies a degree of understanding of the situation as well as involvement and interest in the activity. Responsiveness is associated with a two-way relationship, that is a dynamic interplay between a student and the environment which comprises fellow students, the teacher, the classroom, the task, and the materials. For the student, there is a cycle of thinking, feeling and responding, then further thinking, feeling and responding.

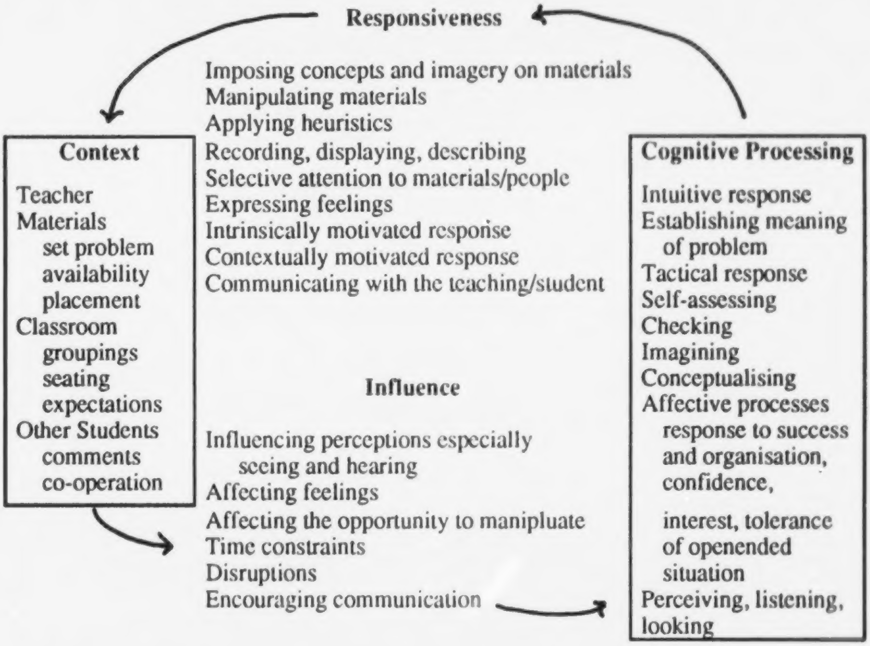

Responsiveness
Imposing concepts and imagery on materials
Manipulating materials
Applying heuristics
Recording, displaying, describing
Selective attention to materials/people
Expressing feelings
Intrinsically motivated response
Contextually motivated response
Communicating with the teaching/student
Context
Teacher
Materials
set problem
availability
placement
Classroom
groupings
seating
expectations
Other Students
comments
co-operation
Influence
Influencing perceptions especially seeing and hearing
Affecting feelings
Affecting the opportunity to manipluate
Time constraints
Disruptions
Encouraging communication
Cognitive Processing
Intuitive response
Establishing meaning of problem
Tactical response
Self-assessing
Checking
Imagining
Conceptualising
Affective processes
response to success and organisation, confidence,
interest, tolerance of openended situation
Perceiving, listening, looking

Figure 2: Aspects of the factors involved in the problem-solving process.

The typical response by the children to a task was one of engagement with the materials. Although there was some degree of uncertainty about what was expected, students were positively motivated to participate, engagement was maintained because of interaction with the materials and with others. This was the case despite differences in confidence, intuition, spatial thinking, and classroom organisation.

The responsiveness of students to the overall environment was a core phenomenon to which other phenomena can be related. Figure 2 lists aspects of each of the concepts used in Figure 1.

Several excerpts from the descriptions of the video-data of some of the Year 2 children will be used to illustrate particular aspects of responsiveness, namely: (a) how children imposed their concepts and images onto the materials that they were manipulating, (b) how they established meaning and tactics for the activity, (c) how they monitored their progress, and

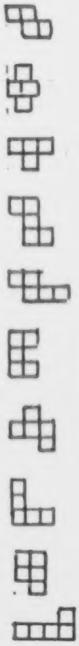
(d) how they expressed their feelings. Each episode captured on video is described in the present tense as it happened. Numbers are used to simplify reference to particular incidents.

Responsiveness - Its Causes and Effects

Each task in the investigation involved the use of materials and, therefore, actions with these materials assisted students to construct concepts and visualisations.

Although James was part of a cooperative group, he began the second spatial activity somewhat competitively. He was thoroughly involved in making new shapes from four square breadclips and then in making pentominoes from five squares. He also enjoyed commenting and in other ways expressing his achievements and feelings of pleasure.

1.01 James continues to count how many he has made, comparing his number with his friend's number.



1.02 Using four squares, he makes a Z, checks that it is all right and then makes a cross avoiding repeating the Z

1.03 His friend points out "it is half here," so he changes it to a T.

1.04 He begins with five squares, deliberately positioning the pieces to make a Z. Then he makes a lineZ.

1.05 He notes his friend's shape saying "yours has three columns, mine has two; she copied me". (The same shapes were made in different orientations.)

1.06 Despite the teacher suggesting that they work together, he keeps making shapes quickly and happily, commenting on how well he is going. He uses a tactic of beginning a new shape with three-in-a-line. He counts his shapes and says "I'm beating her." He knows what he is making before he completes the shape, showing joy before he finishes making the shape. He places three-in-a-row, and claps as he makes a C.

1.07 He cannot recognise the "odd" shape in different orientations despite moving his body to assist orientation. Otherwise he changes the shapes to make easily-recognised shapes such as L3, and the near-square comparing the incomplete shapes with his short-term memory images of those he has made (that is, he is not physically glancing at his shapes).

1.08 He changes his tactic from starting with three-in-a-line to beginning with four-in-a-line.

1.09 He quickly grabs the last five breadclips so that he can make another shape.

1.10 He wants to make a car but ends up with lineZ, considers it, and says "Oh, I can't make any more." His degree of activity wanes as the teacher points out the repeats in the group's work.

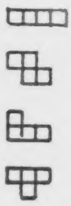
1.11 He recognises the repeated lineZ and L4.

There are several points to note about James' responsiveness. First, a friendly competition existed between the students and this motivated him to participate and achieve (para. 1.01 and 1.06). Certain affective characteristics are evident in his behaviour - his responses to his successes (para. 1.01 and 1.06), his competitiveness (1.01, 1.06, and 1.09), and his loss of interest at the end (para. 1.10). Second, James' use of imagery influenced his responsiveness - not only his manipulation of materials (para. 1.03, 1.04, 1.07 and 1.10) but also his comment to his friend (para. 1.05) and his self-assessments (para. 1.06, 1.10, and 1.11). His imagery helped him to stay on task (para. 1.06 and 1.10). Third, he assessed or monitored his own progress on the task and this, too, influenced his responsiveness. He expressed how he was progressing (para. 1.01 and 1.06), and he changed his tactic in an appropriate way (para. 1.08 and 1.10). Finally, he expressed his understanding and knowledge (para. 1.03, 1.04, 1.05, and 1.11). The changes in his cognitive understanding and his responses (para. 1.03 and 1.10/11) were precipitated by comments to him by his friend and the teacher. Thus we note how his understanding of the problem, his visual imaging together with student and teacher comments, his self-assessment, and his attitudes affected his responsiveness. At the same time, we can note how his visual imaging and tactics improved. In contrast, the imagery used by another student, Sam, was global while he was working on making outlines of the pattern-block and tangram shapes, although he showed some skill in getting sides with the correct slope. He chose simple shapes to start with, and juggled the trapezium and parallelogram shapes as if he was dependent on a global image.

Responsiveness Varies during Problem-Solving Activities

Responsiveness varies throughout the problem-solving activity as illustrated by the following excerpt about Lois. Her initial responsiveness was typical of students beginning a problem; that of unease. But she continued actively, making shapes and changing them until she seemed to work out what she was required to do to solve the problem, and then she systematically continued to make shapes. Lois was asked to make as many different tetromino shapes as possible with four square breadclips. She had her own pile of breadclips and she was working next to her friends.

3.01 Lois makes a line turning pieces; she is uneasy. She looks at her friend who has made a line, then an L but he changes it. She hears her friend say "I've made a pattern. I know, I'll make a square." She makes another line and looks at her friend's work.



3.02 Then she makes two Z's in different orientations and gives her characteristic "erm"- a quiet exclamation of joy.

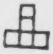
3.03 As usual, she begins a new shape with two squares and says, "I know." She temporarily makes an L but changes it to a T.

3.04 She listens to the teacher ask her friend if he has some shapes that are the same, and she leans over to point out her friend's two T's. When he points out her two Z's, she wants to leave them; "that's the up way", she claims and when he mucks one up, she is cross.

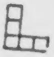
3.05 She makes several L's calling one "a flag" before she has made it.

3.06 She listens to the teacher discussing with her friend why he didn't make the square but she doesn't make a square herself.

3.07 Then the teacher tells them to begin again making shapes with five square breadclips at a time. She quickly falls into the same tactic that she previously used of putting down two pieces, adding a third and then with a piece in each hand systematically moving the pieces around until she sees a new shape. The first shape she makes is a T upside-down.

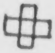


3.08 She puts down two squares, and thinks about where to place the third leaving it so she has the three-in-a-row. She then places a fourth to make an L and then temporarily places the fifth to form the square-like shape before shifting it to make L3.

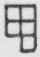


3.09 She looks at her friend's shapes.

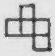
3.10 Next she uses two again, the third in a three-in-an-L and with a piece in each hand, places them to make the square-like shape. However, she is not happy with this and proceeds to place them to form a cross. She gives another expression of joy, "erm".



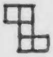
3.11 She quickly collects some more breadclips. She places, two, the third in an L, the fourth to make a square and the fifth to make the square-like shape. This time she leaves it.



3.12 She collects more breadclips; she places two as usual, then the third to make the L, and then two together to make the odd shape, which she decides to keep.



3.13 She continues on as if settled into a routine, occasionally looking at her friend's shapes. She makes the square-like one upside down, contemplates it and moves the last piece to give her a Z which she checks is not the same as the odd shape.



Lois' responsiveness was a little guarded at first (para. 3.01) and she looked at her friend's work for reassurance. Even after discussions with her friends, she appeared to have her own agenda judging by what she kept and what she discarded as inappropriate (para. 3.03, 3.04, and 3.06). However, she continued with a fairly consistent tactic making shapes and checking their acceptance (para. 3.07 and on). There was slight competition between her and one of her friends. Although she worked independently, she appeared comfortable with looking at her friends' work and making comments to them (para. 3.04). Imagery appeared to play a part in her work guiding her manipulation of pieces and her creation of shapes that she named or described (par. 3.03, 3.04, and 3.05). Imagery involved the evocation of holistic pictures, but the imagery became more operational, that is consisted of movements of pieces, and more analytical as the problem-solving continued and she evaluated whether a shape was acceptable. She appeared to be working to images of shapes although she sometimes rejected these shapes after making them (para. 3.03 and 3.05).

Some General Comments

Among the cognitive proceses that developed during the problem solving were visual imagery, conceptual processing and heuristic processing. All the students were keen to play with the materials to make shapes; they tried to meet the problem as set by the teacher; and they interacted with the teacher or other students if they were unsure of what to do or if they were enjoying their success. It was clear that students were manipulating materials deliberately; that is there was some thought directing the actions. As they worked, their manipulations suggested that they were becoming more systematic and more analytical as their imagery and ideas developed. They often monitored their own thinking and progress.

The two different organisations of students did not determine a clear pattern of student-student and student-teacher interaction for there was considerable variation in the amount of interactions with the teacher and fellow students. However, the spreadsheet analyses did show some differences between the groups. For example, there were more comments to the teacher based on holistic imagery and concepts in the classrooms where students were working individually. In addition, student-initiated comments to fellow students and the teacher were high in both groups and were generally either expressing affective factors or thoughts that were more analytical than holistic. There were more comments in the cooperative groups on what to do with pieces and who should be using the pieces.

CONCLUSION

The study indicated that children use a wide range of thinking processes in solving spatial problems and that classroom spatial activities improve and enrich students' spatial thinking processes. Among the thinking processes involved in problem solving are visual imagery, conceptualising, and heuristics. Their concepts, both relating to appropriate actions and to shapes, and visual imagery influenced their attention to details of the materials. Students evaluated their progress through the problem and frequently expressed their self-assessment. Affective processes, in particular enjoyment at success and attitudes to the group, also played a part in their responsiveness. Of particular importance was the effect of interactions with both the materials and other people in encouraging students' responsiveness and their development of spatial and other thinking processes.

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THE USE OF THE REPRESENTATIVENESS AND AVAILABILITY HEURISTIC IN PROBABILISTIC REASONING BY STUDENTS IN THE GRADUATE DIPLOMA OF EDUCATION QUT, 1992

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BACKGROUND

"Probability as the basis for decision making" is a topic of the new Queensland Senior Maths Syllabus. This is as part of what is referred to as "mathematics for intelligent citizenship" and is core for all students.

Misconceptions in probabilistic reasoning are widely reported in the literature. Tversky and Kahneman (1982) report on the use of "representativeness" and "availability" by educated professional people. Shaughnessy (1977, 1981) reports on the use of the same heuristics by university students in the USA and Peard (1991) cites the occurrence of these in secondary school students in Australia. Scholz (1987) examines the misuse of representativeness by a large cross section of German society. He includes an extensive examination of what is referred to as "The base-rate fallacy". Bon-Hillel (1982) reports on follow-up studies to Tversky and Kahneman.

Other misconceptions in probabilistic reasoning include the occurrence of "counter intuitive results" (Peard 1991), misconceptions of regression and illusory correlation (Tversky and Kahneman, 1982).

It is the contention of the author that of all these misconceptions, representativeness and availability appear to be the most widely spread.

Shaughnessy (1981) defines representativeness. This term is used by "those who estimate the likelihood of an event on the basis of how similar the event is to the population from which it is drawn" (p.91). This results in a fallacy by which likely sequences are viewed as unequally likely. This can occur in two ways. For example, a subject observes a "fair" coin land heads on eight out of ten tosses, and then incorrectly reasons that the likelihood of a head on the next toss is less than one half. The subject is reasoning that a balance is expected (i.e. representative) even in the shortrun when in fact no such balance is expected. This is commonly known as "belief in the gambler's fallacy of the law of averages" (del Mas & Bart, 1989).

In the second instance, the subject erroneously reasons that a head is more likely using the preceding short run of outcomes as being a representative base rate.

Tversky and Kahneman (1982) submit that:

> people view a sample randomly drawn from a population as highly representative, that is similar to the population in all essential characteristics. Subjects act as if every segment of the random sequence must reflect the true proportion: if the sequence has strayed from the population proportion, a

> corrective bias in the other direction is expected. This has been called the gambler's fallacy. The heart of the gambler's fallacy is a misconception of the fairness of the laws of chance. The gambler feels that the fairness of the coin entitles him to expect that any deviation in one direction will soon be cancelled by a corresponding deviation in the other.

Whereas, in reality,

> deviations are not cancelled as sampling proceeds, they are merely diluted. (p. 238)

They refer to this as the belief in the law of small numbers. The publication of lotto number frequencies of occurrence in Australian newspapers is evidence of the widespread nature of this misconception. Shaughnessy (1981) refers to availability as happening:

> When people tend to make predictions about the likelihood of an event based on the ease with which instances of that event can be constructed, or called to mind, they are relying on availability. For example, if a person is asked to estimate the local divorce rate or to estimate the probability of being involved in an automobile accident, the frequency of his or her personal contact with these events (perhaps through friends or relatives) may influence the probability estimates. One who has recently been involved in an accident might estimate a high probability for the accident rate. One who has many friends and acquaintances who are divorced has more instances of divorce "available" and may estimate a higher local divorce rate than someone who does not know many divorced people.
>
> Availability causes systematic bias in probability estimates because people tend to believe that those outcomes that can easily be brought to mind will be more likely to occur.

For example, in buying prenumbered lottery tickets, if people tend to avoid numbers such as 123456 or 699999, it is because they cannot easily recall such numbers occurring, whereas they can recall "unpatterned" numbers readily and thus believe that any such number is more likely.

RESEARCH

This research examines the heuristic of representativeness and availability. The subjects were a sample of 24 students in the Graduate Diploma of Education (Mathematics and Science) at QUT, Semester 1, 1992. These all have some formal education in mathematics and most will presumably teach mathematics in Queensland schools - including the topics in probability.

A control group of 22 students was selected from The Graduate Diploma of Education in fields other than mathematics. These have no tertiary mathematics education.

The aims of the research were to determine:

1. how prevalent the misconceptions of representativeness and availability are amongst graduate students in education; and

2. whether or not our future mathematics teachers differ in the use of these heuristics from others.

Both groups of students were asked the same five questions (see Appendix). All were asked to attempt all questions, using intuitive knowledge if no probabilistic knowledge was available. The methods employed in Question 5 were discussed individually with approximately half of each group.

At the recent (May, 1992) QAMT annual conference, question 5 was asked a group of practising secondary mathematics teachers. Their responses are shown also.

ANALYSIS AND INTERPRETATION OF RESULTS

The proportion demonstrating the heuristics in a relatively simple situation (Question 1 and 2) is low. We further note that it is consistently lower for the Maths/Science group than for the others. However, statistical tests for significance of difference (Chi-squared) imply that further research using larger samples is required before any conclusions with a high degree of confidence can be made. Nevertheless, the results are interesting and show that Maths/Science graduates (at least in this sample) tend not to use either heuristic in these simple situations. The use by non-Maths/Science graduates is still relatively low when compared with reports in the literature stating the widespread use. It is further interesting to note that the usage by this group is similar to that of high school students reported in earlier research (Peard, 1991).

In Questions 3 and 4 in which the concepts are more "sophisticated" we see that there is no difference between the two groups. Further, the proportions are in general agreement with results obtained by Shaughnessy (1981) on identical questions asked to a sample of 80 first year college students in the USA. This is cause for some concern, since question 3 can be answered correctly by considering probability from combinations as is done in many senior secondary mathematics cases. These results tend to confirm "mere exposure to the theoretical laws of probability may not be significant to overcome misconceptions of probalility." [Shaughnessey (1981) p. 91]

The results of Question 4 imply that the use of availability in this situation is very evident. Interviews with subjects showed that of seven answering correctly only three (all Maths/Science) were reasoning along the correct lines of a small sample being less likely to reflect the population.

Question 5 is the really interesting one. Here we see that roughly half of each group ignore the base rate. This is consistent with extension research by Scholtz (1987) who reported that the model response of all groups (professionals, graduates, non-graduates, lawyers, teachers, doctors and others) was "80%".

Of our group only one was correct and this one had a good background in statistics and recognised the question as a Bayesian conditional probability and solved it correctly. Six others, with some background in probability (all from the Maths/Science group) attempted a probabilistic solution and arrived at an incorrect (and quite obviously so) answer. Five employed the multiplication principle incorrectly $0{\cdot}15 \times 0{\cdot}8 = 0{\cdot}12$ whilst one attempted to use Bayes' formula but arrived at an answer of 1%. On the other hand, the non-

Maths/Science graduates were more prepared to admit that they were not sure or had no idea, with nearly one third giving this response compared to none of the other group. I think that this has implications for our curriculum classes at uiversity.

We note further that of the four who ignored the witness and relied only on the base rate, three were from the Maths/Science group. Of the six who recognised that the base rate does in fact reduce the probability from 80% (responses to 50 - 80 or <50), four were from the non-Maths/Science group.

Practising secondary teachers of mathematics fared little better, although now 8% correctly used a tree diagram to arrive at the correct answers. A smaller proportion (4%) used the multiplication principle incorrectly and 12% ignored the base rate. One responded >80% demonstrating a response that was more prevalent amongst Scholtz's subjects, in which the respondent's tendency is to "believe the witness".

CONCLUSIONS

I believe that the results have implications for university curriculum units. We must include and give particular attention to the teaching of probability. The concepts required to answer questions 3, 4, and 5 correctly are not beyond basic probability and well within what is required by most secondary mathematics syllabi. Nevertheless, our prospective secondary mathematics teachers are no better prepared to answer them correctly and fall back on the heuristics that are evident amongst the general population.

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QUESTION AND RESPONSES

1(a) In the last 200 Lotto draws the number 17 has occurred more times than any other number. In future draws, is this number:	TOTAL	GRADUATES IN MATHS/SCIENCE	OTHER GRADUATES
(i) more likely to occur than any other number?	4	0	4
(ii) less likely to occur than any other number?	3	1	2
(iii) equally likely to occur as any other number?	38	22	16
(iv) Not sure.	1	1	0
% demonstrating representativeness	**17%**	**8%**	**27%**
1(b)			
A roulette wheel has 18 red and 18 black numbers. The probability of the ball landing on a red is the same as landing on a black. A gambler observes the ball to land on the following numbers in a row of six: RRRRRR. If he decides to be on the next throw is it more likely to be:			
(i) red	2	1	1
(ii) black	7	1	6
(iii) red and black are equally likely	36	21	15
(iv) not sure	1	1	-
% demonstrating representativeness	**20%**	**12%**	**32%**
1(c)			
If you could choose any of the following lottery tickets, in a lottery that has the numbers written on the ticket you buy, would you have any preference for:			
(i) 123456	2	0	2
(ii) 619999	0	0	0
(iii) 615472	9	2	7
(iv) no preference for any of these	35	22	13
% demonstrating availability	**26%**	**8%**	**32%**

2		TOTAL	GRADUATES IN MATHS/SCIENCE	OTHER GRADUATES
(a)	If I throw a single fair die, are the numbers one through six equally likely to occur?			
	No	43	24	19
	Yes	3	0	3
	(availability) %	7	0	14
(b)	If I thow a pair of dice, it is harder to get a pair of sixes than it is to get a pair of twos?			
	No	43	24	19
	Yes	3	0	3
	(availability) %	7	0	14
(c)	A poker hand consists of five cards randomly dealt from the deck of 52. Ace is the highest card. Is it harder (i.e. less likely) to get 3 Aces than it is to get three fives?			
	No	41	23	18
	Yes	5	1	4
	(availability) %	11	4	18
(d)	You and I play a game in which a coin is tossed. Heads I win. Tails you win. Of the last 15 people I played only 5 won. Is the game fair? Is the coin fair?			
	Game Yes	38	22	16
	No	8	2	6
	Coin Yes	38	22	16
	No	8	2	6
	(representativeness) %	17	8	27
3				
A person must select committees from a group of 10 people. (A person may serve on more than one committee). Would there be				
(a)	more distinct possible committees of 8 people?	6	1	5
(b)	more distinct possible committees of 2 people?	21	12	9
(c)	about the same number of committees of 8 as committees of 2?	16	10	6
(d)	not sure.	3	1	2
% demonstrating availability (selecting (b))		**46**	**50**	**41**

	TOTAL	GRADUATES IN MATHS/SCIENCE	OTHER GRADUATES
4			
The chance that a baby will be a boy is about $^1/_2$. Over the course of an entire year, would there be more days when at least 60 percent of the babies born were boys -			
(a) in a large hospital	3	1	2
(b) in a small hospital	7	5	2
(c) makes no difference	33	17	16
(d) not sure	3	1	2
% demonstrating availability	**72**	**71**	**73**
5			
In a certain town there are two types of taxis; 15% are Blue cabs and 85% are Green cabs. A taxi is involved in a hit-run accident. A witness identifies it as a Blue cab. The court tests the witness' ability to distinguish between the two under identical conditions and concludes that the witness is right 80% of the time.			
Question: What is the probability that the cab was a Blue cab?			
%	0	0	0
>80	22(48%)	12(50%)	10(45%)
80	4(9%)	1(4%)	3(14%)
50-80	2(4%)	1(4%)	1(5%)
<50	1(2%)	1(4%)	0
41*	4(8%)	3(13%)	1(5%)
15**	5(10%)	5(21%)	0
12*	1(2%)	1(4%)	0
<1	7(15%)	0	7(32%)
not sure/no idea			
Practising secondary mathematics teachers			
>80		1	
80			
50-80			
<50		3	
41			
15			
12			
<1			
not sure / no idea			

TRANSITION FROM SENIOR SECONDARY TO UNIVERSITY MATHEMATICS: A CASE STUDY

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INTRODUCTION

In Thai secondary school mathematics occupies a central place in the curriculum, and this reflects the vital role the subject is seen to have in the task of preparing more productive citizens. There is a common belief that mathematics is the exemplar of precise, abstract and elegant thought. Indeed, there appears to be an almost universal consensus that the study of mathematics helps to broaden and hone intellectual abilities (Travers and Westbury, 1987). From a societal perspective, mathematical competence is seen as the cornerstone of a numerate society, and that therefore the industrial, technological, and scientific future of a nation depends on more children going on to study higher mathematics than ever before.

The need to establish equitable criteria for the selection of students for entry to mathematics and science programs in Thai universities is the major background issue influencing the direction of this present study. Why do so many school graduates find the entrance examination test difficult? (Makpoon, 1989; Patitas, 1989; Premto, 1987) And why have from 30 to 60 percent of first-year students at Khon Kean University always failed examinations in mathematics courses? (Khon Kaen University, Faculty of Science, 1989). One possible explanation is that students admitted to the scientific and technological courses at Khon Kaen University experience difficulty in the first-year mathematics courses because they are not well prepared in the secondary schools for these courses (see Pongboriboon, Somsoporn, and Krewawarn, 1989).

Morgan, (1989) after analysing the results of national and international surveys, concluded that there is a common message that mathematics appears to be a difficult subject to teach and to learn and that there is great concern about the overall level of mathematical competence. Also there is a concern that a large proportion of the students who enter university come poorly trained in mathematics.

Clearly, there are a number of factors that play important roles in mathematical learning. A study by Lorenz (1982) found that ability is the most important factor determining mathematics achievement, with the second most important factor being motivation. The attainability of higher education is also affected by the way in which individual attributes act in combination with characteristics of a young person's situation. The performance of Thai students in mathematics should relate to their academic environment, ability background, study habits, and the teaching competence of their mathematics teachers.

This paper addresses the question of why so many first-year students at Khon Kean University (hereafter referred to as KKU) experience difficulties in coping with their mathematics courses. Factors apparently directly related to difficulties associated with students' transition from senior secondary school to university mathematics are a special focus of the study. Also, the varying extents to which school teachers, university lecturers, and first-year students differ in their perceptions of how school and university mathematics courses should be related are studied.

PROCEDURE

In order that a deeper understanding of the importance of variables affecting performance on first-year mathematics courses at KKU might be obtained, an investigation involving the collection and analysis of both quantitative and qualitative data was carried out. The investigation focussed especially on the difficulties experienced by first-year mathematics students at KKU.

THE SAMPLE

A small group of senior secondary school mathematics teachers, university mathematics lecturers, and first-year mathematics students from six faculties at KKU were interviewed during the first semester of 1990. This group consisted of:

1. twelve school mathematics teachers located in twelve schools - six in the Khon Kaen city area and six in rural areas.

2 ten mathematics lecturers who taught first-year mathematics courses in the first semester, 1990, at KKU; and

3. twenty-four first-year KKU mathematics students, comprising four students from each of six faculties (Medicine, Nursing, Agriculture, Engineering, Science, and Education). The students from each faculty were selected by using mid-term mathematics examination performance as the criterion, one student having high achievement, two average achievement, and the fourth, low achievement

RESEARCH INSTRUMENTS

Measures of the mathematics achievement of students beginning mathematics studies in six faculties were obtained by using a school mathematics achievement (SMA) test. This pencil-and-paper, multiple-choice instrument was constructed by the present writer with the assistance of a group of experienced senior secondary school mathematics teachers. It was intended that scores on the SMA test would not only reflect individuals' understanding of senior secondary mathematics, but also how well schools had enabled students to acquire the basic skills needed for effective mathematical study in tertiary institutions.

Other information on the extent of the mathematics known by beginning first-year KKU students was provided by students' scores on one of the standard entrance examinations in mathematics, namely the Direct Entrance Examination Mathematics (DEEM) test, administered by KKU, and National Entrance Examination Mathematics (NEEM) test, administered by the Thai Ministry of University Affairs.

In order that a deeper understanding of the difficulties in mathematics experienced by first-year KKU students might be arrived at, interviews with senior secondary mathematics teachers, and first-year KKU mathematics students and lecturers were conducted by the author during the first semester of 1990. These interviews were based on a structured questionnaire schedule, the aim being to identify causes of student difficulties in first-year mathematics. In the interviews special attention was given to identifying difficulties that might be directly associated with the transition from senior high school to university mathematical study.

DATA ANALYSIS

The students' mathematical abilities at the beginning of their university study were measured by scores obtained on the SMA (N=946), DEEM (N=451), and NEEM (N=471) tests. These scores were analysed using standard techniques in which mean scores and standard deviations became the basis for comparison. The Mathematical performances of student taking first-year programs in six different faculties at KKU, together with interview data, were used for the purpose of examining the students' mathematics abilities and their willingness and preparation for university mathematical study.. The interview data were analysed and classified into two main areas: (a) those which threw light on the source of students' difficulties; and (b) the views held by the school teachers, the students, and the KKU lecturers on relationships between school and university mathematics, and on other issues that might be expected to influence performance on, and attitude towards, first-year mathematics at KKU.

RESULTS

Analysis of the quantitative and qualitative data indicated that there were six major influences affecting the first-year KKU mathematics performance. These major influences were:

1. the students' mathematical abilities at the beginning of the University year;
2. curriculum content;
3. course organisation;
4. the students' study habits;
5. instructional styles; and
6. assessment procedures.

The influence of each of these factors will now be summarised.

THE STUDENTS' MATHEMATICAL ABILITIES AT THE BEGINNING OF THE UNIVERSITY YEAR

A number of selected test items from the DEEM, NEEM and SMA instruments were given to the interview sample to consider from the point of view of test validity. Approximately one-fourth of the items on each instrument (12 from 50 on DEEM, 10 from 48 on NEEM. and 10 and 7 from 30 items on the SMA test) were chosen for this purpose, the criterion for item selection being the set of items chosen should cover the major themes in senior secondary mathematics in Thailand.

The interviewees reported that the test items had content validity, that is to say the three instruments were strongly related to the mathematical content and skills emphasised in senior secondary schools in Thailand. Also, according to the interviewees, most of the selected test items were of suitable difficulty, although a few of the items on the entrance examination tests (DEEM and NEEM) were deemed to be quite complex.

The mean DEEM and NEEM scores obtained by the students in the main sample were 43.8 and 32.2 respectively (with a maximum possible score of 100 for each examination), and 13.3 for the SMA test (with a maximum possible score of 30). Since it appeared to be the case that the three instruments had strong content validity, so far as senior secondary mathematics was concerned, and few if any of the items were regarded by teachers as unreasonably difficult, these results can hardly be regarded as satisfactory. Indeed, it could fairly be concluded that many students in the sample lacked the basic mathematical skills and knowledge needed to cope with first-year mathematics courses at KKU.

At the examinations taken at the end of the first KKU semester in 1990, 27.2% of the first-year university mathematics students in the main sample for this study obtained an F grade, and 32.4% obtained a D grade (F was the lowest possible grade and is regarded as a "Fail"; and D was the second lowest grade, and was regarded as a borderline Pass). Therefore, there is strong evidence that at the first semester examinations in 1990, more than half of the students in the main sample failed to demonstrate a sound understanding of the mathematics that they had studied during the semester.

The data finding suggested that most of the students in the main sample were not well prepared, from the point of view of having necessary and desirable mathematical knowledge and skills, to take first-year mathematics courses at KKU. Consequently, many of the students were unable to cope with first-year mathematics courses at KKU, and they did not perform well on the first semester mathematics examinations In other words, students who are not well mathematically well prepared for first-year mathematics courses will not gain much benefit by taking the courses - mere participation in the course did not seem to remedy their mathematical weaknesses or improve their mathematics competence.

Results from the interviews with the school mathematics teachers, the university mathematics lecturers, and the mathematics students revealed further insights into the difficulties students experienced in coping with first-year university mathematics. In particular, these brought out the relationships conceived by each group between the school and university levels in terms of the curriculum content, the course organisation, and the study habits acquired by students in school and university settings.

CURRICULUM CONTENT

There was agreement from all groups interviewed that the mathematics curriculum for senior high school covered most of the content areas which are necessary for further mathematics at the higher education level. However, some content, such as *vectors* and *calculus*, was felt to be in need of clearer explanation in the mathematics textbooks provided for the school curriculum. Limitations on the time available for instruction also meant that it was difficult to cope sufficiently with crucial concepts related to these topics. The teachers also reported that students frequently manipulated the mathematics in a superficial way rather than gain a depth of understanding of the content, and that this is a possible cause of some of the difficulties experienced by students at the university.

There were, however, two topics which are not taught at each of the school and university levels. *Groups*, a part of the mathematical content in Year 10 does not directly relate to any content in the first-year university mathematical course, while *Mathematical induction* which is a component of 314 111 Mathematics 1 for Science and Education faculty students was not considered at the school level. In addition, it was felt that the school mathematics textbooks provided very simple explanations and straightforward approaches to the content in such a way that these probably contributed to the development of conceptual blockages in students. Consequently, most students discovered that the mathematics syllabus at the university level was pitched at a level that appeared to be unreasonable in terms of their knowledge and experience and far higher conceptually than they were able to accommodate.

In summary, the interview data revealed that there was a reasonable degree of articulation between the mathematics curricula at the school and the university at the topic level. However, a lack of precise details and complete explanations provided insufficient depth of understanding of basic concepts, knowledge and skills sufficient for the *conceptual preparation* required for university study in mathematics. Further, at the university, lectures were based on the assumption that the school graduates were familiar with all the knowledge and skills provided in the school mathematics curriculum. Yet the test data revealed that, generally, this was far from the case. Discrepancies between the two curricula at senior secondary school and first-year university in terms of content and conceptual understanding were a further source of difficulties for students trying to cope with their first-year mathematics courses.

COURSE ORGANISATION

Class sizes were felt to be too large at both the school level, with 45 to 50 students per class, and in university lectures with 135 to 225 students. There were also commonly reported difficulties caused by the text material from which the mathematics was to be learned, although the Thai mathematics textbooks used in all high schools are published by the Ministry of Education. Teachers reported difficulties in comprehending the meaning due to inconsistencies in the use of the bold and italics words used to highlight important points and the unsystematic use of punctuation. For instance, the same number of spaces was used between clauses or sentences. As the Thai language does not have commas or full stops, but uses spaces to indicate punctuation, writing mathematics in Thai language requires particular care to avoid ambiguities and misunderstandings. Misprinted words were also found in both theorems and exercises.

At the University level, most students indicated that the structure of the written language in the course material produced by the Department of Mathematics was very difficult to comprehend without also attending the lecture classes. The only examples involved given procedures and the use of rules or concepts, rather than the concepts themselves, and some of the examples were not clear. In addition the students felt that examples could not be generalized to provide solutions for most of the exercises, so that they were unable to do many of the exercises. Some lecturers pointed out that reading the mathematics texts assumed a mental or cognitive structure in the students' mind corresponding to the mathematical context to be learned. They justified this by arguing that learning from texts in which all aspects were immediately recognizable by the students would prevent the students from learning to handle texts independently.

Tutorial programs were also provided by the Department of Mathematics during worktime and by the University after office time in the late afternoon and during weekends. However, it appeared that only a small number of students,15-35 percent, attended tutorials regularly. The students claimed that many did not attend because they expected to be asked questions which, if they had not prepared themselves by studying the previous mathematics lessons or by completing the exercises, they would be unable to answer. A particularly strong reason for non-attendance was that many students reported that they felt too tired after attending five-seven hours per day of normal classes. The university tutorial program which was intended to remedy students' mathematical deficiencies failed because it did not involve those who required assistance. The difficulties in attending were, in fact, generally made even harder because first-year students were recommended to register for seven-nine courses during each semester, requiring attendance at classes for about 19-24 hours per week.

THE STUDENTS' STUDY HABITS

Teachers reported that the senior secondary school students developed undesirable study habits at school mathematics classes. In particular, the students' concentration during mathematics classes was inconsistent. If the topic was understandable they would pay attention and try to do exercises by themselves, but if the topic was too difficult for them, their attention would be lost and they would give their attention to other matters. Most students also completed their homework exercises by simply copying the work of friends or from a mathematics guide. The students indicated that mathematics teachers directly taught every detail of the mathematical processes and that they practised mathematics exercises by simply following these steps. Consequently, they could not solve other mathematical problems which required different approaches.

Differences in the demands of mathematics courses between the school and university required students' to adjust their study habits, but they experienced difficulties in seeing the need for and achieving these changes. Lecturers and students felt that mathematics deficiencies of first-year university level resulted from:

1. Differences in students' study activities between high school and university level:
   (a) acting skills and dispositions necessary for autonomous study; and
   (b) a shift from superficial study to productive study practices

2 Undesirable study habits developed in university study level:
   (a) do not attend class and tutorial consistently;
   (b) allocate insufficient time to study and review content; and
   (c) do not enough practice on mathematics exercises.

INSTRUCTIONAL STYLES

Instructors at both school and university levels typically taught the whole class as a unit using a teacher-centred approach. The mathematics school teachers planned carefully for mathematical lessons, strictly following mathematics class textbooks to teach conceptual and procedural knowledge. The feedback on students' submitted school assignments focussed on correct or incorrect answers, but did not give adequate attention to any missing mathematical problem solving steps. The lecturers used the interrogative format as the instructional mode of choice during all phases of lessons, but did not expect and indeed moved on without receiving students' responses.

The first-year KKU students reported that the style of teaching mathematics at university was similar to the style they had experienced at high school level, that is, a content-focused lecture approach. At university, little or no class time was given to reviewing prerequisite mathematical concepts. University lecturers expected students to understand the mathematical topics which were listed in the senior high school syllabus. University lecturers would not review or clarify such concepts to provide a link to the new knowledge. The pace of university lecturing was very fast. Many students lost the factual ideas if they tried to interpret what was being said and integrate the new knowledge into their experience. In a mathematics lecture class, the students claimed that they were unable both to listen and to take notes, most settling for one or the other.

ASSESSMENT PROCEDURES

Each secondary school sets its own examinations, tests and assignments. The Ministry insists that the ninety percent of the total assessment come from mathematical achievement performance as indicated by testing and examination (three examination during the semester and the final examination), leaving ten percent of the total assessment for the teacher's perception of students' value to the subject. Most of the teachers used objective test styles, particularly multiple-choice test format, for the three examinations during and the examination at the end of the semester. In practice, for the remaining assessment, the school teachers used their scoring of work submitted by students instead of giving their perception of the ways students value the subject of mathematics, mostly awarding all students close to ten percent.

Students claimed that their school mathematics experience of looking for a correct answer in a multiple-choice item question, did not prepare them for the university requirement of demonstrating mathematics processes in a written test where all working had to be shown. In KKU mathematics courses, achievement was assessed by summing scores of two only performances: 40 percent from a mid-semester test and 60 percent from the final examination; no account was taken of performance on class exercises or assignments.

DISCUSSION

The students at the upper secondary school level learned details of mathematical knowledge, but could not comprehend and generalize to operational procedures. This finding confirmed some difficulties reported in the literature concerning the teaching of mathematics. Teaching materials and aids are needed to build concepts and abstract mathematics content.

In addition, it revealed a need to change the classroom situation from teacher-centred to student-centred so as to focus students' interest on the lessons.

Examining the perceptions of the first-year mathematics students, and the university mathematics lecturers on the instructional styles at university mathematics classes, it was felt that the traditional lecture, content-focused approach seemed to be appropriate for a large class of more than one hundred students. However, while the content-focused approach was considered appropriate to develop students' factual knowledge, it only brought about superficial understanding (Marty, 1988). The teaching style at the university could not improve students' proficiency in mathematics. The first-year mathematics students who carried their mathematical deficiency from schools needed remedial courses before entrance into university or before studying a university mathematics course.

In addition, the students asserted that their aim in studying mathematics at high school was to pass the school examinations and then more specifically to pass the university entry examinations, both of which were composed of multiple-choice items tests. Accordingly, the students learned to develop their operational skills directed to finding a path straight to the correct answers. They considered themselves as lacking deep understanding in many mathematics concepts, and lacking skills of mathematical operational processes. These mathematics deficiencies from the students' view were the results of their lack of practice in the mathematical operation and the results of schools using inappropriate evaluation tools.

Some suggestions on senior secondary mathematics course were made by both school teachers and students. They agreed on the following suggestions for reform in the secondary school mathematics curriculum:

1. Some curriculum topics should be reviewed. Content areas should be given more detailed explanation, and a range of examples giving applications of the theorems in some of these topics, such as *vectors*, and *calculus*. Overlapping of mathematics content among different years of study should be examined and remedied, for an example, *exponential function and logarithmic functions* in Year 11 and *exponents or indices* in Year 9.

2. Some topic contents should be rearranged:

    (a) In Year 10 mathematics curriculum, the topics are ordering from lines, functions, and conics should be changed to *lines*, *conics*, and *functions*, because the content in conics is rather related to lines and it should be taught continuously;

    (b) In Year 11 mathematics curriculum, it was suggested that the content in *complex number* should be reduced. Then the topic of *calculus* which needed to be taught more precisely could be moved from chapter 3 to replace chapter 1; and

    (c) All of the topics related to *elementary statistics* were taught in the second semester of Year 12. It was suggested that, instead, the content be grouped into a few main topics and then put separately into other mathematics year levels. In turn, this content could be replaced with other topics from the mathematics curriculum of Year 10 and Year 11.

Suggestions were also made for the first-year mathematics curriculum. In each content area, more exercises should be given and some of these should relate to every-day-life mathematical problem solving. However, the mathematics problems set in the examinations were much more difficult because they were not directly related to the type of exercise provided for practice. More difficult mathematical problems are also needed in the exercises provided in the text to *assist students to move towards the expectations* of the examinations.

In conclusion, the present author believed that this study makes three contributions. First, mathematics curricula in both senior secondary school and first-year levels should be reviewed and some rearrangement of topics carried out. Second, University mathematicians must play an active role in precollege education. Some articulation programs with co-operative partnerships between KKU mathematics department and high school mathematics teachers should be designed to improve school mathematics curriculum and instruction. The first-year mathematics student needs to be prepared for autonomous study in lecture note-taking, mathematics reading skills, and awareness of self- directed study at the university level. Third, the assessment criteria and procedures affect students' beliefs and they orient their study habits to meet the particular forms of assessment. At the school level, the teachers should not only give their students practice in solving mathematical problems in the form of exercises or homework, but also items where full working is expected, rather than solely a multiple-choice format, should be set for the examinations. At the university mathematical level, lecturers need to be aware that assessing an individual's achievement needs to be based on more than examinations and should include different learning experiences. Course assignments in individual or small group work will lead students to be more involved in reading, studying, and practising mathematics. Appropriate criteria for assessing mathematical achievement and performance should derive partly from examinations and partly from their own practice with mathematical activities.

The findings of this study also relate to the policies at KKU. The Department of Mathematics lacks sufficient academic staff to service mathematics as a required subject for all faculties of study in the science program. Lecturing or tutoring in mathematics in smaller class sizes produces more effective teaching and learning outcomes. Finally, given the students' reported workload of seven to nine courses a semester in first-year KKU, it is clear that a school graduate who comes to the different academic environment and harder conditions of the university, is almost certain to experience difficulties. The required number of courses expected of first-year students should be reconsidered by the faculty to allow a more reasonable workload. Failure in first year courses not only causes difficulties for a beginning student, it carries further penalties in succeeding years when these courses have to be made up and added to the load for subsequent years.

This study has revealed the situations and causes of difficulties in studying mathematics experienced by first-year mathematics students at the university. If some of the suggestions derived from the study could be implemented at KKU and/or at institutes which are responsible for senior secondary schools, this should alleviate some difficulties experienced in their first-year of university mathematics.

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COGNITIVE DIFFICULTIES IN EARLY ALGEBRA

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Over seven months, one group of Year 7 students showed little, if any, progress with algebra while another group improved markedly. The latter were significantly better in developing substitution skills, in identifying when two algebraic expressions were equal, and in understanding algebraic symbols as generalized numbers and/or variables and as representing numbers rather than objects. The former were significantly more inclined to persist with incorrect pre-algebra ideas and to interpret conjoining (e.g., in '2n') as addition. They were not coping with the new ideas being presented to them.

THE KEY TO PROGRESS

An important question, especially from a teacher's point of view, is the question of why some students do not show much or any progress despite the fact that they are present for the same lessons as other students who do progress, some of them quite rapidly. What is it that students really need to learn before they can show signs of general progress? Do they need to reach certain steps in a hierarchy of concepts and/or skills before they can make secure progress in other aspects of learning? One of the ways in which concern about the "slow learners" was addressed was to study their performance sequentially from one test to another in comparison with the "fast learners".

This paper is based on the results of four testing stages for 208 Year 7 students who completed the same test instrument three times during their first three weeks of algebra in 1990, and for the 186 of these who completed it a fourth time some six months after the teaching intervention sessions.

Analyses based on standards. To set criterion levels for following changes in performance from one test to the next, four equal ranges up to and including the maximum scored by these Year 7 students were chosen: 0 to 13, 14 to 27, 28 to 41 and 42 to 55. Scores within these ranges were called Standards One to Four and will be referred to as S1, S2, S3 and S4 respectively. The frequency distribution of students in each of these standard brackets is displayed in Table 1.

Table 1: Frequency Distribution of Year 7 Students in Standards Brackets

Standard	Range of Scores	Test			
		1	2	3	4
S1	0 - 13	158	71	24	23
S2	14 - 27	42	103	111	67
S3	28 - 41	8	30	59	76
S4	42 - 55	0	4	14	20
Totals		208	208	208	186

An investigation was carried out to find out if any students stayed at the lowest level (S1) in all four tests, and it was found that 15 actually did. There were two more students who were absent for the fourth test but remained in the S1 group for the first three tests. These 17 students, then, had shown very little, if any, improvement during the time period of the testing program, even though it included at least three weeks of introductory algebra teaching. Another 7 students stayed in the S2 bracket for all four tests. There were 18 students who, in contrast, had moved from either the S1 bracket (in 8 cases) or the S2 bracket (in 10 cases) to the top S4 bracket during the same period. The question that called for an answer was whether or not some vital aspect of learning allowed the latter two groups of students to progress so markedly while the former two groups made little headway.

DEVELOPMENTAL CONTRASTS BETWEEN TWO EXTREME GROUPS

An investigation of the contrasts in learning patterns of the two extreme groups shed light on the problem of why some improve and others do not. The groups were the 17 students who stayed in the S1 bracket (referred to as the SS11 Group) and the 8 who moved from the S1 bracket to the top S4 bracket (the SS14 Group). Table 2 presents a summary of the relevant statistics for Tests 1 and 2.

Table 2: Summary of *t*-tests for Groups SS11 and SS14 on Tests 1 & 2 Responses

Test	Scale	Comment	Max	Mean SS11	Mean SS14	*t* value	*df* V	*p*	Favours
1	Total	Test Total	65	5.12	9.25	3.86	23	***	SS14
2	GNV	Gen.No.and/ or Variable	17	1.33	5.83	5.49	5.70	**	SS14
2	Total	Test Total	65	8.88	20.50	4.51	8.08	**	SS14
2	SUBS	Substitute & Solve	7	0.38	5.00	4.41	7.22	**	SS14
2	EQL	Equals Scale	9	0.11	1.67	3.57	5.70	*	SS14
2	VBL	Variable	11	0.10	2.17	3.39	5.28	*	SS14
2	NBR	Number View	4	0.43	1.67	2.49	18	*	SS14
2	AD	"*a, d*" Scale	2	0.18	1.00	2.34	17	*	SS14
2	CON	Conjoining	7	2.31	0.43	-3.38	21.00	**	SS14
2	PRE	Prestructural View	19	15.50	8.67	-3.10	10	*	SS14

Note: Test 2 entries sorted in order of *t* values. Max. = maximum possible score (= no. of items for scales). V *df* : decimal point if using separate variances; otherwise, pooled variance. *** $p \leq .001$, ** $.001 < p \leq .010$, * $.010 < p \leq .050$.

Table 2 opened the door for a view of what aspects of learning made the difference between these two groups after their first week and a half of algebra. In Test 1, the SS14 Group scored significantly better on test total than the SS11 Group, although all members of both groups were at Standard 1 (test scores less than 14), and the difference between the mean scores was only about 4 points. There were no other significant differences at this pre-algebra stage. The *t*-tests based on Test 2 responses identified that those who were destined to progress to the top Standard 4 level significantly out-scored those who stayed in the bottom Standard 1 level in the following ways:

1. Their average test score was more than double that attained by Group SS11 and was about 12 points better (out of 65);
2. They had progressed significantly further in their understanding of algebraic symbols as generalized numbers and/or variables (GNV Scale), e.g., symbols allowed at least two values in equations '$2x + y = 9$' and/or '$c + d = 10$', and when comparing '$t + t$' with '$t + 4$';
3. They were more rapidly developing the skills for substituting a numerical value into algebraic expressions and for solving a simple equation (SUBS Scale), e.g., If '$y = 3$, $2y + 5 = \ldots$, $2(y + 5) = \ldots$, $2(5y) = \ldots$ ', and 'Solve $3a= 36$';
4. They were more successful in identifying when two algebraic expressions were equal (EQL Scale), e.g., '$a + b + c = a + x + c$', '$2a + 3b + 7 = 5a + 7$', '$a + 2b + 2c = a + 2b + 4c$';
5. They were developing the variable concept more rapidly (VBL Scale), e.g., comparing values of '$2n$' and '$n + 2$' and/or '$t + t$' and '$t + 4$';
6. They had the greater tendency to regard the letter symbols in early algebra as standing for numbers rather than objects (NBR Scale), e.g., for the Professors-and-Students problem, interpreting 'S' as "number of students" in the equation '$S = 6P$';
7. They were more able to allow the symbols 'a' and 'd' to stand for numbers without restriction (AD Scale) in the question "If 'a' and 'd' are any two numbers, which, if either, is the bigger? Give a reason for your answer.";
8. They were less inclined to conjoin symbols for addition (CON Scale), e.g., If $y = 3$, $2y = 6$, not 5, and "Add 4 onto $3n$" gives '$4 + 3n$ ', not '$7n$'; and
9. They were breaking away more rapidly from some of their prestructural views of algebra (PRE Scale), e.g., claiming that '$t + t$' is always greater than '$t + 4$', ordering symbols alphabetically, and/or needing numerical values for all symbols.

Findings 2, 4, 5, 6, and 7 all, in some degree, recorded that those in the group on the verge of greater improvement had started to develop the concept that the alphabetic symbols of early algebra represented numbers and that the numbers could vary. Those not destined to improve out of Standard 1 over the time period used for the collection of research data lagged significantly behind them and had failed to appreciate that the symbols that had been introduced to them were standing for numbers which could vary. This seems to be empirical evidence that it was the student views of the meaning of the symbols which identified significant differences between those who were on the way to improvement and those who were not. This outcome made logical sense considering the fact that most of the test items were designed to measure the level of understanding of symbols. Findings 2, 4 and 5 indicated that those who were to progress the more had started to understand that algebraic symbols represented numerical variables, and they had begun to apply this concept to some, at least, of the problems presented in the test.

Finding 3 above recorded that those on the way to higher scores were faster in developing the skills needed for solving a simple equation (‘$3a = 36$’), and for substituting a given value for ‘y’ in given expressions containing y (such as ‘$2y + 5$’), showing that they had a clearer grasp of the meanings of algebraic expressions. Findings 8 and 9 above reported that it was helpful for the prospect of improving in algebra to understand the convention that conjoining was used for multiplication in algebra, and to start to understand what the test questions were actually asking. Those who were not on the way to progress made very little intelligent headway at the time of Test 2 on most of the items in the Prestructural Scale, as shown by their high mean score (15.50 out of a possible 19) on the PRE Scale, which tallied the number of times the meaning of a problem was missed.

Data obtained in Test 3 allowed the continuation of the investigation of what aspects of learning discriminated between those who proceeded to Standard 4 from those who stayed at Standard 1. The differences between the means of these two groups were significant on 22 scales, the large number emphasizing the growing gap between the rates of development within the two groups. In every case the differences favoured the SS14 Group. At the time of responding to Test 4, the SS14 Group were still well ahead of the SS11 Group, as there were 21 significant differences between the means.

Comparison of Groups SS11 and SS14 Over Four Tests. Figures 1 and 2 summarize the scores per item for Groups SS11 and SS14 respectively on scales which registered significant differences, using t-tests, for Tests 2, 3, and 4.

Figure 1 records that Group SS11 showed very little change in their views about algebra. The average scores in the four tests for each scale are quite close together, indicating that there was not much change from test to test, except for an improvement on the Substitute and Solve (SUBS) Scale and the ‘a, d’ (AD)Scale. They also recorded just minimal changes in their incorrect views, as measured by the Conjoin (CON) Scale and the Prestructural (PRE) Scale.

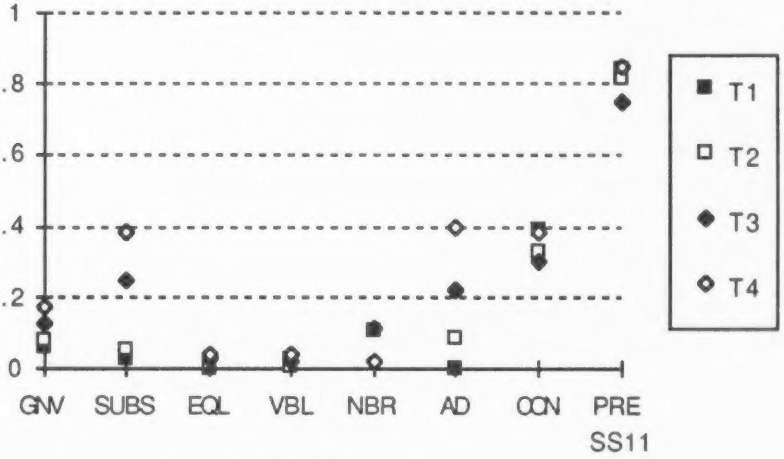

Figure 1: Average scores/item for SS11 Group

Figure 2 reports that the SS14 Group, on the other hand, showed a rapid improvement on the first six scales, especially the Generalized Number and/or Variable (GNV) Scale, the Substitute and Solve (SUBS) Scale and the '*a*, *d*' (AD) Scale. They showed a growth in their ability to solve equality problems (EQL Scale), and in understanding algebraic symbols as variables (VBL Scale), and as representing numbers (NBR Scale). They moved away from incorrect views, as recorded by the noticeable decrease in average scores per item on the Conjoin (CON) Scale and the Prestructural (PRE) Scale.

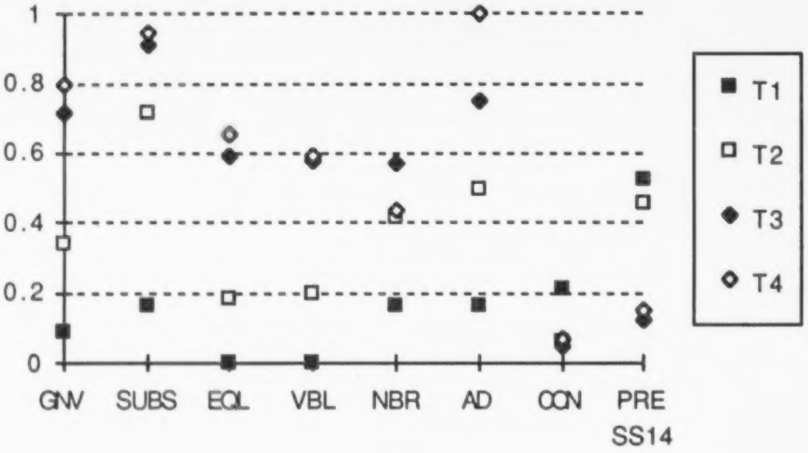

Figure 2: Average scores/item for SS14 Group

The data used in the search for the key to progress were found to be applicable to two of the propositions which were supported by analyses of the frequencies of successful responses from all 517 students in the study. These are discussed in turn.

Proposition 1. Empirical support was given to Proposition 1 which claimed that *success in interpreting algebraic expressions is a prelude to success in comparing the values of two expressions or comparing the values of two variables within the one expression.*

In order to make substitutions successfully, students had to know the conventions for writing the expressions so that they could interpret what the expressions meant. An investigation was carried out to see how the two groups compared on a subset of the SUBS Scale, namely the SUB Substitution Scale which was available as a measure of skill in substitution. As was expected, *t*-tests using scores on the SUB Scale showed that there was no significant difference between the two groups in Test 1 but that the differences were significant in the other tests, just as was the case for the SUBS Scale. A summary of *t*-test analyses for the SUB Scale responses is presented in Table 3.

Table 3: Summary of *t*-tests for Groups SS11 and SS14 on SUB Substitution Scale Responses

Test	Scale	Max	Mean SS11	Mean SS14	*t* value	*df* V	*p*	Favours
2	SUB	6	0.24	4.38	5.09	8.31	***	SS14
3	SUB	6	1.76	5.75	7.33	18.92	***	SS14
4	SUB	6	2.07	5.75	6.64	15.35	***	SS14

Note: *Test 2 entries sorted in order of t values. Max. = maximum possible score (= no. of* items for scales). V d_f^r : decimal point if using separate variances; otherwise, pooled variance. *** $p \leq .001$.

These analyses clearly showed that the SS14 Group progressed to almost 100% efficiency in substitution skills after just three weeks of algebra, as could be judged by their high average scores (5.75) on Tests 3 and 4. Those in the SS11 group achieved only about one-third of that success rate, reaching an average scale score of about "2" in Tests 3 and 4. If Proposition 1 applied to Group SS11 students, then their failure to succeed in interpreting algebraic expressions sufficiently well to enable them to substitute correctly should have been accompanied by failure to succeed with the test items which formed the VBL Variables Scale, a scale which gave a measure of their degree of success with items requiring them to compare the values of two expressions (in Items 10, 12, and 15 (iii)) or two variables within one expression (in Item 6 (c)).

Figure 3 illustrates the way the differential rates of development for Groups SS11 and SS14 followed the expectations expressed in Proposition 1, by graphing the average scores per scale item on the scales in question. The SS14 graphs seem to indicate that progress on substitution is a prelude to progress with problems involving the notion of a numerical variable, in the form required by the items in the VBL Scale. At the same time, the graphs for the SS11 Group show that improvement on the VBL measure does not necessarily follow once there is some improvement on the SUB Scale. These outcomes support the reasoning that one needs to reduce the memory load required for simply understanding the meaning of an algebraic expression before one can carry out cognitive tasks involving relationships between algebraic expressions, as was required in items belonging to the VBL Scale.

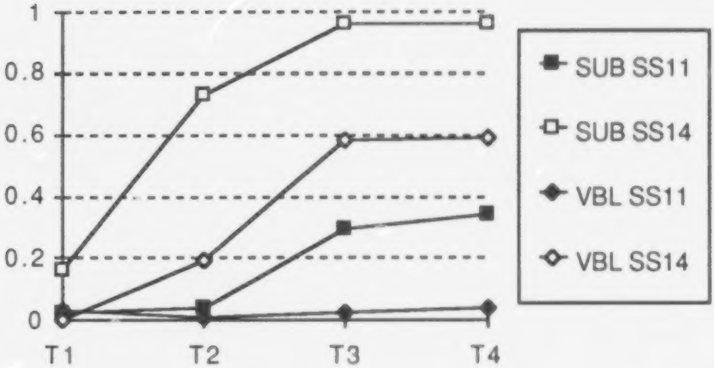

Figure 3: Average scores/item for SS11 and SS14 Groups on SUB and VBL Scales

Proposition 2. Proposition 2 stated that: *Success in solving a simple linear equation is a prelude to success in identifying the conditions for equality of two algebraic expressions or for the equality of two variables within the one equation.*

The EQL Equality Scale found its place in Table 2 and Figures 1 and 2 as one of the measures which identified significant differences in the learning rates of the beginning algebra students in Groups SS11 and SS14 in their first week and a half of algebra. This scale supplied a measure of success in dealing with the equality notion in the contexts mentioned in Proposition 2 and so provided one suitable assessment for testing the proposition with respect to the rates of cognitive development shown by the two groups. One item, namely, Question 8 (b) ("If $3a = 36$, what would be the value of a?"), measured ability to solve a simple algebraic equation and was chosen as the other measure needed for testing Proposition 2. As Question 8(b) was a subset of the EQL Equality Scale, the scale scores were adjusted to form the EQL* Adjusted Equality Scale by subtracting scores on Item 8 (b), so that there would be no overlap between the two measures being considered.

Figure 4 displays the differential rates of development on these two measures for each of the groups SS11 and SS14. Support was evident for Proposition 2. Those who improved their scores on the Equality Scale had, in most cases, previously improved their success with solving a simple equation in Item 8 (b). It was established, however, that success on this item was not a sufficient condition for success on the Equality Scale items. It seemed that the memory load needed for solving the easier cases of algebraic equality had to be reduced before more complex equality problems could be solved. Solving an equation involving one arithmetical operation and one variable was apparently less demanding cognitively than finding the condition for the equality of two algebraic expressions.

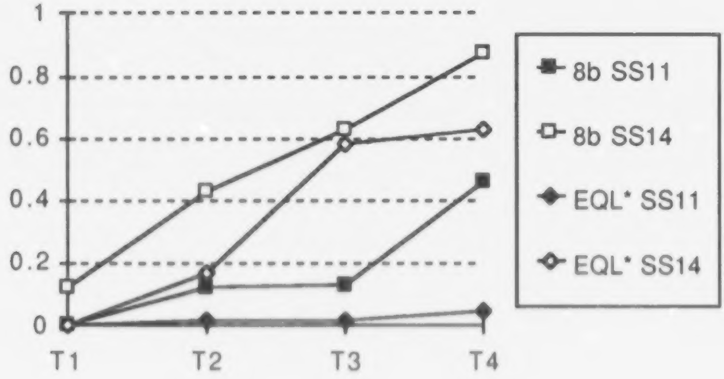

Figure 4: Average scores/item for SS11 and SS14 Groups on EQL* Scale and Q.8b

Persistence of Incorrect Ideas. Two other variables from Table 2 deserve close attention in this exploration of the problem of why some students progress while others do not. They were scores on the CON Conjoining Scale and the PRE Prestructural Scale, both measures of incorrect thinking about early algebra. There were no test items in common between these two scales and, between them, they covered 26 items. The scale averages for the two groups, as graphed in Figure 5, indicated that members of the SS11 Group were more persistent in their incorrect views than were their counterparts in the SS14 Group. This is an additional insight into the factors which influenced the vastly different rates of development recorded for these two groups.

Those in the SS11 Group hardly changed, on average, in their acceptance of the incorrect view of conjoining: Only six of these 17 students kept their CON Scale score under "3" (out of 7) for Tests 3 and 4. They also showed very little change in their prestructural approaches to the 19 problems registered in the PRE Scale. Many had missing data classification for the scale and all the registered scores were "11" or more, except in two cases. It seemed that the class activities were making very little impact on their way of thinking.

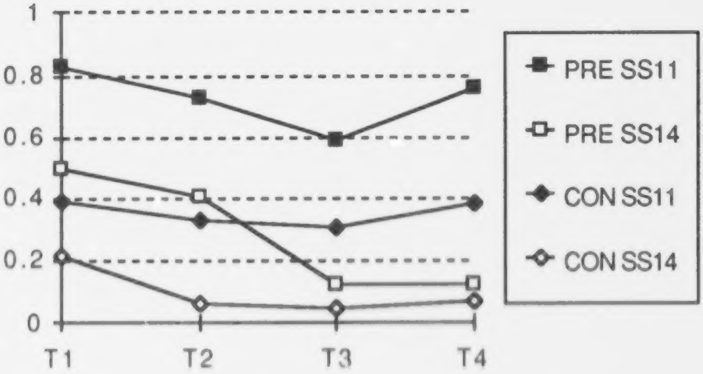

Figure 5: Average Scores/item for SS11 and SS14 Groups on PRE and CON Scales

In contrast, those who made rapid strides in mastering the basic concepts of early algebra, the members of Group SS14, distanced themselves from the misconceptions measured by these two scales. None of them, for instance, scored greater than "1" on the CON Scale after Test 2. Only two scored more than "4" on the PRE Scale after Test 2 and all but one of the rest scored "2", "1" or "0" after Test 2.

Numbers View Versus Objects View. The remaining feature of Table 2 to merit comment is the entry which recorded that students in Group SS14 were significantly more inclined to view algebraic symbols as representing numbers than were the members of Group SS11. The contrast between the two groups in terms of their tendencies to regard the symbols as standing for numbers or as standing for objects (or people) is brought out by the graphs in Figure 6. By Tests 3 and 4 the difference between the groups on the OBJ Objects View Scale was statistically significant and the difference on the NBR Numbers View Scale continued to be statistically significant.

Figure 6 displays the facts that the SS11 Group scarcely changed their point of view at all about what algebraic symbols basically represented: they persistently kept, on average, to the Objects View. The graphs for the SS14 Group display the dramatic drop in average item score for the OBJ Objects Scale and the corresponding rise in the preference for the Numbers View, as measured by the NBR Scale, over the period of the three weeks' intervention teaching which introduced these students to algebra. In the six months following this period there was some regression towards an Objects View.

The contrast between the groups on whether or not they favoured an Objects View of symbols or a Numbers View emphasized that the development of an understanding of algebraic symbols as representing numbers rather than objects appeared to be beneficial for ensuring substantial progress in the algebraic tasks assessed by the research instrument.

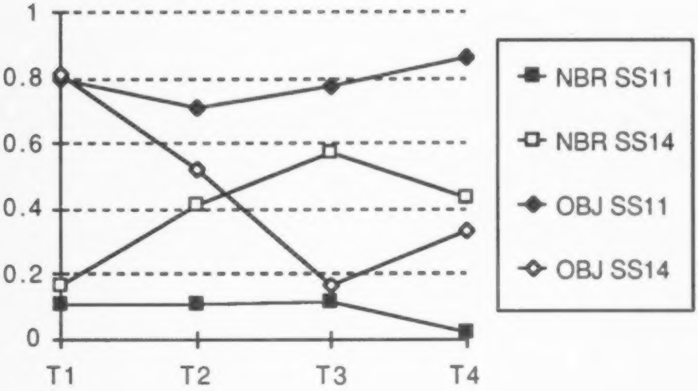

Figure 6: Average scores/item for SS11 and SS14 Groups on NBR and OBJ Scales

SUMMARY OF FINDINGS

An important problem for many teachers of students beginning algebra is the question of why, in the same class, some students do not show much or any progress while other students do. In Test 1, Group SS14 scored significantly better on test total than Group SS11, although the difference between the mean scores was only about 4 points. There were no other significant differences at this pre-algebra stage. By Test 2, significant differences on test total and eight cognitive measures had developed.

Group SS14 progressed more rapidly towards acquiring the concept that algebraic symbols stood for numbers which could vary than did their counterparts. Thus, empirical evidence indicated that it was student views of the meaning of the symbols which identified significant differences between those who were on the way to improvement and those who were not. The SS14 students also acquired more quickly the basic skills required to substitute in simple expressions and solve simple equations.

Group SS11 misunderstood more persistently such conventions as the conjoining process in algebra, and tended more to miss the point of problems set. This indicated that they were not coping with the new ideas that were being placed before them. Significant differences between the groups were registered in each following test on more than 20 cognitive measures.

The failure of Group SS11 students to succeed in interpreting algebraic expressions (e.g., '$3y + 5$') sufficiently well to enable them to make substitutions correctly was accompanied by failure to succeed with test items requiring them to use the variable concept, as in comparing the values of two expressions (e.g., '$2n$' and '$n + 2$'). The overall outcomes for Group SS11 pointed to the likelihood that one of the reasons why they had not progressed in the development of an understanding of algebraic symbols as numerical variables was that they had not learnt to interpret algebraic expressions. A theoretical argument supporting this finding is that one needs to reduce the memory load required for simply understanding the meaning of a single algebraic expression before one can carry out cognitive tasks involving relationships between algebraic expressions.

Success in solving an equation with one operation on one variable (viz., '$3a = 36$') was a prelude to success in finding conditions for the equality of two algebraic expressions. It seemed, moreover, that the memory load needed to solve the easier case of equality had to be reduced before more complex equality problems could be solved. Group SS11 recorded a gradual improvement in solving the given equation, but this did not necessarily result in increased success with the other equality problems.

The contrast between the groups also indicated that the development of an understanding that algebraic symbols represent numbers rather than objects appeared to be beneficial to ensure progress in completing algebraic tasks successfully.

PRE-SERVICE PRIMARY TEACHERS' ATTITUDES TO TEACHING MATHEMATICS A REAPPRAISAL OF A RECENTLY DEVELOPED INSTRUMENT

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Steven Nisbet (1991) recently published an article which reported on the development of an instrument for measuring pre-service teachers' attitudes to teaching mathematics. Although this was a worthy project we were not convinced by some of the conclusions drawn and suspected that the resulting twenty-two item instrument did not include the best indicators of attitude.

Definitions of attitude generally include the idea that attitudes manifest themselves in one's response to the object or situation concerned. One such definition states:

> "Attitude is a mental and neural state of readiness, organised through experience, exerting a directive or dynamic influence upon the individual's response to all objects and situations with which it is related".
>
> (Allport in Kulm, 1980, p 356)

When exploring the attitudes of preservice teachers toward mathematics it is necessary to not only consider their attitude towards the subject itself, but also their attitude regarding the teaching of mathematics.

The attitudes of pre-service teachers are of particular importance because of their potential influence on pupils. Although the research evidence is certainly not conclusive, it has been sufficient to suggest that positive teacher attitudes contribute to the formation of positive pupil attitudes (Aiken, 1972, 1976; Phillips, 1973; Sullivan, 1987). Some studies have indicated that teacher attitudes towards a subject and the teaching of that subject influence the instructional techniques they employ and that these, in turn, may have an effect on pupil attitudes (Brush, 1981; Carpenter and Lubinski, 1990; Williams, 1988).

Although there is little hard evidence that holding positive attitudes towards mathematics is actually beneficial, it is difficult to argue against their desirability. The widespread belief in the relevance of positive pupil attitudes towards mathematics is reflected in the following extract from the *National Statement on Mathematics for Australian Schools*,

> "An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their involvement in it."
>
> (Australian Educational Council, 1991, p 31).

One argument presented in support of the need of positive attitudes is that such attitudes can enhance achievement in mathematics at primary, secondary and tertiary level (Dungan and Thurlow, 1989). Most studies on the relationship between attitude and achievement have revealed a low but significant correlation (Aiken, 1976; Kulm, 1980). However, the nature and direction of this relationship is yet to be unravelled (Daane and Post, 1989; Kulm, 1980; Suydam, 1984).

Research into attitudes has explored various components of attitude such as anxiety, enjoyment, self-concept and belief in the usefulness or value of mathematics. One component that has received much attention is that of "mathematics anxiety". There is no doubt as to the existence of 'mathematics anxiety' and a number of instruments for measuring levels of anxiety has been developed and implemented (Richardson & Suinn, 1972). However, Sovchik, Meconi, and Steiner (1981) suggest that the construct "mathematics anxiety" is not as well defined and measureable as assumed by some mathematics researchers. There is some doubt as to whether anxiety is in fact a separate construct. It may just be a reflection of some deeper attitude (Wood, 1988). Anxiety's relationships to other factors such as enjoyment, general attitude towards mathematics and performance are unclear. There is growing evidence that self-concept is a better measure of how people feel about themselves as teachers of mathematics, and that self-concept has an influence on the formation of attitudes. Studies have also found a consistently high positive relationship between self-concept and mathematics achievement (Hackett and Betz, 1989; Marsh, Cairns, Relich, Barnes, and Debus, 1984; Relich, Conroy, and Webber 1991; Reyes 1984).

Comparisons of anxiety levels in males and females at primary, secondary and tertiary levels, have reported females to be generally more anxious about mathematics than males (Aiken, 1976; Kelly and Tomhave, 1985). Of particular concern is the notion that female teachers, who constitute the majority of primary school teachers, may be perpetuating negative attitudes with the girls in their classes (Kelly and Tomhave, 1985; Relich et al., 1991).

If it is accepted that it is highly desirable for teachers of mathematics to exhibit positive attitudes then the high proportions of pre-service teachers found to hold negative attitudes towards mathematics is somewhat alarming (Becker, 1986; Kelly and Tomhave, 1985; Sullivan, 1987).

Recent research with pre-service teachers has begun to reveal a series of links between mathematics attitudes, the choice or avoidance of mathematical studies, self-concept and attitudes towards the teaching of mathematics. It appears that students (both male and female) with low self-concepts in mathematics are less likely to pursue mathematical studies. Not surprisingly then, studies have revealed that most pre-service teachers who exhibit negative attitudes towards mathematics have not chosen to study mathematics in their final years of high school (Aiken, 1976; Relich et al., 1991; Sullivan, 1987). Some studies have gone as far as to recommend that appropriate minimum levels of mathematics study may need to become course entry requirements in order to raise the attitude levels (Mansfield, 1981; Relich et al., 1991).

The potential of teacher training courses to change the negative attitudes of pre-service teachers towards mathematics needs to be considered. Sullivan (1987) found that almost half of the students entering a teacher training course possessed negative attitudes towards mathematics. He states:

> "The course improved their attitudes overall, but those who started with negative attitudes still had the most negative attitudes at the end"
>
> (Sullivan, 1987, p.1)

He concluded that if these initial attitudes are so significant, teacher education courses may need to establish entry criteria based on the mathematics background of the applicants.

Several studies suggest that beginning teachers are especially concerned about methodology as well as content and therefore training courses should cater for this (Aiken, 1976; Blunden & De La Rue, 1990; Mansfield, 1981; Sovchik et al., 1981; Watson, 1987). Relatively high correlations have been found between mathematical achievement, the enjoyment of mathematics and the perception of mathematics being useful, which also carries the implications for the design of teacher education courses (Watson, 1987).

If the attitudes of preservice teachers are to be improved, there first needs to be a reliable instrument with which to measure levels of attitudes and perhaps to identify groups of students with special needs (Aiken, 1976; Nisbet, 1991; Watson, 1987). The original instrument used by Nisbet (1991) consisted of the Fennema-Sherman (1976) "Mathematics Attitude Scales" plus the parallel scales constructed by Nisbet to cover the 'Attitudes to Teaching Mathematics' aspect. Neither of these scales included items regarding self-concept, so they were supplemented by items developed by Marsh (1988).

METHOD

The information reported by Nisbet did not include complete correlation tables on factor loadings therefore it was not possible to reanalyse the data. Instead we opted to re-administer the battery of subscales used in the initial developmental stages as well as some additional self-concept items and to factor analyse the new data set to see whether a similar pattern of results would emerge.

Subjects

The battery of tests was administered to 345 pre-service students in the Diploma of Teaching/Bachelor of Education program at the University of Western Sydney, Nepean. All students in the three years of undergraduate study were invited to participate in the study in their normal class time for lectures in mathematics education. Approximately 20 students declined to participate. The subjects ranged in age from 17 years to 43 years with 39.6% classified as mature age students, over 21 years of age. Table 1 gives details of the age distribution of students with a mean of 23 years, a median of 20 years and a standard deviation of 6.9 years.

Table 1: Age distribution of the sample

Age	*n*	% of sample
17-21	212	62.5
22-25	42	12.4
26-30	20	5.9
> 30	65	19.2

NOTE: 6 missing cases

The majority of the sample was female (79.1%) and was represented in similar proportions within the mature age group (73.8%) and recent school leavers (81.6%). The majority of these students had studied maths to Year 12 (88.5%) (see Table 2).

Table 2: Gender distribution of the sample by age group

	Age							
Gender	**17-21**		**22-25**		**26-30**		**>30**	
	n	%	*n*	%	*n*	%	*n*	%
Male	39	18.4	14	33.3	4	20	11	18.6
Female	173	81.6	28	66.7	16	80	48	81.4

NOTE: 12 missing cases

Table 3 gives a breakdown of level of study by age and gender. These results represent a group similar to Nisbet's sample of students. However, a larger proportion of students (10.3%) had completed tertiary mathematics discipline studies apart from units included in the teacher education program.

RESULTS

The Sample

In order to determine how the sample matched Nisbet's with reference to training in high school mathematics, we crosstabulated age group with study in high school mathematics, here defined by the NSW syllabus as Maths in Society, 2 unit, 3 unit and 4 unit at Year 12 level. For this group 10.8% studied Maths in Society and 51.7% and 12.9% two unit and three unit respectively. No students entered the program having studied four unit maths. Table 3 outlines the number of students with Year 12 maths in each age group.

Table 3: Level of study of high school mathematics by age group as a percentage of the total sample.

	% of sample			Level of Study	
Age		**Year 10**	**MS$^{(1)}$**	**2 unit**	**3 unit**
17-21	63.7	5.6	61.0	73.3	79.1
22-25	12.6	22.2	15.9	9.9	9.3
26-30	6.0	16.7	6.1	4.7	2.3
31 +	17.7	55.6	17.1	12.2	9.3

MS$^{(1)}$ Maths in Society

These results clearly indicate that mature age students in general have studied significantly less (Chi-Square = 70.06, $p<.001$) mathematics than their younger counterparts and conforms with Nisbet's findings. This further emphasises the serious implications for this group as entrants into primary teacher education programs without the requisite background in mathematics as recommended in the *Discipline Review of Teacher Education in Mathematics and Science* (1989).

Materials

The range of items described by Nisbet were included in a 65 item questionnaire designed to measure general attitudes to the teaching of mathematics. These scales were derived from the Fennema and Sherman (1976) attitude scales and the Marsh (1988) Self-Description Questionnaire.

The subscales include the *Confidence in teaching mathematics* (items 1, 9, 11, 17, 26, 41, 49, 53, 60); *Mathematics teaching anxiety* (items 2, 18, 27, 32, 37, 38, 42, 51, 54, 61); *Attitude toward success in teaching mathematics* (items 3, 12, 19, 43, 55, 62); *Mathematics teaching as a male domain* (items 4, 13, 21, 28, 22, 44, 56, 63); *Usefulness of teaching mathematics* (items 6, 14, 22, 29, 34, 46, 57, 65); *Effectance motivation in teaching mathematics* (items 7, 16, 23, 31, 36, 39, 47, 52, 58, 64); *Perception of teacher's/lecturer's attitudes towards one as a teacher of mathematics* (items 8, 24, 48, 59); and, *self-concept in mathematics* (items 5, 10, 15, 20, 25, 30, 35, 40, 45, 50).

Nisbet argued against the inclusion of mother's and father's *attitudes towards one as a teacher of mathematics* on the grounds that a large proportion of the sample (up to 50%) indicated that they were undecided or did not respond to these items. He does not include these items in his final scale. In the interest of limiting the number of items included in our questionnaire and in order to include the Marsh self-concept items without lengthening the questionnaire unduly, we decided to eliminate this set of items from our questionnaire.

Design

The sixty-five items were factor analysed using principal axis factoring with varimax rotation (using SPSS-X). A six factor solution was identified as being the most appropriate in isolating six distinct scales to identify attitudes.

Regression analyses using stepwise procedures were then used to determine whether differences in attitudes could be predicted from the set of variables identifying participant characteristics such as age, gender, and educational background. Analyses of variance and t-tests were also used to determine whether differences in attitudes could be identified based on these characteristics and subsequent subscale scores derived through the factor analysis.

Factor Analysis

A six factor solution using principal axis analysis and varimax notation resulted the clearest distribution of items into identifiable subscales. Two major factors emerged which measure two similar but distinct aspects of self perceptions related to mathematics: individual attitudes of pre-service teachers as *teachers of mathematics* (ATM) and their self-concept *as mathematicians* (MSC). In addition, we found evidence for a distinct *mathematics teaching as a male domain* (MTMD) scale, *usefulness of teaching mathematics* (UTM), *excellence as a teacher of mathematics* (ETM) scale and an *other's perceptions of me as teacher of mathematics* (OTM) scale. Table 4 lists the factor loading of each item on each of these scales.

Table 4: Factor loading of 65 item questionnaire on six factors

	Factor					
Item No	ATM$^{(1)}$	MSC$^{(2)}$	MTMD$^{(3)}$	UTM$^{(4)}$	ETM$^{(5)}$	OPTM$^{(6)}$
7	84					
32	81					
2	75					
16	74					
37	73					
27	72					
60	71					
36	70					
1	66					
61	65					
42	65					
53	65					
58	64					
17	64					
39	63					
18	61					
51	59					
49	59					
8	59					
64	56					
52	56					
5	55					
38	55					
41	54					
50	53					
26	51					
11	49					
9	46					
47	46					
54	40					
35		73				
45		68				
25	44	65				
20		60				
40		60				
10		55				
15	40	54				
30		53				
28			68			
33			67			
4			58			
31			57			
44			57			
13			53			
21			49			
56			47			
63			43			
23			-			
22				79		
34				77		
14				72		
6				63		
57				54		
29				46		
65				45		
43					66	
55					59	
12					47	
62					.	
3						57
19						42
46						-42
48						.
59						.
24						

Note: Decimal points are omitted. Values less than .40 are not included

(1) ATM: Attitude to teaching mathematics
(2) MSC: Mathematics self-concept
(3) MTMD: Mathematics Teaching as a male domain
(4) UTM: Usefulness of teaching mathematics
(5) ETM: Excellence as a teaching of maths
(6) OTM: Other's perceptions as a teacher of maths

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The first scale which consists of an amalgamation of 30 items from a variety of the original subscales tends to reflect a general attitude towards the teaching of mathematics. In fact all bar two items specifically mention the act of teaching as part of the content. The coefficient alpha reliability was .96.

The second factor (coefficient alpha = .89) on the other hand, reflects eight of the ten Self-Description Questionnaire items and is specifically relevant to personal performance on mathematical tasks. All of these items load to some extent on the first factor but form a distinct separate factor.

The third factor (coefficient alpha = .79) includes all of the items from the *mathematics teaching as a male domain subscale* and two items from the *effectance motivation in teaching mathematics* scale but the loading of one of these items is very low.

The fourth factor (Coefficient alpha = .78) contains seven of the eight *usefulness of teaching mathematics* scale.

The last two factors which emerge are not as clearly delineated, nor as reliable as the first four, but may be identified as *excellence as a teacher of mathematics* (coefficient alpha = .57) scale and *other's perceptions of me as a mathematics teacher* (coefficient alpha = .29) scale.

Further Analysis

The total scores on the subscales which emerged were correlated with each other and the set of independent variables which described the general characteristics of the group of participants; age, gender, years of study of high school mathematics and tertiary mathematics. These results are reported in Table 5.

Table 5: Correlations among the full set of independent variables and the emergent factorial subscales

	Age(1)	Gen(2)	HSC(3)	TER(4)	ATM(5)	MSC(6)	MTDM(7)	UTM(8)	ETM(9)
GEN	00								
HSC	-35***	-05							
TER	-01	-16**	15**						
ATM	04	-06	03	-07					
MSC	-21***	-12*	35***	08	75***				
MTMD	-06	23***	-03	-23***	27**	12**			
UTM	13	10	-13*	-06	05	01	38***		
ETM	-03	01	-01	-05	20*	26***	31***	17*	
OPTM(10)	09	-08	06	08	17*	11	23**	33***	17*

Note: Decimal points omitted

* p < .05
** p < .01
*** p < .001

(1) Age: Age
(2) Gen: Gender
(3) HSC: Years of study (maths) at high school
(4) TER: Years of study (maths) at tertiary level
(5) ATM: Attitude to teaching mathematics
(6) MSC: Mathematics self-concept
(7) MTMD: Mathematics teaching as a male domain
(8) UTM: Usefulness of teaching mathematics
(9) ETM: Excellence as a teacher of mathematics
(10) OPTM: Other's perceptions as a teacher of maths

One of the other major differences in the findings of this research in contrast to Nisbet's is the inclusion of *mathematics teaching as a male domain* subscale in the item pool to be analysed. Not only did this set of items clearly stand out as a separate construct but further analyses of the subscale showed that there are significant differences ($t=4.17$, $p<.001$) in response with females scoring higher than males (Female: M=74.4, SD= 6.2 vs Male: M=70.7, SD = 6.6). It is debatable whether such a statistical difference is important as both groups score very high overall (maximum score possible 80). The only other significant difference ($t = 3.9$, $p<.001$) in response between males and females was found for the *mathematics self-concept* subscale with males (M= 45.1, SD= 10.4) scoring higher than females (M=39.5, SD=11.9) (maximum score possible = 64). This is consistent with all the current literature which does indicate a consistent trend for males to rate themselves more highly than females on this construct.

The regression analysis did not lead to any major findings. While attitude to teaching mathematics was not predictable from the set of independent variables, mathematics self-concept is clearly predictable from level of study of high school mathematics. Attitudes towards *teaching mathematics as a male domain* can be predicted from gender (males are more likely to see it as a male domain than females) and level of tertiary study (students with greater tertiary training are more likely to see it as a male domain than students with *no* tertiary training). Tables 6 and 7 outline the regression equation.

Table 6: Regression analysis of hypothesised predictions on self-concept

Source	**Multiple R**	**ΔR^2**	**$\Delta R^{2\#}$**	**beta**	**F^F**
HSc	.345	.119	.119	.345	36.2*

*p < .001

$^{\#}\Delta R^2$ = the added percentage of predicted variance accounted for by the inclusion of the variable to the regression equation.

F_F determines the significance of the contribution of the individual variable.

Table 7: Regression analysis of hypothesised predictors on mathematics teaching as a male domain

Source	**Multiple R**	**ΔR^2**	**$\Delta R^{2\#}$**	**beta**	**F^F**
GEN	.234	.055	.234	.203	17.027*
TER	.304	.043	.038	-.197	14.900*

*p < .001

# ΔR^2 = the added percentage of predicted varince accounted for by the inclusion of the variable to the regression equation.

F_F determines the significance of the contribution of the individual variable.

DISCUSSION AND CONCLUSION

This factor structure does not conform to the Nisbet factor pattern which defines both a confidence and an anxiety factor. He states:

> "anxiety and confidence in teaching mathematics are independent factors. They are not opposite extremes of the one continuum. The most confident students are not necessarily the least anxious" (p. 45).

The factor structure we propose does not differentiate between anxiety and confidence rather it combines these two scales along with the restructured Sherman and Fennema (1976) *Effectance Motivation* scale, into a conglomerate scale which seems to reflect general attitude to the teaching of mathematics and personal feelings towards this activity. Interestingly, the self-concept items derived from the Marsh (1988) Self-Description Questionnaire constitute a distinct factor but one which shares considerable variance (55%) with the initial factor, suggesting that attitudes and confidence (here derived from the self-concept variable) are inextricably associated. In contrast to Nisbet's results, these results suggest that personal perceptions of one's adequacy as a mathematician should impinge on attitudes to teaching mathematics and are part of a continuum.

The evidence for mathematics anxiety as a separate construct, particularly as a construct different from confidence, is not convincing as has been argued in the literature review. The factor structure which emerged from the analyses of this set of data provides additional evidence for this point of view.

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TOWARDS A PROFESSIONAL DEVELOPMENT ETHOS

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INTRODUCTION

Professional development for teachers of mathematics has assumed increasing importance over the past decade, for it is recognised that teaching practices will not change substantially without vastly improving the quality of pre-service and in-service education. As pointed out in *A National Statement on Mathematics for Australian Schools* (1990), teachers will have to be engaged in professional development throughout their careers in order to keep abreast of ongoing developments in mathematics education. In the past few years, a great deal of time and money has been invested in professional development programs for mathematics teachers, so it seems timely to consider their long term effectiveness.

This paper reports on the general impact which professional development has had on mathematics teachers in Victoria, gauged from evaluations into the effectiveness of three different models of staff development. It is argued that professional development programs should aim to foster in teachers a professional development ethos which is ultimately self-sustaining and empowering.

RESEARCH ON PROFESSIONAL DEVELOPMENT

During the past 20 years, there has been an accumulation of research literature on in-service education, leading to increased knowledge about the art of teaching and the problems associated with change in schools. Indeed, Gage (1984) maintains that we now know enough about teaching practices to make it possible to change them in ways which make an educationally important difference. He asserts that "we need approaches to teacher education that walk the path between unacceptable regimentation and unacceptable anarchy" (p. 91). The work of Doyle and Ponder (1977/78), Joyce and Showers (1980), Fullan (1982), Huberman and Miles (1984) and others has clearly indicated that there are a number of factors influencing teacher change which had not previously been taken into account by those pressing for reform. These factors relate to the nature of teachers as people, schools as organisations and change processes themselves, including the many variables which facilitate or constrain change. In an important study of four change projects located in schools, Pinks (1989) identified twelve barriers to innovation effectiveness. Such knowledge has led to the development of in-service models which take these factors into account.

Much more is known also about the relative effectiveness of particular staff development models. Stalling's research, for instance (cited in Fullan, 1990) identified a number of critical factors which facilitate the success of a program. Meta-analyses of research in this area (see Showers, Joyce and Bennett 1987, Wade, 1985) reflect a high degree of consensus about what constitutes effective professional development practices. Sparks and Loucks-Horsley (1990, 13.1) have summarised these practices as follows:

(a) programs conducted in school settings and linked to school-wide efforts;

(b) teachers participating as helpers to each other and as planners, with administrators, of in-service activities;

(c) emphasis on self instruction, with differentiated training opportunities;

(d) teachers in active roles, choosing goals and activities for themselves;

(e) emphasis on demonstration, supervised trials and feedback: training that is concrete and on-going over time; and

(f) on-going assistance and support available on request.

As Ingvarson (1987) and Sparks and Loucks-Horsley (1990) detail, there are a number of ways in which teachers can engage in professional development and a number of models to choose from. These range from structured training models to school-based, teacher-controlled programs. Of course, not all teachers learn in the same way and not all schools have the same needs as far as staff development is concerned. In fact, Robinson (1989, p.274) asserts that there is no prescription for "exemplary practice which fits all teachers in all situations, (so) any model of change that relies on determining ends, then devising means will be inadequate, including all those within the management paradigm".

Although there are now some models for professional development which reflect this thinking and have provided teachers with the opportunity to empower themselves with regard to change, they are not necessarily able to do so. Despite extensive investment in professional development, there are still problems associated with the fact that teachers' attitudes to and beliefs about professional development begin to form long before they are confronted with the complexities of teaching in classrooms. Their own schooling, initial teacher training and subsequent in-service education has tended to be prescriptive and spasmodic in nature, leaving them with a perception of themselves as technicians. There has been a belief that if educational reformers can prescribe what it is that teachers should do and then have them do it, change will necessarily follow. This process, over a long period of time, has deskilled teachers in the sense that it has not encouraged them to think and act in self-directed, empowered, collaborative ways.

PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS

This situation is particularly pertinent as far as the teaching of mathematics is concerned. Changing teaching practices in mathematics is compounded by the fact that many otherwise able teachers lack confidence and competence in this aspect of their teaching. As Good and Grouws (1987, p.778) assert, there are "critical weaknesses in the way teachers teach mathematics" because of their "misunderstandings of basic mathematical concepts and their misconceptions about which skills need to be emphasised".

Because of this and because of the need to inform teachers of new approaches in mathematics, professional development for teachers of mathematics has received unparalled attention in the last decade. A number of initiatives have been taken to assist teachers to improve their mathematics teaching. For example, the activities developed for the Reality in Mathematics Education (RIME) project and the Mathematics Curriculum and Teaching

Project (MCTP) reflected quite a different approach to teaching mathematics. They were designed to be exemplary activity-based lessons which teachers could use as springboards for new ideas. According to Stephens, Lovitt, Clarke and Romberg (1989, 229) they "represent snap shot images of quality learning environments" which "can serve as vehicles which illustrate new understandings and reshaped practice". The publication of The *Mathematics Framework* and *A National Statement on Mathematics for Australian Schools* provides curriculum frameworks on which teachers can base their classroom programs.

Some of the existing models of professional development for mathematics teachers have arisen out of the Basic Learning in Primary Schools (BLIPS) program which was a project funded by the federal government in 1984. Its aim was to increase the literacy and numeracy skills of children in infant grades by raising the confidence, competence and interest levels of teachers. The most popular programs are the school-based models where teachers are encouraged to increase their understanding of the processes of teaching and learning through reading and discussion groups and by researching their own practice in collaboration with their colleagues. Two such models are EMIC (Exploring Mathematics in Classrooms) and Key Group. Teachers involved in EMIC attend ten weekly after-school sessions where they learn and discuss new ideas to trial in their classrooms. Key Group is somewhat more involved for participants, as they attend a three day live-in conference where they plan a course of action for change which takes account of up-to-date research findings, then return to their schools to implement their action plan. For further details about each of these models, see Robinson (1987) and (1989).

OUTCOMES OF EVALUATIONS

Different models of professional development may have different objectives or emphases, but the over-riding aim is to improve teaching practices in ways which result in more effective learning environments for students. The three forms of professional development discussed in this paper are Exploring Mathematics in Classrooms (EMIC), Key Group and involvement in a Deakin course, ECT413 - Mathematics Curricula. All three reflect the effective practices outlined earlier. They are school-based and encourage teachers to take responsibility for their own professional development. Support and advice is provided by tutors or consultants as teachers reflect on their practice, trial new ideas and generally explore processes of change. In the course of evaluating these three approaches, I have interviewed 52 teachers about their professional development and the extent to which their practice had changed as a direct result of their involvement in a program. Detailed results of each of the formal evaluations are reported in Rice and Mousley (1989), Charlton and Holmes-Smith (1987), Rice (1990) and the Barwon - South Western Region EMIC Evaluation Report (1990-91). Whilst some outcomes are applicable to individual models, there are a number of common outcomes and generalisable trends which have emerged and which are outlined below.

Impact on teachers

By and large, teachers' perceptions of the respective programs were very positive. They appreciated the fact that they were treated as professional people capable of identifying and improving aspects of their mathematics programs which needed to be changed. From a philosophical point of view, teachers were accepting of new ideas. As a result of their involvement in one of the professional development programs, a majority of teachers reported the following:

- perceptions of the nature of mathematics have been broadened
- they know more about how children learn mathematics
- they are more confident about trialling new ideas in their classrooms
- they have a broader range of teaching/learning strategies to use in the classroom
- understandings of the process of change have increased
- there is much more sharing of ideas and collaborative planning with colleagues
- interaction with teachers from other schools is more common

Impact on classrooms

Teachers' responses indicate that since their involvement in professional development, classroom environments have changed in the following ways:

- programs are more flexible and for some, more integrated
- there is more use of concrete materials
- there is more child-directed, discovery learning and problem solving
- children talk and write more about their own mathematical strategies
- children record mathematics in a variety of ways
- there is more interaction between children and more group work
- children enjoy mathematics more and are more willing to take risks.

At the classroom level, a number of teachers experienced difficulties when trying to implement new teaching strategies. The most frequently mentioned concerns were:

- lack of time to trial and reflect on so many new ideas
- difficulties with classroom organisation and management
- competing priorities in other curriculum areas

It appears that many teachers tried to incorporate new ideas and strategies into existing programs, rather than reformulate their program. When questioned about changes in children's learning, a majority of teachers referred to attitudinal and affective rather than cognitive factors.

Impact at the wider school level

It is evident that there are still a number of problems associated with changing mathematics at the whole school level. Many teachers reported that their involvement in professional development had made no difference to the way other teachers in the school were teaching mathematics. The most frequently mentioned concerns were:

- inflexible school wide programs
- traditional attitudes and beliefs of other staff at the school
- a lack of support from the principal and other senior staff
- parental and community expectations

However, there were some positive outcomes reported by some teachers at some schools:

- an increase in collaborative activity between teachers
- information sessions were held for parents
- an increase in the level of resources at the school

- a minority of teachers had been able to instigate or contribute to a review of their school mathematics policy.

Major trends

Overall, it is clear that as a result of recent initiatives in mathematics education, including involvement in professional development programs which reflect practices known to be effective, the following can be stated with some efficacy:

- many more teachers are *aware* of recent trends and desirable practices in mathematics education
- more classroom environments *are* changing to varying extents and in various ways, as teachers endeavour to implement new ways of working.
- some individual teachers have developed a strong, self-directed professional development ethos
- implementation and institutionalisation of change at the whole school level are still extremely difficult to effect.

Fullan (1990) argues that for staff development to be more effective, it must be carried out in conjunction with institutional development, that is whole school improvement. Working with individual teachers or small groups of teachers may be effective in the short term, but as Fullan (1989, p.13) asserts, "small groups of teachers working together ... represent only a small subpart of the total culture of the school". In the long term, "the culture of the school is much too strong to be influenced for any length of time by single passing projects".

Despite these continuing problems, involvement in professional development programs can be a catalyst for new professional directions for a significant number of teachers. In evaluating these different models it has become clear that, providing there is the opportunity for teachers to exercise some control over the momentum and directions for change, the crucial factor is not the model, but how teachers respond to it. In other words, irrespective of the form of staff development undertaken, and notwithstanding the culture of schools, the professional lives of some teachers are significantly changed whilst others remain largely unchanged.

Teachers involve themselves in change processes to varying degrees: some will invest a great deal of time and energy into improving classroom practices, whilst others do little more than 'dabble at the edges'. In a recent study of curricular change in mathematics, Wilson (1989) isolated three critical variables that correlate with the success of a change project but which are *not dependent* on a particular situation. These are the extent of self investment by the teacher, the level of support offered by the school and an on-going commitment to nurture, evaluate and revise. Success was linked with the presence of one or more of the variables while lack of one or more of the variables correlated with lack of success.

According to Doyle and Ponder (1977/78), in responding to a change proposal, most teachers are guided by the 'practicality ethic', the three central components of which are:

instrumentality - clarity of the procedures, depicting classroom contingencies.

congruence - how closely it matches teachers' personal preferences and existing classroom situation.

cost - teachers weigh up the effort required to implement the change against the potential outcome

PROMOTING A PROFESSIONAL DEVELOPMENT ETHOS

Notwithstanding the problems associated with school cultures, it is crucial that teacher education aim to foster an ethos which is self-sustaining and empowering and which regards the act of teaching as a dynamic, ever changing phenomenon. I believe there are three identifiable behaviours which are most likely to be associated with the development of such an ethos. These are *introspection, reflection and self-direction.*

Introspection is a way of thinking which involves an examination of one's own beliefs, thoughts and feelings, a process which leads to a greater awareness of the self in relation to others in the social context. Teachers who are able to situate themselves in a cultural milieu are more likely to be able to recognise possibilities and constraints. This is an important consideration in terms of implementing change in schools.

Reflection provides constant review of thought and action. It enhances understanding of a situation, promotes self-confidence and enables more efficient learning and problem solving to occur. There is already a well established body of literature on the importance of reflection in the teaching situation. As Schon (1971) has asserted, practitioners who neglect opportunities to reflect on what they are doing and why, can only attend selectively to phenomena which shape their practice. This idea is still valid today.

Self-direction. The notion of self-direction has been studied over a long period of time, from philosophical, educational and psychological perspectives. According to Candy (1990, p.15), 'central to the notion of 'self-direction' is the learner's sense of personal control, which is defined as the learner's belief that he or she is responsible for, and capable of, directing his or her own learning'.

I am suggesting that teachers who continually engage in introspection and reflection and who are confident in self-direction are most likely to adopt an enquiry approach to teaching which helps build a professional development ethos in a school. The characteristics of such an ethos are that teachers perceive themselves as professional teacher researchers, capable of researching, instigating and controlling the directions and momentum for change which is designed to enhance the welfare of their students.

The implications of this are that professional development programs need to fundamentally change teachers' perceptions of themselves as technicians who follow the lead of others. Professional development programs should be starting points which aim to provide support for teachers to become skilled in each of the abovementioned behaviors which in turn will assist them in the struggle to control their own professional destinies. As Fullan (1990, p.22) points out, the agenda for professional development is "to work continuously on the spirit and practice of life-long learning for all teachers". Such learning in his view has four related components, reflective practice, an enquiry approach to teaching, collaboration and the mastery of technical skills. It is important that "teachers come to internalise these ways of being, to the point where it becomes second nature to be a perpetual learner". I contend that the qualities of introspection, reflection and self-direction would enable teachers to do just that, though clearly, there is a need for more extensive research in this area.

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INTEGRATING COMPUTERS INTO THE TEACHING OF CALCULUS: DIFFERENTIATING STUDENT NEEDS

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The study undertaken here looked at difficulties associated with the first principles approach to the derivative of a function and concentrated in particular on the first five lessons in calculus as experienced by a typical group of nineteen year 10 students who were preparing to take calculus at year 11. A traditional teaching approach was contrasted with an alternative computer teaching approach and both approaches were analysed for success in terms of conceptual understanding, skill acquisition and student perceptions of whether the work was easy to understand. The traditional first principles approach was found to be too cognitively demanding for the students who demonstrated a 'rush to rule' for meaning. Students undergoing the computer treatment also demonstrated this rush to rule and therefore very gradual development is recommended for students in their first encounters with calculus.

CALCULUS AS A GATE-KEEPER IN MATHEMATICS

Calculus continues to be a significant component of senior secondary school mathematics and is still most often recommended as study for those students who wish to gain entry to a university mathematics course. The presentation of the early lessons of calculus appears to have changed little throughout this century but there is now a radical re-think with regard to how students should approach the ideas of calculus.

Since the intrinsic difficulties of calculus are usually part of the first encounter for the beginning student, it is no wonder that students (even those who have prided themselves on being on top of the subject) are somewhat bewildered by the intricacies of the underlying concepts:

> There is no part of mathematics for which the methods of approach and development are more important than the calculus, partly on account of the novelty of notation, but chiefly on account of intrinsic difficulties. These occur at the start, or more acutely at the start than at any later stage. (Mathematical Association, 1951, p.4)

Other mathematics educators too have exhorted teachers to take time with the early concepts (Austin, 1982; Orton, 1985). However, teachers and textbook writers have continued to introduce differential calculus to students with scant or token reference to underlying concepts followed by a relatively fast move to concentrating on facility with the processes (Ryan, 1990).

As some believe that calculus is "an obvious dividing line between those who claim to know a little about mathematics and those who do not" (Graham, Read, and King, 1973) and, as it is a significant or high status component of school mathematics, it can provide the final closed gate to the potential success and commitment of many secondary school students to mathematics. Because it "is a significant barrier on the road to a professional career" (Peterson, 1987), mathematics educators need to look closely at the intrinsic difficulties and the teaching methodology associated with beginning calculus. Allowing more time for gradual development may also have long term dividends for both student performance and satisfaction. Rather than a barrier for the majority of our students, calculus may prove to offer an exciting new direction in their mathematical growth - the stage where they are challenged to think in "new and different ways" (Orton, 1986).

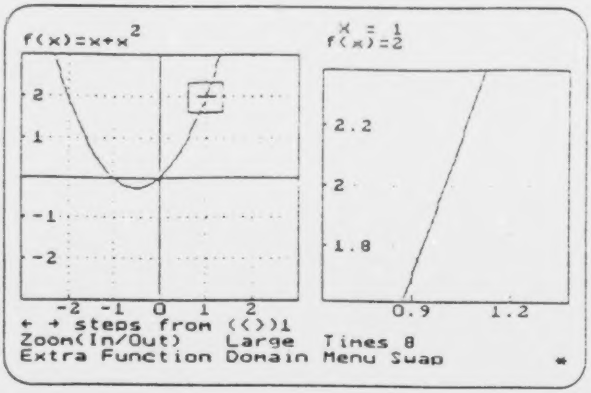
To improve the presentation of calculus is a challenge, too, to mathematics educators - the potential of new technology offers the chance to radically re-design the introductory stages as well as expand the applications. We now have computer software to re-direct our approach so that, potentially, more students can gain greater understanding and confidence with it and not believe that this is where their mathematical education must end.

USING THE COMPUTER TO DEVELOP CONCEPTUAL UNDERSTANDING

Use of computer technology is certainly fashionable in mathematics education at the present time. For many educators the computer appears to offer a more interactive, exciting, real-world and individual learning format for students. In mathematics, the promise of greater access to the 'treasures' for more students who have historically rejected mathematics, through boredom or failure, is just too great to ignore. For the classroom teacher, with limited entry time to new technology, the challenge can sometimes only be taken up as an add-on to current practice; former teaching practice is not necessarily altered in the light of the new tool and students' preferred style is rarely matched with the new technology. Resistance by either student or teacher to new technology may not be a problem here - the type of interaction required by the 'best' software needs to be studied. I suspect that the whiz-bang technology can blur the underlying ideas just as much as the formal chalk-and-talk delivery if some respect for time for reflection and questioning is not built into the learning process.

New interactive computer packages. David Tall's *A Graphic Approach to the Calculus* was designed to "offer student and teacher alike the possibility to visualise and experiment with the fundamental concepts of calculus" (Tall, 1991, p.3). He believes that the traditional approach, where elementary ideas are presented first, actually sets up later cognitive difficulties for the student. He advocates a curriculum approach that allows the student to experience "the powerful ideas of the calculus from the beginning" (Tall, 1991, p.44).

His package, used in the study reported here, includes the MAGNIFY program (Figure 1) and the GRADIENT program (Figure 2). MAGNIFY allows students to investigate the *local straightness* of graphs of functions "thus the gradient of the graph can be given an immediate global impression, with the limiting processes implicitly contained in the magnification idea. No explicit discussion of limits is necessary" (Tall, 1991, p.48). The notion of the gradient of the tangent to a curve being a measure of the gradient of a curve may then be established in an intuitive sense initially, in preparation for the formal ideas of later analytical approaches.


f(x)=x+x²
x = 1
f(x)=2
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-1
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-2 -1 0 1 2
2.2
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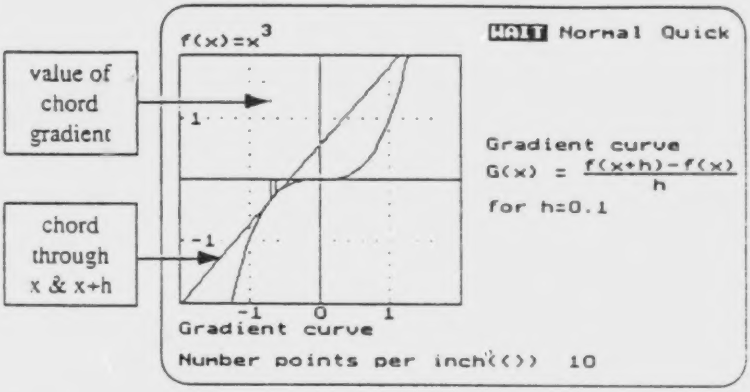
1.8
0.9 1.2
← → steps from (<>)1
Zoom(In/Out) Large Times 8
Extra Function Domain Menu Swap

Figure 1: The Magnify program

The GRADIENT program (figure 2) point-plots the approximate *gradient function* and requires the student to name the derived function. GRADIENT also allows the student to investigate the limiting process at a point on the graph.


value of chord gradient
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WAIT Normal Quick
1
-1
-1 0 1
Gradient curve
G(x) = f(x+h)-f(x) / h
for h=0.1
Gradient curve
Number points per inch(<>) 10

Figure 2: The Gradient program.

The exploratory nature of the activity is the key feature of Tall's *A Graphic Approach to the Calculus* programs. For students who have used the programs the underlying ideas of calculus are discovered quite intuitively and rules eventually developed *as the student is ready to organise them.*

4.3

BEST COPY AVAILABLE

THE STUDY

The study undertaken looked at difficulties associated with the first principles approach to the derivative of a function and concentrated in particular on the beginning lessons in calculus as experienced by a typical group of year 10 students who were preparing to take calculus at year 11. A traditional teaching approach was contrasted with an alternative computer teaching approach using Tall's package and both approaches were analysed for success in terms of conceptual understanding, skill acquisition and student perceptions of whether or not the work was easy to understand. As calculus has served as a critical filter for further study in mathematics, teaching methodology and student attitudes to the topic were also a focus of the study.

The study undertaken was in the form of a case study of the event "the first lessons in calculus" and used information gathered over a seven-day period encompassing a pre-test, a class lesson sequence of five lessons using two different teaching styles for contrast and a post-test. As the study aimed to focus in detail on students' early encounters with calculus, the first five class lessons were studied closely, both for what was learned and what the students believed they learned, under different learning and teaching styles. The two styles of teaching were contrasted for similarities and differences in student learning.

It is in the first few lessons that most teachers hope to establish a conceptual base for what comes later in differential calculus. It is already known that rule acquisition in calculus is the leading long-term outcome for most student learning, and that conceptual understanding is most often a casualty. Hence, the short-term concentration on the crucial first five lessons aimed to establish the source of early specific misconceptions and difficulties or the grounds for what appears to be 'concept avoidance' in students taking calculus for the first time.

The class. At the end of the academic year a class of twenty-five Year 10 students in a co-educational secondary school was divided into two groups, matched on the basis of their teacher's experience of their overall mathematical ability and test results throughout the year. Friendship groups or pairs were maintained to allow for mutual support in the studied groups, to maximise discussion and to minimise student anxiety. This class had never studied calculus, but all but one of the students hoped to go on to take the Year 11 mathematics subject "Change and Approximation" in the following year where calculus was a key component of the subject. They were seen to be a typical pre-calculus class with the normal mathematics preparation to that stage and were 'ripe' for the introduction of calculus.

SCOPE OF THE STUDY AND SOURCES OF DATA

The two groups formed were treated to a different style of class teaching over the five class lessons: traditional or non-traditional teaching using computer software.

The traditional group. The class teacher who had over twenty years' teaching experience undertook her teaching in what will be called *the traditional style*. This matched her teaching of introductory calculus in Year 11 up until this time. Her method can be characterised as a formal approach which was skills-based even though attempts were made to explain the underlying ideas to students. The teacher was at pains to resist newer techniques being presented to the second group even though she had recently become aware of them and hoped to vary her teaching in the future as a consequence. She was recognised in her school by students and colleagues as a leading and successful mathematics teacher. This presentation was therefore seen to be an example of 'best' or expert teaching in the traditional style.

The computer group. The second group was taught by the writer in the same class period in what will be called *the computer style* making use of Tall's *A Graphic Approach to the Calculus* for the IBM (Tall, 1991) in the school's computer laboratory. The method of teaching used here purported to establish a strong *intuitive* base for some of the basic concepts of differential calculus and incorporated visual learning and discussion leading to the students' establishing a rule for the derivative of a polynomial function. The method was underscored by an informal numerical approach and formal definitions were not used at any stage. The students were unfamiliar to the writer but appeared relaxed about taking part in the study - they had used the IBM computer to a limited degree in their Year 10 mathematics class. The methods used here had been trialled by the writer with groups of students at the tertiary-intake level and the writer was familiar with the materials used. This presentation was therefore seen to be an example of 'best' or expert teaching in the non-traditional style.

Students in the groups. Nineteen students attended seven class sessions for this study - nine students were in the traditional group and ten students were in the computer group.

Group	Students
Traditional (T)	9
Computer (C)	10

Table 1. Number of students in the study by treatment group.

Even though the class was the streamed 'top' group in the school, both groups formed had a similar number of upper (U), middle (M), and lower (L) ability students as classified by their teacher. The attempt to have similar groups from the same class subjected to the two different treatments allowed for reasonable similarity of background experience and range of student ability. Pre-test items verified these similarities.

Scope of the teaching. Both teachers initially came to agreement on the end-point of their lessons: the students were to be able to differentiate simple polynomial functions of the form $f(x) = ax^n$ by rule. Student understanding of the following concepts was to be established (differently) by both methodologies:

- *gradient of a curve at a point*
- *tangent to a curve at a point*
- *derivative function*
- *derivative at a point*

The traditional teaching would state and use the limit concept and limit notation explicitly and would present the formal definition of the first derivative of a function but the non-traditional computer teaching would use the limit concept intuitively or informally only and no definitions would be presented. Necessary notation (in particular, function notation) would be introduced formally to the traditional group but introduced only informally or incidentally through usage in the computer programs to the computer group.

Instrumentation. A pre-test questionnaire was administered to all nineteen subjects prior to the five class lessons. Records were kept by each student of any written class work in an exercise booklet provided to each student together with all distributed class material. At the end of each lesson the students were asked to fill out a short questionnaire which sought their reflections on the lesson. Both teachers' lesson notes were kept. A post-test was administered the day after the five class lessons.

Pre-test questionnaire (Q1). Items on this questionnaire had been trialled with groups of students at Years 10, 11 and the pre-tertiary level and included questions which sought to uncover students' notions of measures of curves. The results from this piloting showed some consistency between the three groups and with Tall's (1985) results which demonstrated students' difficulties with tangent to a curve. Another item from Hart (1981, p.130) was included to test the students' understanding of the gradient of a straight line - this item was also used to validate the matching of the two groups.

Teachers' notes. The notes of the teachers were kept to check what was taught in order to match student notes and to assist in the description of the methods used.

Student class notes. All notes taken by students and work undertaken in a written form were collected for verification of what was taught and for a search for possible sources of error either in the misreading of teacher notes or of consistent errors in set-work. Students were also encouraged to jot down in their booklets, during the lesson, notes on anything which was troubling them - this was to assist them in their reflections at the end of the lesson.

Reflections on the lesson. At the end of each lesson the students in both groups were given five minutes to fill in a short form asking them to describe what had happened in the lesson, what they found hard to understand, and what they found easy to understand. These items were used for comparison with what the teacher had meant to deliver and to trace student confidence and belief in their own progress.

Post-test questionnaire (Q2). The items on the questionnaire administered after the lesson sequence included questions:

- directly testing acquisition of the rule for the differentiation of $f(x) = ax^n$ with verbalisation used as a check of the rule; (skill)
- relating to ability to transfer the rule beyond their lesson experience to date; (transfer)
- seeking explanations for the meaning of the gradient of a curve; (concept)
- directly testing global understanding of derivative of a function; (concept)
- comparable to items on the pre-test to check for growth of understanding; (growth)
- seeking perceptions of the students relating to what was understood or not understood; (beliefs)

The information from the post-test was used to establish differences and similarities between both groups.

Limitations. A learning factor beyond the scope of this study relates to student predisposition to teaching or learning style. It was hoped that the group division would be random in this regard and match usual class distribution.

It may be that some students prefer a visual presentation rather than an analytical presentation and that this could be a key factor in effective individual student learning but, it was felt that as variety of teaching methods is currently promoted in whole-class teaching, and that it is not easy to access student-preferred learning styles or to match them to teaching style in normal practice in schools, it would be reasonable to ignore the factor in this study.

A recent study on visual versus analytical thinking favours "developing every topic with its analytical *as well as* with its visual aspects thus allowing each student to grasp the material in the way which is closer to his cognitive orientation" (Eisenberg and Dreyfus, 1986, p. 158) (italics added).

The possible novelty factor of the non-traditional teaching using computer software was disregarded as it was believed that the students were reasonably familiar and adept at using the particular computers and were in a familiar classroom setting.

FINDINGS AND DISCUSSION

Subtleties of the activities. There was evidence that the case study students in the computer group (C) were missing the subtleties of the activities. Use of different technology may require the teaching process as well as the learning process to be changed.

Teacher's aim (Computer group, lesson 2): to use the magnifying option of *A Graphic Approach to the Calculus* to demonstrate that for points on the graph of a polynomial function the curve approximates a straight line as we zoom in closer and closer to the point.

S1(U,C):	On the computer we put in a function and we magnified a point so we could *see the point more easily*. (Italics added)
S3(U,C):	We used the computer to magnify places on the computer to *find a curved line*. (Italics added)

Student number one (S1), from the upper ability group (U) of the computer treatment (C), while wanting to "see the point", has not picked up that the neighboring 'straightness' of the curve around the point is the focus here. Student number three, S3, also from the upper ability group, wanting "to find a curved line", has not picked up the local straightness theme at all.

Other students in the group failed to mention the magnifying at all, whereas other students (from all ability groups, U,M,L) reported differently.

S5(U,C):	Magnify shows where the line is straight.
S8(M,C):	The magnifying was seeing which part of the graph was straight.
S10(L,C):	Magnifying, magnifies the line where it goes straight.

It may be that some of the upper ability students are operating on a more complex level and are actively seeking understanding rather than just picking up on key instructions.

Monitoring their own learning. There was also an indication that some students in the upper ability group (U) of students were able to identify when they were having a learning difficulty. At the same time, the middle and lower ability students (M, L) did not report any problem even though their descriptions of the lesson content did not indicate that they had taken in the lesson ideas fully.

Activity (Computer group, lesson 3): determining gradient at (1,1) on $y = x^2$ using P(1,1) and successive Q(2,...), Q(1.9,...) ....Q(1.001,...) .

Question:	What was hard to understand?
S2(U,C):	The tangent lines and what P and Q find.
S10(L,C):	Nothing, I found it easy.

(Traditional group, lesson 1): limits of discontinuous functions.

Question:	What was hard to understand?
S3(U,T):	Undefined problems e.g. 1/0. How do we know when an equation is undefined?
S6(M,T):	Undefined problems e.g.1/0. I thought it would equal = 1. Limits were easy.
S9(L,T):	I found the limits easy.

Short-cuts: Meaning vs. formula. There was almost relief, as students recognised that they were struggling with ideas, when a rule or "short-cut" was provided.

Activity (Traditional lesson 3): First principles derivation leading to the pattern for the derivative of a polynomial function.

S1(U,T):	I found the long way difficult to understand. The short cut (was easy).
S6(M,T):	Trying to remember the formula the long way (was hard). The quick way (was easy).

Most students found this lesson hard to understand. The complexity of "the formula" appears to have been a problem.

S2(U,T):	I didn't understand any of it really.
S3(U,T):	How to find the gradient of a curve itself (was hard to understand).
S1(U,T), S4(U,T), S5(U,T), S6(M,T), S9(L,T):	The formula m = f(x+h) - f(x)/ h (was hard).

This was the pivotal lesson in the sequence where the first principles derivation was meant to give meaning to the gradient of a curve at a point. Only S1 appears to have picked up some ideas here but she too reported that

S1(U,T):	*Getting used to* the formula ... (was difficult). (Italics added)

Use of rule. Skill acquisition in taking the derivative of a simple polynomial function was not markedly adversely affected by the computer treatment. While successful use of rule is eventually a priority it was surprising to see *how quickly* it had a tendency to take over as the meaning for what was being done.

Global understanding of derivative. The computer group only marginally outperformed the traditional group in recognising the nature of the gradient (positive, zero or negative) of a curve along its 'track'. This was surprising given that the major focus of the treatment was on f ' (x) being a *function.* The benefits might be longer-term, however.

SUMMARY AND RECOMMENDATIONS

Limited concept image for gradient. It was found that most students in the study had a limited concept image for gradient ("measure rise over run") and it is recommended that the development of the global ideas associated with the gradient of a straight line be a focus of learning *before* the idea of gradient of a curve is introduced in beginning calculus. Some students are still indicating fundamental problems with slope as a rate of change and are keying into the x-axis rather than the nature of the slope to state whether it is positive, zero or negative.

Tangent to a curve. It was found that the use of a tangent to a curve at a point to measure gradient of the curve was not a spontaneous intuition and it is recommended that more time be given to this notion in the first principles approach to differentiation. The traditional first principles approach was found to be too cognitively demanding for the students who demonstrated a 'rush to the rule' for meaning. Students undergoing the computer treatment also demonstrated this 'rush to rule'. It appears that it is very easy to foster instrumental understanding even with the use of dynamic and self-directed software where relational understanding is the teaching goal. The question is whether or not this suits the needs of some students; perhaps, *for some reason*, some students prefer to be able to *do something* early on rather than spend time on exploration. Decisions for teaching need to be well informed about student needs if new methodology is to be properly exploited.

In this study of student experiences in their first few days of calculus it appears that there are considerable learning difficulties in both modes of learning and that the introduction of "the rule" too soon may in fact be distracting the students from the underlying processes *even at this early stage*. Rather than the concepts being left behind, as facility with rule improves, perhaps the concepts are never taken on board at all. For many students the ability to use the rule appears to have offset their early conceptual struggles and the rule becoming the meaning (what you do is what it means) just takes over.

The computer group appears to have reached a reasonably similar position in terms of skill acquisition to the traditional group with less feeling of finding the work hard to understand - if anxiety can be reduced then future work in calculus may be more accessible.

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CHILDREN'S APPROACHES TO MATHEMATICAL PROBLEM SOLVING

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This paper reports on one aspect of a study designed to explore the role of metacognition in fourth grade children's mathematical problem solving. A response mapping technique was developed to represent individual problem solving episodes in terms of Flavell's (1981) model of cognitive monitoring. The consistency of the patterns observed in the individual models suggested a relationship between conceptual and procedural knowledge of a cognitive and metacognitive kind. A meta-model was proposed to describe and explain the nature of this inferred relationship. The paper provides an outline of the response mapping technique, a summary of the meta-model, and an example of the response map data. The analysis supported the view that children's approaches to mathematical problem solving can be described in terms of the four generic approaches identified by the meta-model, that is, a Solver's approach (high conceptual-high procedural), a Diver's approach (high conceptual-low procedural), a Player's approach (low conceptual-high procedural), and a Survivor's approach (low conceptual-low procedural).

BACKGROUND

The work reported here derives from two teaching experiments involving upper primary grade children and their teachers. The first, conducted from an information-processing perspective, was designed to investigate the effects of metacognitive training on children's mathematical problem solving performance over a ten week period. Two questions prompted the first study, (1) to what extent was metacognition a "driving force" in children's mathematical problem solving, and (2) to what extent could children's cognitive awareness and ability to monitor and regulate their actions be improved through training. A ten-week teaching experiment involving three grade four classes and one grade six class was designed to address these questions (Siemon, 1986). While the results of the initial study supported the view that metacognition as it was defined, is a "force" governing mathematical problem solving behaviour, and that for some students enhanced metacognition can be achieved by training, the complexity of the problem solving behaviour observed suggested that metacognition is not a single or even bi-valued "driving force", but a multiplicity of complex "forces"and relationships not all of which may be operating in the same direction at the same time. The second, and more major of the two studies was conducted from a constructivist's perspective. A year long teaching experiment was designed to explore the role of metacognition in children's mathematical problem solving, specifically, the sufficiency and viability of a theoretical meta-model of children's approaches to mathematical problem solving derived from a *post hoc* analysis of the individual interview data obtained from the first study. A summary of the meta-model and the response mapping technique which helped generate it will be addressed in this paper.

THE FIRST STUDY

The uniqueness and significance of children's existing knowledge, beliefs, goals, and motivations, in relation to their subsequent learning and problem solving attempts was powerfully demonstrated by the first study. As a result, problem solving was seen as an exercise in negotiating meaning, not as a "process" to be taught and exercised in schools because it was the "focus" of the decade. It was apparent that a theory of learning as opposed to a theory of instruction was needed, and a view of metacognition which recognised the importance of existing metacognitive knowledge, beliefs and experience in the construction of mathematical meanings and solution strategies. The underlying theory of learning was found in *constructivism* (for example, Cobb, 1987). The more encompassing view of metacognition was found in Flavell's (1981) Model of Cognitive Monitoring, in which he includes the two recognised aspects of metacognition, *metacognitive knowledge* and *metacognitive experience*, and adds two other components, *cognitive goals* and *cognitive actions*. From a constructivist's perspective of problem solving, this model seemed to provide a better explanation of what had been observed in the protocols than the two component models so prevalent in the problem solving literature conducted from an information processing perspective (for example, Schoenfeld, 1983; and Garofalo and Lester, 1985).

THE RESPONSE MAPPING TECHNIQUE

On the basis of this re-evaluation of research goals and meanings, a protocol analysis scheme was developed which recognised the interactive role of metacognitive knowledge, metacognitive experiences, cognitive goals and cognitive actions. The scheme was adapted from the diagrams used by Biggs and Collis (1982) to characterise the various levels in the SOLO Taxonomy. While the basic system of recording used to generate the maps is similar to that used by Collis and Watson (1989), there are some important differences. For instance, the response mapping technique does not assume that information or procedures not given in the problem stem but essential for solution, are available to the solver if they have been taught or can be deemed to be in the "universe of discourse" (Collis and Watson, 1989, p.182). The technique is not concerned with identifying the particular SOLO level of a child's response nor with classifying problems in terms of SOLO levels on the basis of written test data. It was developed to accommodate clinical interview data obtained from 9 to 10 year olds. A major difference in the technique, at least in its formative stages, is the inclusion of a summarised record of interview which can be used to support an inference or to elaborate a response. The following symbols are used to construct the response maps developed for this study: **I** - information provided in the problem stem or by the researcher, which may or may not be relevant to solution, and which may or may not be attended to by the solver; and **o** - information observed or inferred on the basis of transcript evidence or the child's actions in the course of a problem solving attempt, that is, information provided or generated by the solver which may or may not be relevant to solution.

In this context, *information* includes: numerical data, problem conditions, questions, cognitive goals, items of cognitive or metacognitive knowledge (including beliefs, concepts, skills, processes and strategies), and the products of cognitive actions and metacognitive experiences. Two forms of information were proposed: *presented* information and *generated* information.

Two types of presented information were identified, These are listed on the left hand side of the response map. The first type is information provided in the problem stem or by the researcher (indicated by the symbol **I**). The second type is information provided by the solver in the form of an inferred *cognitive goal* or item of *cognitive* or *metacognitive knowledge* (indicated by the symbol **o**), for example, the decision to represent a problem by drawing a diagram, or the belief that because it is a mathematics problem, an algorithm is required.

Generated information is information provided by the solver in the course of the problem solving enterprise as a result of a *cognitive action* or *metacognitive experience*, for example, a drawing, an answer to the first step of a multiple step problem, or suddenly recognising that a cognitive goal or action is inappropriate. Items of generated information are also represented by the symbol **o**, but they appear in various locations to the right of the list.

Classifying cognitive goals and items of metacognitive knowledge as items of presented information recognises that the solver's knowledge, beliefs and prior experience play a powerful role in interpreting and utilising the result of any cognitive action or metacognitive experience. That this distinction is worth making is strongly supported by the transcript data from the first study which suggests that differences in children's approaches to problem solving are related to differences in the ways and extent to which cognitive goals and cognitive actions are generated, retrieved and monitored.

The maps are constructed from the transcripts of interviews and any written record that the student might have made during the problem solving episode. In general, a separate map is constructed for each problem attempted. Where necessary, an abbreviated transcript is presented with the map. "Expert" maps may be generated for the purposes of discussion or elaboration, but no assumptions are made about the uniqueness or "appropriateness" of such maps. An example of an expert map for the *Incy Wincy p*roblem, more widely recognised as the frog in the well problem, appears in Figure 1. The items of presented information are listed on the left hand side of the map. They include the data and question from the problem as well as the items presumed to be brought to the situation by the expert. That is, the recognition (metacognitive experience) that this problem is similar to the frog in the well problem (metacognitive task knowledge), and that it can be solved by representing the situation in some way and by acting it out (metacognitive strategy x task knowledge).

The decisions, to represent the situation and to act it out, are cognitive goals. The implementations of the decisions are cognitive actions which produce items of generated information, a diagram in the first instance, and a solution in the second. These appear to the right of the list. Items observed or inferred as being responsible for prompting a particular cognitive action or item of generated information are linked by lines to the generated item(s) concerned.

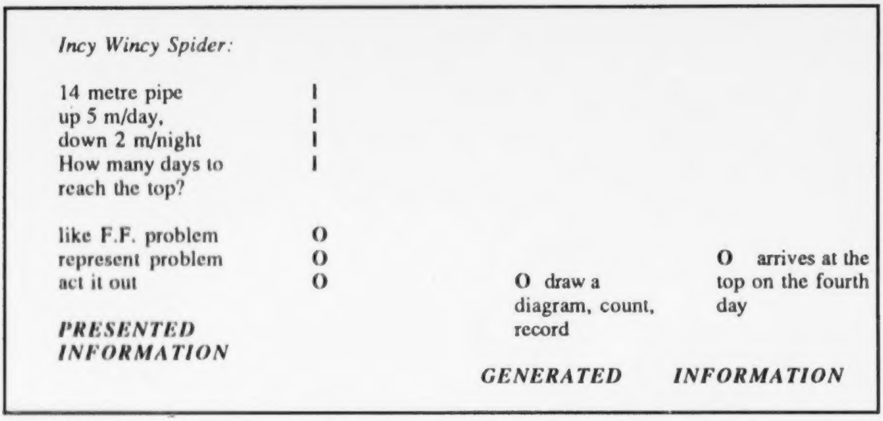

Figure 1: An 'expert' response map for the *Incy Wincy* problem

As each response map was constructed by the researcher in an attempt to explain the child's observed and/or inferred behaviour, no claim is made for the independence and reliability of this technique. Rather, a claim for generalisability is made on the basis of the number of interviews conducted and analysed in this way and on the basis of the broad patterns that were observed over time and problem type.

A META-MODEL OF CHILDREN'S APPROACHES TO PROBLEM SOLVING

The relatively consistent patterns observed in the individual models of problem solving provided by the response maps suggested there was some value in identifying and comparing individual children's problem solving attempts on the basis of the actual solution attempt made rather than on the basis of some pre-set criteria about the perceived nature of the problem or task itself. For example, observable patterns in Julia's response maps suggested that it was her knowledge and beliefs, both metacognitive and cognitive, that was driving her problem solving efforts more so than the particular demands of the task. Cobb's (1987) observation that our "primary objective is to develop explanations that either account for novel, unanticipated observations or resolve conceptual inconsistencies and contradictions within the theory or model." (p.31), prompted the decision to construct a meta-model that was consistent with a constructivist's perspective of problem solving.

From the *post hoc* analysis of the individual interview data it was apparent that different children appeared to attend to and value qualitatively different aspects of the problem solving experience. Julia, for example, seemed overly concerned with procedures and algorithms (procedural knowledge of a cognitive and metacognitive kind), as opposed to understanding and representing the problem meaningfully (conceptual knowledge of a cognitive and metacognitive kind). Julia's response maps revealed that she was more likely to implement a range of cognitive actions on a trial and error basis than to stop and think about what she was attempting to do or what her actions might mean. While she demonstrated considerable skill in monitoring the implementation of her actions, she did not display anywhere near the same commitment to the formulation and evaluation of cognitive goals in relation to the problem's conditions and/or questions. What appeared to be driving her actions was her beliefs about the nature and purpose of school mathematics (prior metacognitive knowledge), for example, her view that doing mathematics is not concerned with meaning: "we don't do meanings in maths". Other case studies suggested a predisposition to value and attend to the construction of meaning. For example, Annie and Nancy (Siemon, 1991), demonstrated a preparedness to thoroughly analyse problem statements before making any decision about what needed to done. This valuing was reflected in their response maps which indicated that both girls tended to formulate and evaluate cognitive goals which in turn informed their selection and execution of a range of cognitive actions. The apparent role of conceptual and/or procedural knowledge, both of a cognitive and metacognitive kind, in the formation and operation of these predispositions, suggested that Julia's predisposition or approach to problem solving could be described as *low conceptual/high procedural and* Annie's approach as *high conceptual/high procedural*. Further analysis suggested that Annie's earlier approach might be classified as *high conceptual/low procedural* which led to the conjecturing of a fourth possible approach, *low conceptual/low procedural*.

It was postulated that a *high conceptual/high procedural* approach would be typified by a high instance of effective goal setting linked to the implementation of an appropriate range of cognitive actions. Instances of cognitive monitoring, both with respect to goals and actions, would also tend to be associated with this approach. A *high conceptual/low procedural* approach was hypothesised to be typified by a relatively high instance of appropriate goal setting, but a correspondingly low level of implementation of appropriate procedures and actions. Where there were instances of cognitive monitoring they would be more likely to be directed at cognitive goals than at cognitive actions. By contrast, a *low conceptual/high procedural* approach was believed to be typified by infrequent instances of appropriate goal setting, but a correspondingly high level of implementation of appropriate procedures and actions. In this case, where there were instances of cognitive monitoring, they would be more likely to be directed at cognitive actions than at cognitive goals. A *low conceptual/low procedural* approach would be typified by low levels of appropriate goal setting and implementation and few, if any, instances of cognitive monitoring.

A systematic examination of all of the response maps generated from the first study transcripts and records of interview revealed that virtually all of them could be described in terms of either one or other of the four approaches hypothesised. The transcripts were then examined qualitatively with respect to such notions as impulsiveness and reflectivity, field dependence-independence, and approaches to learning based on motivation and type of strategy usage (Biggs and Telfer, 1987; Marton and Saljo, 1976). The meta-model which resulted from this empirical and theoretical analysis is shown in Figure 2. It reflects the major dimensions along which the response maps varied and lists the attributes believed to be characteristic of each approach as a result of the qualitative analysis of the individual interview data with respect to the literature on cognitive style. The suggestion of orthogonality is deliberate as the approaches are not meant to imply discrete, mutually exclusive entities, but *tendencies* towards particular behaviours rather than others, in relation to a specific task at a given point in time. To simplify further discussion, it was decided to name the four approaches as the Solver's Approach *(high conceptual/high procedural)*, the Diver's Approach *(high conceptual/low procedural)*, the Player's Approach *(low conceptual/high procedural)*, and the Survivor's Approach *(low conceptual/low procedural)*.

CONCEPTUAL KNOWLEDGE

metacognitive knowledge and cognitive goals

Diver:	***Solver:***
High conceptual/ Low Procedural	***High Conceptual/ High Procedural***
cognitive goals attended to more than cognitive actions	attends to cognitive goals and actions
comprehension strategies: checks, monitors, plans, predicts, links, reflects, introspects,	comprehension and regulation strategies: checks, predicts, monitors, plans, links,
more on knowledge than on actions	reflects, on knowledge and actions.
access to a variety of strategies, not always well used	uses a variety of strategies knowingly
identifies goals	identifies goals and appropriate actions
tends to synthesis and analyse data	synthesises and analyses data
some tendency to conceptually-driven premature closure	strong tendency to persist until reasonable solution obtained
questions directed more at goals than actions	questions directed at goals and actions
some undirected actions	directed manipulation
uses labels, understands structure	uses numbers, labels, structure
deep approach to learning	deep-achieving approach to learning
extended locus of control for actions	internal locus of control
Survivor:	***Player:***
Low Conceptual/ Low Procedural	***Low Conceptual/ High Procedural***
unlikely to attend to either cognitive goals actions cognitive goals	cognitive actions attended to more than
tendency to remember and replicate, but experiences difficulty	remembers and replicates, often quite effectively
strong tendency to premature closure closure	tendency to procedurally-driven premature
tends not to check, monitor, reflect or predict on actions or goals goals	regulation strategies: checks, monitors, predicts, reflects, more on actions than on
experiences difficulty identifying goals	tends to assume goals
experiences difficulty identifying appropriate actions	tends to try a range of actions
more likely to synthesise than analyse	more likely to synthesise than analyse.
little or no questioning	questions directed more at actions than at goals
surface approach to learning	surface-achieving approach to learning
external locus of control for knowledge and actions	external locus of control for knowledge

PROCEDURAL KNOWLEDGE

metacognitive experiences and cognitive actions

Figure 2: A meta-model of children's approaches to problem solving

THE SUFFICIENCY AND VIABILITY OF THE META-MODEL

The second study was similar to the earlier study in that the teacher conducted at least one problem solving session per week which was video-taped. The program explored a variety of problem types and strategies and provided an accessible model of cognitive monitoring in terms of the ASK-THINK-DO problem solving cycle (see Barry, Booker, Parry and Siemon, 1985) . Key components of the program were (1) the mathematics content of the particular problems considered was selected by the teacher to support her curriculum objectives, (2) the problem solving process was specifically talked about in terms of the ASK-THINK-DO cycle and modelled by the teacher whenever a strategy was reviewed or introduced, and (3) key questions, strategies and observations about problem structure were discussed and recorded on a large ASK-THINK-DO problem solving chart which was on constant display in the classroom. Problems were worked on individually, in small groups or as a class. Twelve children were selected to be interviewed on a regular basis. The interviews consisted of a reflective review of a problem considered in class followed by an attempt at a similar or related problem. Response maps were constructed for all interviews conducted (155 interviews over 15 tasks). The meta-model was judged to be sufficient and viable if the observations of each child's problem solving attempts could be explained by the meta-model without excessive contradictions or glaring omissions.

The analysis of the response map data indicated that while there were important qualitative differences in individual response maps, both within and among children, there were some fundamental similarities in the ways in which some children approached problem solving. Essentially, the patterns observed in different children's responses to the same problem and in each child's response to a range of problems could be described by the nature and extent to which the child was observed to be actively engaged in monitoring his or her cognition, that is, in terms of the four approaches identified in the meta-model. For example, some children were more likely to monitor their cognitive actions than their cognitive goals (Player's). Others were more likely to monitor their cognitive goals and their cognitive and metacognitive knowledge than their cognitive actions (Diver's). An example of a response judged to be characteristic of a Survivor's Approach, that is, *low conceptual/low procedural*, is provided below for the *Incy Wincy* problem referred to earlier. The response map is supported by an abbreviated transcript of interview (Kellie's comments are in bold).

Kellie. . (problem presented orally). **.7 days** ... How did you get that? ... **half of 14 and I thought that might be the answer because if you count by twos to 14 ... you'll get 7** ... Do you remember the problem? ... **Yes** ... Where did the 2 come from? ... **Oh, it was 2 she slid down** ... Do you still think it is 7 days? You'd be fairly happy about that? ... Is there any other way you could do it? ... **You could go by 3's to 14, but then you'll have to say when you get to 12 ...if you add another 3 that would be 15, so I'd have to do an extra** ... 3 from? ... **oh, here ... from 5 down to ... there was 3 left.** ... How many days will it take Incy Wincy to get to the top of the pipe? .... **4 and a half days.** ... How did you do that? With the threes was it? ... **yes . ...** so how did you get 4? ... **well, there are 4 threes in 12, and when I said "and a half", that was including, that meant that he had 2 more jumps to go** *(note link to Freddie context, "jumping")* . ... Would you use a diagram? .... **I might ...** but in your head is the first way you would do it? ... **yes, and then, to check it I might do a picture** ... Do you have a picture in your head like you told me you had for Freddie? ... **yes** ... What did you have ... imagine in your head? ... **well ... I was thinking every night he went back 2 ... 2 of the steps went away** ... So how many steps did he go up each day? ... **4 each day ... but in 24 hours ... 2 steps** ... That was for Freddie was it? ... **yes** ... What about Incy Wincy? ... **there were 14 steps and then going up 5 and sliding back down 2 and that meant ... it took him one day, then up 5 and down 2 that took him 2 days, and he kept on going like that until he got to the top** ... So, how many did he do in each day? ... **Altogether 3.**

Incy Wincy:			
14 metre pipe	I		
up 5 metres/day	I		
down 2 metres/night	I	O halving, use number facts, "7 days"	
How many days to reach the top?	I		
2 metres/day (F.F.?)	O	O recalls problem conditions, "oh, it was 2 she slid down"	
Where did the 2 come from?	I		
Any other way?	I		O subtract, 3 metres/day, count by 3's to 14, "4 and a half days"
metres per day	O		
threes in 14	O		
Response Map for:	**Kellie**	Approach:	**LC/LP - Survivor**

Figure 3

DISCUSSION

Quite clearly, the link between the conceptual and procedural knowledge is a dynamic and complex one affected by contextual setting, specific content knowledge, beliefs, motivations, and values. The orthogonal relationship proposed in the model was useful in describing and explaining the children's problem solving behaviour. This feature of the model highlighted the fact that many problem solving efforts failed not necessarily because of a lack of monitoring ability *per se* or even a lack of knowledge as to what strategies to use and when, but because of a lack of access to specific content, and the inability to apply monitoring strategies to what is known as opposed to what is done. For example, many children knew that a diagram was useful for the *Incy Wincy* problem but failed to implement this strategy, not because of their inability to draw, but because they did not have access to a well structured knowledge base which could generate and withstand the negotiation required to establish *what* diagram was needed. This confirms an almost identical result reported by Resnick and Nelson-Le Gall (1987) and Lesh's (1983) observation that "the activities that facilitated successful solutions were those which focussed on conceptual rather than procedural considerations" (p.6). As an exploratory study, the work reported here is believed to have important theoretical and methodological implications for future research and classroom practice.

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BELIEFS ABOUT MATHEMATICS AND MATHEMATICS EDUCATION

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Problem solving is recognised as an essential goal in mathematics education. The performance of primary pre-service teachers, however, appears to be adversely affected by anxiety and other negative emotional and social factors. A previous study by Southwell(1990) indicated that some mature-aged students hold beliefs which prevent them from tackling certain kinds of problems and from being successful in others.

It is possible that if pre-service primary teachers hold beliefs to the effect that mathematics is a very difficult subject, that they are not good mathematicians and that mathematics has limited appeal and use, their teaching will reflect these beliefs and each new generation of learners will be locked into similar beliefs and negative attitudes.

Ball (1988) lists a number of "hunches" she has about what prospective teachers believe about mathematics. They include the following:

> Mathematics does not have much relationship to the real world and most mathematical ideas cannot be represented any way other than abstractly, with symbols. Knowing mathematics means 'knowing how to do it'. Teaching mathematics involves telling (or showing) the students how to do different kinds of problems. Teachers ask questions to elicit right answers; if a teacher questions your answer, it means you made a mistake. Learning mathematics is scary. Good teachers make mathematics fun for students. Elementary school mathematics teaching does not require much knowledge of math - everyone who can add, subtract, multiply and divide knows enough mathematics to teach little kids. Learning to teach, therefore, is mainly a matter of acquiring techniques. Love of children, not knowledge of subject matter, is the basis of elementary school teaching. Young children are eager to learn and trusting, but are not yet capable of thinking about complicated mathematical ideas or solving real problems.
>
> (Ball, 1988)

Several of these "hunches" are supported by studies conducted by Schoenfeld (1989), Thompson (1984) and Cockcroft (1981). Schoenfeld found three beliefs directly related to mathematics learning.

1. The processes of formal mathematics ... have little or nothing to do with discovery or invention,
2. Students who understand the subject matter can solve assigned mathematics problems in five minutes or less.
3. Only geniuses are capable of discovering, creating, or really understanding mathematics.

The Cockcroft Committee commissioned a study of the uses of mathematics in daily life. The results of this study reinforced the belief that it is permissible to not be able to do mathematics whereas very few were willing to admit they could not read. Furthermore the committee contended that the apparent dislike of the subject was attributed to some single event in the past, such as the over-expectations of parents, change of teachers, emphasis on examinations.

These findings led the authors to undertake a study into the beliefs of students and teachers concerning mathematics and themselves as learners and teachers of mathematics. If prospective teachers hold certain beliefs about mathematics and themselves which mitigate against them teaching mathematics with enthusiasm and real understanding then it is the responsibility of the institution preparing these students for their teaching role to undertake procedures to break the cycle which has developed. While there is considerable evidence that a person's beliefs about mathematics, about themselves and about the teaching of mathematics are interrelated and affect performance in mathematical tasks, the authors set out to explore whether these same phenomena exist in the western suburbs of Sydney and whether they persist across different age levels.

The questions the authors raised for study were:

1. What do students and teachers believe about mathematics?
2. Do beliefs about mathematics alter as students progress through the school system?
3. Do beliefs change when a person assumes a teaching, as distinct from a purely learning, role?
4. Do beliefs about ability to learn mathematics affect the learning and teaching of mathematics?
5. To what do students attribute their success or failure in mathematics?
6. What are the implications of teachers' reactions to students' responses to questions?
7. How do students perceive differences between mathematics learning and learning in English and science?
8. What contextual considerations are important in learning mathematics?
9. What is the motivation of students to learn mathematics?
10. What are students' perceptions of their own mathematical ability?

THE SURVEY

Survey instruments were constructed using as a basis the instrument reported by Schoenfeld (1989). Some additonal items were included on the respondents' perceptions of collaborative learning taken from a study by Way (1990). Three similar documents were prepared in order to make them suitable for administration to samples of primary and high school students and teachers. Each survey contained a number of sections each covering a part or all of one of the research questions. The differences between the three surveys were necessary to make the questions and ideas more relevant for the age level and role for which it was intended.

The participants were asked to respond to each item on a four part Likert scale ranging from "1 = very true" to "4 = not at all true".

The sections set out in the survey were as follows:

1.	Beliefs about achieving good grades in mathematics
2.	Beliefs about bad grades in mathematics
3.	Beliefs about the nature of school mathematics
4, 5.	Beliefs about teachers' questioning in mathematics
6,7.	General beliefs about mathematics, people's ability in mathematics, comparisons with English and science, good teaching in mathematics and problem solving in mathematics
P8, S11, T10	Reasons for trying to learn mathematics (P = Primary, S = Secondary, T = Teachers)
	The High School survey also included questions related to geometry.
P9, S12, T11	Grade levels, self-perceptions of ability in mathematics, perceptions of ability and beliefs of parents, and ethnic origin.

The remaining section included some open-ended questions to which the students were asked to provide any other additional comments. This was designed to extract information about beliefs on a less structured basis.

THE SAMPLE

The sample was selected on the basis of availability and convenience. The teachers in the study were fourth year students in the BEd program at the University of Western Sydney, Nepean. Responses were received from a sample of 75 of these teachers which was approximately 80% of the enrolment. The primary students were students in Years 4-6 classes of the teachers in the study. Responses were received from 185 of these students. Because of this situation, it could be said that the sample was biassed towards the positive beliefs since the students were in classes of teachers obviously wishing to upgrade their qualifications. This may be so and needs to be kept in mind when interpreting the data as any negative results may reflect worse general beliefs.

The high school sample came from an independent girls' school in a middle class area of the western suburbs of Sydney. The initial motivation to carry out the study came from a consideration of the affective aspects involved in the problem solving behaviour of mature-aged women. The data collected from a girls' school readily accessible to the researchers were considered to be relevant to the general population. The respondents from this school numbered 310 students in Years 7-10.

TIME

The surveys were administered to the high school students in November, 1990. The other surveys were completed in April, 1991.

RESULTS

1. From an examination of Table 1, a general belief exists that good grades in mathematics were the result of hard work and not because of any element of luck or teacher attitudes. The teachers reported that liking the pupils (2.5) was more likely to influence the pupils' results compared with the pupils' perceptions which indicated these are not related at all.

Table 1: Reasons for Good Grades

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
1	Worked hard	1.6	1.6	1.5
2	Teacher liked me	3.5	3.7	2.5
3	Luck	2.6	2.8	2.7
4	Good at maths	2.2	2.8	2.0
5	never knew	2.9	3.0	2.8

2. In both the secondary and teacher responses, the perception is that lack of hard work contributed to poor grades. No relationship between teacher attitude to students and grades was found as far as primary pupils were concerned. The results were not so definitive as far as teachers were concerned.

   For all respondents, we found that careless mistakes contribute to poor grades. This was more evident in the responses of the high school students and the teachers. It is probably the result of complicated procedures which require several steps. Mistakes can occur at any stage of the operations. (Table 2)

Table 2: Reasons for Bad Grades

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
6	Lack of hard study	2.3	1.7	1.8
7	Teacher didn't like	3.6	3.7	2.8
8	Bad luck	2.5	3.1	2.7
9	No good at maths	2.8	2.7	2.6
10	Careless mistakes	2.1	1.7	1.7

3. All respondents generally perceive that the mathematics children learn in school consists mainly of facts and procedures that have to be memorised but that mathematics is thought provoking.

A general belief appears to exist that students have to think hard to provide answers to questions and that occasionally there would be a number of answers. However, there appeared to be agreement that questions do have right answers and the students who understand the work require less time to answer questions. (Table 3)

Table 3: Perceptions of the Nature of Mathematics Learned

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
11	Facts and procedures	1.7	1.9	2.2
12	Though provoking	1.6	1.9	1.7
13	Space, numbers and problems	2.3	2.4	1.8
14	Remember right answer	2.6	2.4	2.3
15	Lots of right answers	2.5	2.7	2.1
16	Think hard	2.3	2.5	1.8
17	Few seconds for those who understand	1.9	1.9	2.0

4. There appears to be general agreement by the three groups about the nature of classroom interactions. When teachers ask questions pupils cannot answer, they try to keep the lesson moving by asking someone else. Respondents indicated that an attempt is made to allow a "short" period of time for pupils to answer then the teacher moves on. All respondents generally agreed the teacher tries not to answer all the questions. (Table 4)

Table 4: Possible Alternatives when Students Cannot Answer Questions

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
18	Someone else asked	2.4	2.6	2.2
19	Teacher will answer	3.1	3.0	2.8
20	Everyone waits	2.3	2.3	2.3

5. Students generally believe that people are just good at mathematics and that there are correct procedures to follow in order to get the right answers.

   The perception was just as valid for English and for science but slightly fewer students believe that answers in science and English are either right or wrong. (Table 5)

Table 5: Perceptions of the Nature of Maths Compared to Other Subjects

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
21	Some are good at maths	1.4	1.6	2.0
22	Right or wrong	1.4	1.7	2.4
23	Good teachers - different ways	1.4	1.7	1.3
24	Good teachers - exact way	1.9	2.0	2.8
25	Good at English	1.5	1.7	2.1
26	English right - wrong	1.8	2.9	3.1
27	Good English teachers - different ways	1.8	1.8	1.5
28	Good English teachers - exact way	1.9	2.4	2.9
29	Good at Science	1.6	1.7	2.3
30	Science right - wrong	1.7	2.1	2.7
31	Good Science teachers - different ways	1.9	1.9	1.4
32	Good Science teachers - exact way	1.9	2.2	2.8

All respondents indicated that teachers try not to show students exactly what they have to do in order to answer the question. This perception was stronger amongst teachers than students. In addition, teachers believed more so than the students that in mathematics you can be creative and discover things.

Some evidence exists to suggest that students' perceptions of good teachers were related to being shown how to solve problems. Teachers on the other hand did not think this way.

Primary and secondary students appeared to seek the teachers' guidance to obtain solutions. Both groups felt that "good" teachers showed their students the exact procedures to solve problems.

This belief was not supported by the teachers who appeared to indicate that students should not be shown the exact procedures but should be encouraged to explore different ways of solving problems. This is supported by the teachers' response to (T, 34).

There were some differences in the way the respondents perceived the solving of problems. Generally, all respondents felt there could be a number of ways to solve the problem, but primary and secondary students emphasised the remembering of the rules much more so than the teachers. All respondents felt that common sense played a role in problem solving. (Table 6)

Table 6: Perceptions of Learning Strategies Associated with Maths

ITEM No	ITEM	PRIMARY PUPILS MEANS	SECONDARY PUPILS MEANS	PRIMARY TEACHERS MEANS
32	Everything important is already known	na	2.1	3.2
33	In maths you can be creative	1.7	2.1	1.4
34	Maths problems done	2.6	2.9	3.4
35	Solve by common sense	1.9	2.4	2.3
36	Taught the right method	2.1	2.1	2.8
37	Remember all rules	1.6	1.8	2.8
38	Maths - everyday life	1.8	1.8	1.2
39	Work in groups	1.6	1.9	1.4

All respondents also felt that mathematics was important for everyday life.

6. Secondary students appeared to reflect the conventional teaching approach which required students to check their answer with the answer in the back of the textbook and if there was a difference the student was more likely to be wrong. When students got the wrong answer they generally "started" again. (Table 7)

Table 7: Possible Courses of Action when the Answer is Perceived to be Wrong

ITEM No	ITEM	PRIMARY PUPILS MEANS	SECONDARY PUPILS MEANS
42	Wrong - no argument	2.4	2.7
43	Different to back of book	2.1	3.1
44	Start again	1.3	2.4
45	Mathematician already verified it	1.8	2.4
46	Statements/reasons in proper form	1.6	2.4
47	Better understanding maths thinking	2.0	1.8
48	Can't remember next step	2.8	3.1
49	Discover things about geometry	2.2	1.7
50	Nothing to do with real world	2.8	3.1
51	Doing something useful	2.0	1.5
52	Constructions	2.3	na
53	Memorise	1.9	na
54	Not related	2.5	na
55	Easy to figure	2.5	na

Responses to questions 45 - 55 indicated that the girls in the secondary school generally had a slightly negative attitude to mathematics. Mathematics was not easy to figure and had nothing to do with real life although the latter perception is not as evident (mean 2.8). Some discrepancy exists about this - e.g.

doing something useful	2.0
not related	2.5
easy to figure	2.5

7. Reasons for doing mathematics included the following:

Students generally wanted to do well in mathematics even though some reported they were only doing mathematics because they have to.

They did not appear to be overconcerned with "feeling stupid", "looking dumb" or "trying to make the teacher think they were good."

There was general agreement about this issue among the three groups. (Table 8)

Table 8: Reasons for Learning Maths

ITEM No	ITEM	PRIMARY PUPILS	SECONDARY PUPILS	PRIMARY TEACHERS
		MEANS	MEANS	MEANS
56	Help to think clearly	1.7	2.0	1.3
57	I have to	2.0	1.8	2.3
58	Want to do well	1.5	1.5	2.1
59	Was interesting	2.0	2.4	1.5
60	Get into trouble	2.7	2.8	2.4
61	Feel stupid	2.1	2.3	2.5
62	Don't want to look dumb	2.1	2.3	2.5
63	Make teacher think I was good	2.7	3.1	3.0

DISCUSSION

1. The belief that good grades are the result of hard work is in keeping with other studies on attribution theory (Barnes, et al, 1984). In these studies, females were found to attribute their success to hard work rather than luck or ability. As all the subjects in the secondary sample in this study were female, as was also the majority of teachers in the sample (78.5%), this result was to be expected. The inference can be made, however, that at the primary level, the male-female attributions of success are not yet very strongly developed. It is at this early stage that work needs to be carried out to obviate the danger of later attributions. The prevalence of this belief among teachers is viewed with concern, particularly when one realises that the subjects in the study were all graduates of a teacher preparation course and were aspiring to complete a BEd (Primary) a degree. A further consideration is that 40.5% of the teacher sample perceived their mathematical effort in school to have been above average compared to their effort in other subjects and 48.1% perceived it as average, leaving only a small percentage (11.4%) perceiving it as below average. Contrasting with these percentages, 38% perceived their actual ability as a mathematician to be above average, while 51.9% considered themselves average. This downward trend continues with only 27.8% seeing themselves as above average in teaching mathematics and 60.8% see themselves as average. These results need to be viewed cautiously as 63% of the teacher sample have been teaching for less than five years.

   The discrepancy between the perceptions of the primary pupils and the teachers on the effect of the teacher liking the pupil implies an emotional response to the teachers' attitudes reflected in mathematical achievement may be expected on the part of the teachers. An investigation of the interactions within a classroom and the effects on student learning at different levels is needed as the secondary students also appeared to share the perceptions of the teachers.

2. Again, the two predominantly female groups in the sample believe that lack of hard work results in poor grades, while no similar relationship was found amongst the primary pupils.

3. The belief that mathematics learnt in schools is mainly facts and procedures to be memorised is an alarming result in view of the many statements of a much more inclusive nature that have been issued in recent times (*National Statement on Mathematics for Australian Schools, NSW K-12 Statement of Principles for Mathematics in K-6 Syllabus*). The one heartening feature about this is that the teacher sample viewed mathematics in this way marginally less than the primary and secondary students.

   This belief that mathematics is mostly facts and procedures implies that it is being associated with arithmetic only. This is a view which has been very prevalent in the past with most syllabi in primary mathematics emphasising the number aspect of mathematics rather than geometry and other branches of the subject. An attempt to overcome this was made in the *NSW K-6 Mathematics Syllabus* by placing the Space strand at the beginning of the syllabus.

   There was a general belief that mathematics is thought provoking but one wonders whether this is because it is viewed as difficult or because it is a subject to be seriously considered. Unfortunately, the former of these views appears from other responses in the survey to be the more applicable one.

   That mathematics is a subject in which one has to think hard does not seem to be linked in the minds of the primary and secondary students with the idea that mathematics is thought provoking. This seems to indicate that mathematics is a subject where effort is necessary. Teachers on the other hand indicate a belief that one can do mathematics with less effort, implying that there could be greater emphasis on structure, making connections and processes.

   An obsession with 'the right answer' is indicated on the part of all subjects. There is also, however, some indication that all the subjects recognise the possibility of more than one answer to a question. The emphasis on remembering the right answer is particularly alarming when there is so much emphasis in modern curricula on processes. This aspect of mathematics education is particularly relevant for assessing student learning. If memory is to be the chief determinant of success in mathematics then the affective implications are horrific. Under stress as in test situations, students' memories may be very unreliable.

4. That teachers do not attempt to answer all the questions asked by students may relate to the way in which teachers have been 'trained' by 'redirecting' questions to other pupils. Trying to involve the total group is indicated by the fairly consistent response by the three groups in the sample, all of whom reported that the teacher does not usually answer the question. (3.1,3.0, 2.8). The Cockcroft Report makes a comment on what has come to be called 'teacher lust' where teachers give students more than prompts or clues either by telling them the method to be followed or the answer to a question. Lovitt and Clarke (1988) also discuss the issue of 'wait time' and stress the importance of allowing students time to focus on a question and think through it.

5. The belief that mathematics is a 'cut-and-dried' subject comes through again in beliefs that people are just good at mathematics and there are certain procedures which must be followed in order to get the right answer. The belief that in mathematics something is either right or wrong raises an number of issues, not least of all the effect on one's emotions of always being wrong, or of believing that one is not good at mathematics and can never be any better.

   Similar beliefs in connection with English and science are evident though these do not appear to be as strong as for mathematics. This reinforces the public perception that mathematics is the most difficult and narrowest subject students encounter.

   The importance of the teacher showing students different ways to look at the same question is seen to be greater in mathematics than in English or science. This may be because when students have difficulty with a particular question, the teacher explains another method is possible. This situation may not occur as often in English and science. Despite this, there still seems to be quite an emphasis on teachers showing students the 'exact procedure' to obtain an answer.

   For the primary and secondary samples, remembering rules is considered the most important thing required in order to do well in mathematics. This supports a previous observation.

   The belief that one can be creative in mathematics as indicated by the primary sample and the teachers and less so by the secondary sample may be negated by the examination system which exists. When one is constrained by the demands of external examinations and set syllabi there could develop the belief that mathematics is only what fits into these categories. This means that secondary students are denied the opportunity of exploring mathematical situations and making discoveries.

6. Attitudes of secondary students to getting 'wrong' answers are interesting from the point of view of the implications which can be drawn from them. That they accept the decision they are wrong without argument could mean one of two or three things. It could be that the student has already detected an error in her working and the teacher is simply confirming this or the student believes that her ability is low, therefore the teacher must know better. A third possibility is that other emotional factors prohibit the students from debating the answer. The tendency to start the question again rather than to consider the incorrect effort and amend it reinforces the view that the second of these theories is more likely to be the relevant one.

7. Reasons for learning mathematics differ slightly between the school students and the teachers. The primary and secondary students tended to learn mathematics because they wanted to do well, whereas teachers leaned more to the benefit of thinking clearly. This could be because the teachers consider themselves to have already done well and were now striving to improve their mathematical thinking. The existence of some of the rather negative reasons for learning mathematics - because they have to - calls into question the levels at which mathematics should cease to be compulsory. This also relates to beliefs about the usefulness of mathematics in everyday life. As reported before, most respondents agreed that mathematics is very important in everyday life. That being so, some greater attempt may need to be made to incorporate real life situations into mathematics or to at least help students make the connections between mathematics and the real world.

IMPLICATIONS FOR MATHEMATICAL PROBLEM SOLVING

Certain beliefs about mathematics are shared by students and teachers alike. When these are restricted, narrow and unimaginative, the way in which mathematics is taught and assessed will also be restricted, narrow and unimaginative. This kind of approach is entirely opposed to the view that considers problem solving as the key to mathematics. In mathematical problem solving emphasis is placed on thinking skills, the use of both inductive and deductive reasoning, on being creative and adventurous in approach to the mathematical situation.

Some of the beliefs thus held appear to be the following:

1. You are either good at mathematics or not.
2. Answers in mathematics are either right or wrong.
3. If you do not get the answer, you just start again.
4. Mathematics is important for everyday life.
5. Memorising facts and procedures is the way to learn mathematics.
6. Mathematics is arithmetic.

These beliefs, with the exception of Number 4, are likely to lead to failure in mathematics education where mathematics is not viewed in the way specified in the *National Statement on Mathematics for Australian Schools.* They are reinforced by the methods which have in the past been used to assess student learning in mathematics.

Greater efforts need to be made to enlarge the students' concept of mathematics. While this requires a general public education program since certain beliefs are acquired long before children get to school, some things can be done on a local level. One is to ensure that spatial and measurement concepts are given equal importance as number concepts in planning and teaching mathematics. Another is to ensure that varied teaching styles are used with considerable emphasis on the role of language and concrete materials.

Problem solving and mathematical investigations can play a significant role in this regard. Both problem solving and mathematical investigations require a reflective approach if they are to be effective in facilitating mathematics learning. Because the problems and the situations to be investigated can be focussed on geometry or some other branch of mathematics, emphasis on them should help students to see mathematics as far more than arithmetic and far more than rules and procedures. Such processes as specialising, generalising, enumerating, conjecturing and testing conjectures all form an important part of mathematics and help the student recognise the great scope of the subject.

Methods of assessment which emphasise reflective procedures for students should also be used. These could involve the preparation and implementing of personal checklists, practical tasks, the keeping of mathematics journals, preparing written test items, oral tests/interviews, projects and other written reports.

While not wanting to imply that hard work is not necessary, some emphasis should be put on willingness to tackle problems and on the importance of thinking rather than remembering rules. Again problem solving and mathematical investigations are good ways in which to make this emphasis.

Just as considerable emphasis has been placed on the desirability of matching teaching styles to the students' learning styles, so greater effort needs to be made in matching affective considerations to teaching styles. For this purpose some assessment procedures should ensure that students' attitudes and beliefs are known to the teacher and that the teacher has the skills to bring about change where necessary. Assessment procedures which have a reflective component are best suited for this purpose.

Reflective procedures are ones which allow the student and the teacher to return to their previous experience, eliminate negative feelings, emphasise positive ones and integrate new knowledge and understanding with their existing understandings. This type of cycle can do much to obviate the negative aspects of assessment and enhance the students' opportunity to learn mathematics effectively and happily.

While there are no great differences between the three groups on any of the items, there are several in which slight differences indicate certain trends. The perception that the lack of hard work and careless errors contribute to poor results does not appear to be as strong with primary students as it is with secondary students and the teachers.

There are some differences in the way the respondents perceive the solving of problems. Primary and secondary students emphasise the remembering of rules much more than the teachers.

While this study does not have the capacity to indicate that beliefs about mathematics affect mathematics learning, there are some related issues to be considered. When a student holds a belief such as that one is either good at mathematics or one is not and then decide on the basis of a single failure or difficulty to understand what is going on that he or she is no good at mathematics, then there is little if any, motivation to try the mathematics presented to the class. Also, if one convinces oneself that one cannot do mathematics, this will become a self-fulfilling prophecy.

Motivation is a critical issue in learning and its role in mathematics learning cannot be underestimated. The emphasis on investigating mathematical situations or solving problems, especially if they are ones which are relevant for the students, could well be the key to motivation in mathematics. If students are encouraged to pose their own problems and find ways of solving them, they will have a sense of achievement which will be invaluable in developing positive and enthusiastic approaches to mathematics.

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USING OPEN QUESTIONS FOR TEACHING: A CLASSROOM EXPERIMENT

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INTRODUCTION

This is the report of an investigation of one classroom teaching strategy which has the potential to facilitate the achievement of the stated curriculum goals within the constraints of the classroom. There are three criteria which are necessary for any classroom strategy to be effective. First, the strategy must acknowledge the way that children learn the discipline. Second, it must accommodate the sociological constraints operating in classrooms. Third, the strategy must address the content goals of the discipline. These are discussed in the following sections.

THE LEARNING OF MATHEMATICS

The first criterion for a successful classroom strategy is that it must acknowledge the way that students learn. The view of learning as "a process of interactive adapting to a culture through active participation" (Bauersfeld, 1991, p.6) is useful. Learning which is an outcome of a child's exploration of a mathematical task, which highlights connections to existing knowledge, which is dependent in part on the child's own strategies, and which is linked to the child's experiences, is more likely to be useful in the future than learning based on rules or taught procedures.

The classroom interpretation of this is contentious. Obviously it will not be productive if the teacher just "tells" children how to do procedures. There is simply too much mathematics to learn it all by rote, and in any case the mathematics we need is continually changing. If mathematics teaching emphasises rules and procedures, this can create the impression that mathematics is about remembering, rather than working things out for one's self. On the other hand, it is not clear that pupils will experience the full range of mathematics needed for participation in our society solely as an outcome of undirected explorations. For example, we do not expect that all pupils would construct the system of operations with decimal numbers by themselves. There is also considerable risk that structures constructed through undirected exploration may not be adequate (Kilpatrick, 1987). We believe that pupils will need some teacher guidance for their learning to be effective.

This view of learning suggests that teachers must have an active role in planning and reviewing classroom mathematical activity while at the same time allowing flexibility for students to explore mathematical situations and to record their own solutions in their own ways.

THE CONSTRAINTS IN THE CLASSROOM

A second criterion is that the classroom teaching strategy must acknowledge that context can influence both the response of the students and the subsequent actions of the teachers. The implication for teaching is that students reorganise beliefs about the classroom, and about mathematics in particular, to resolve problems which are primarily social rather than mathematical. When asked a question in the classroom, the pupil's short-term goal is to give an answer so that the spotlight will shine elsewhere. Students who primarily seek to satisfy the perceived demands of the teacher have neither access nor motivation to appeal to relevant models or to inconsistencies in their own understandings. Such students could well be reluctant to listen to explanations given by their peers, for example.

It has also been argued that the pupils, in turn, have a major effect on the way that teachers teach. Doyle (1986) argued that the demands of academic work are shaped by a complex negotiated process between teachers and students. On one hand, students try to reduce the risk of failure by seeking to increase the explicitness of task requirements and to reduce the level of accountability, thereby narrowing the demands of the task. Teachers, for their part, tend to react to the response of the pupils by selecting tasks which are familiar and easy. Desforges and Cockburn (1987), in a study of primary classrooms in the U.K., noted the students' capacity to avoid thinking about classroom tasks wherever possible. In other words, any classroom strategy needs to address the pupils' view of the purpose of schooling and how mathematics is learned.

THE CONTENT GOALS OF MATHEMATICS

A third criterion by which any classroom approach can be evaluated is the extent to which the students learn the content which was the focus of the program. The NCTM (1989) described three categories of mathematical content which students should learn. These are empirical concerns, abstract concepts, and higher order reasoning skills. So, while on one hand it is necessary to see the content of mathematics as more than facts and skills, and to recognise that the mathematics used by our society is changing, on the other hand the content, as defined here, forms a critical component of mathematics teaching. It follows that the success of any teaching strategy can be determined, in part, by the extent to which the students demonstrate proficiency at the content of the program.

THE TEACHING STRATEGY

One way to seek to implement such a classroom regime, and to change the students' view of mathematics learning, is by focussing on the questions which teachers ask. This is relevant because questions are the main way in which teachers and pupils communicate (Sullivan and Leder, 1991) and because questions transmit the classroom requirements explicitly. A change in the type of questions asked will, by necessity, change most components of the mathematics classroom.

The particular type of questions which are the focus here are both content specific and open. Sullivan and Clarke (1988) called such questions 'good' questions. 'Good' questions are defined as those which are more than recall, educative and open. Examples of 'good' questions are as follows:

A number has been rounded off to 5.8. What might the number be?
Draw some triangles with an area of 6 sq cm.
Find two objects with the same mass but different volume.
Describe some boxes whcih have a surface area of 94 sq. cm.

Each of these questions requires pupils to do more than recall a set procedure. They provide opportunities for students to learn from the act of working on the question and they allow for a range of correct answers. These features each contribute to the educational potential of the questions, as is discussed in the following sections.

More than recall. Pupils can answer most classroom questions merely by recalling a procedure (Sullivan and Leder, 1991). Education for the future requires more than this.

Classroom questions have, at times, been categorised as high or low level, knowledge or evaluation, convergent or divergent and subjective or objective (Ornstein, 1987). Sullivan and Leder (1991) used nine cells in a two dimensional grid with structured to non-structured on one axis, and from recall to extension on the other. More common is the simple dichotomy of fact questions and higher order questions (Gall, 1984). While research on the effect of higher cognitive questions is not conclusive (Samson, Strykowski, Weinstein and Walberg, 1987), it seems hardly likely that teachers could stimulate higher order thinking if they ask only recall questions.

The level of questions asked also reflects the quality of understanding which the teacher seeks. Basically, pupils will be unlikely to aim higher than their teacher does. Likewise if the context of schooling appears to emphasise the accurate repetition of procedures, that is what the students will concentrate on (Cobb, 1986). Of course, procedural knowledge can be an important component of mathematical learning (Nesher, 1986). Nevertheless, the capacity to develop and expand the base of mathematical knowledge, to link concepts effectively, and to learn in the future will depend on well formed mental structures or schema which are fundamental to learning.

Educative. Perhaps the most important aspect of 'good' questions is that learners can become engaged in the tasks and so learn particular mathematics concepts as a result. Consider, for example, the question:

The perimeter of a rectangle is 30 cm. What might the area be?

Many students see the perimeter and area as linked. Working on this task can allow children to see that even if the perimeter of a rectangle is constant, the area can vary. The students may either find this out for themselves or they might learn it in the review of the questions from the responses of other pupils. In either case, this key measurement concept is addressed through the activity of the pupils rather than the instruction of the teacher.

Open. The third component of 'good' questions is that they are open. Open questions are used in a variety of contexts in education (Cliatt and Shaw, 1985; McGinty & Mutch, 1982; Necka 1989). Open questions can facilitate divergent thinking, imply that a range of possible answers exist, and provoke students to respond at a variety of levels. They require pupils to base their response on their own thinking rather than to mimic their teacher. Owen and Sweller (1985) argued that open questions facilitate learning because the learner is not required to attend to directed goals and so has more processing capacity available to explore the overall situation.

In short, the focus on content allows the content goals of mathematics teaching to be achieved. The openness of the tasks assists in the learning and enhances their applicability to classroom teaching.

THE STUDY

The focus of the study was to determine whether pupils learn substantive mathematics content through the use of content-specific open questions, with no direct teaching. The hypothesis was that pupils would develop the skills and orientation towards seeking multiple answers to open questions if these formed the basis of instruction. It was assumed that this, in turn, would facilitate the learning of substantive concepts by stimulating the development of more robust mental structures. These mental structures would be more accessible because they had been constructed by the learners themselves.

The components of the implementation of the study, namely the subjects, the instruments, the teaching program, and the observations of the teaching, are described in the following sections.

SUBJECTS

The study was conducted in a Catholic primary school in a suburban area of Melbourne, Australia. The school has a high proportion of students from non-English speaking backgrounds and served a predominantly lower socio-economic community. There were two composite grade 5/6 classes. Only the grade 6 pupils participated in the study. These pupils were randomly allocated to two groups prior to any testing. The grade 5 pupils were grouped together and completed a separate program during the experiment.

The experimental group was taught a unit on length, perimeter and area over seven one hour lessons. The control group was taught the same topic. The teacher of the control class was instructed to follow the program presented in the most commonly used text. This was to simulate a standard approach to the topic.

INSTRUMENTS

The purpose of the experiment was to measure the impact on the learning and approach of the students as an outcome of instruction. There were a number of instruments used. These were:

(a) An attitude instrument, prior to and after the teaching.

(b) A content test on the topic of length, perimeter and area. This consisted of 10 items. Three items were written by the class teacher. There were some open-ended items and one 'good' question. .

(c) A further instrument of four 'good' questions using a broader response request statement, after the teaching.

The attitude instrument and content test were also administered to six other grade 6 classes to provide some background on typical responses for pupils of that age. After the content test, four pupils from each group were interviewed to allow further examination of the strategies used in responding to the items.

THE TEACHING PROGRAM

Each lesson started with a number drill or game. This was the format requested by the class teacher. The program for the experimental group consisted totally of 'good' questions which addressed aspects of the topic. Examples of the tasks used in each of the lessons are as follows:

- Suppose that we used body parts to measure things. Name something which is the same length as your height . ... the length of your foot.
- John measured the basketball court and said that it was 20 rulers long. Maria measured the same basketball court and said that it was $19\frac{1}{2}$ rulers long. How could this happen?
- What would we use a ruler 1 metre long to measure? ... a ruler 1 cm long?
- What is longer than 1 metre but less than 2 metres in this room? ... not in this room?
- Write down all the ways to describe "how long" that you know.
- *The pupils are given a piece of string.* What shapes might this be the perimeter of?

These tasks formed the basis of the instruction. In each lesson, a task was posed and the students given the materials needed to commence working on the task. Generally the pupils worked in pairs. The teacher moved from pair to pair checking responses, and facilitating the work. Each task was reviewed with the whole class. The students were invited to suggest a response or to write/draw their contribution on the board. Other children were then invited to suggest other responses. When the responses were displayed, the students were asked to comment on the overall pattern or to suggest a way to describe all of the answers. There was no direct teaching in the experimental group.

The format for the control group lessons consisted of an introductory game (15 minutes), teacher demonstration (15 minutes), written applications (15 minutes) and correction of written work (5 minutes). The content included use of formulae to calculate perimeter and area, using regular and irregular shapes. Questions were asked both ways: given a shape with dimensions, find the perimeter and area, and given the area of a shape, to find the length and width. There was a focus on hands-on activities such as measuring and applications such as finding the area of the playground.

THE LESSON OBSERVATIONS

A number of the lessons for each teacher were observed. There were two aspects to each observation. First, an attempt was made to record the teacher's actions. The focus of this was on explanations given, questions asked, and tasks set. A schedule using a similar format to the SCAN notation (Beeby, Burkhardt and Fraser, 1980) was used. A naturalistic record was also made of the lessons and a diary was completed after the lessons observed.

Second, an attempt was made to record the activities of the pupils. The activities of four pupils in each class were recorded at two minute intervals. These provided a record of the extent to which pupils were working on the task or engaged in other activities.

In the five lessons coded in the experimental class, there were no instrumental explanations and only three relational explanations. Eighty-six per cent of the questions in the experimental class were open and 61% were higher thinking questions. Eighty per cent of all set tasks were open and all required more than recall of information. The level of engagement of pupils in the experimental group was high. The students were engaged in listening to the teacher or working on the tasks for over 70% of the observations. An analysis of the students' work showed that they responded to the questions well, and gave multiple responses to the questions. The reports of the naturalistic observations indicated that the pupil adapted to the style readily and that they were engaged in the tasks productively.

In summary, the observations indicated that the implementation of teaching program was as had been designed. There were few explanations given in the experimental group and these were relational. There were few questions asked and these were predominantly open questions which required thinking. The tasks set were open. The pupils were task-oriented. The work produced by the children was good and they seemed receptive to this style of teaching.

RESULTS

The impact of the teaching on both groups was measured by a content test and an attitude instrument before and after the intervention, and by interviews with selected students after the teaching.

Content. The content test was completed by students in both classes. Three of the items were written by the teacher of the control group. There were two components of the content test: eight conventional or closed items on length, perimeter and area and one 'good' question on both the pre-test and post-test. Four additional 'good' questions were completed after the teaching program.

The conventional items on both tests required a single answer. It should be noted that some of the items were difficult for this grade level. There was no difference between the scores of the experimental group and the control group on the pre-test (t= 0.8, df = 32, p=0.43) and no difference on the post-test (t=1.0, df=33, p=0.32). A comparison of the gain in the scores of both groups on these items is presented in Table 1.

Table 1: Comparison of the gains of the experimental and control groups.

	n	mean of gains	s.d. of gains
experimental	16	1.6	2.2
control	16	1.2	1.7
		$t = 0.6$, $p = .54$	

These scores are not statistically different. This is an interesting result since it indicates that the pupils in the experimental group have learnt the skills and concepts as well as those in the control group. As seen earlier, there was no attempt to teach the skills directly in the experimental group. There was less that one explanation per lesson, only one closed recall question overall and no recall level tasks set, yet the students in the experimental group have improved their scores on skill items to the same extent as those in the control.

Since this is an important outcome of the experiment, it is relevant to examine the responses of the students to particular items. Table 2 presents the percentages of correct responses for each group on selected items from the pre-test and post-tests. The responses of 188 pupils at the same grade level from other schools tested at the same time are also presented to provide a context against which the test scores can be interpreted.

Table 2: Comparison of the correct responses (%) between the groups on selected closed items

	Experimental		Control		Other
	Pre n=18	Post n=17	Pre n=16	Post n=18	Schools n=188
Name something which is 10 cm long	56	94	63	72	53
What is the perimeter of this shape? (rectangle drawn here)	33	88	44	83	59
What is the area of that shape?	50	76	25	72	60
A table is the shape of a rectangle 3m long and 50 cm wide. What is the perimeter?	22	41	25	33	43

It is clear that there are children in both groups who have learned to calculate the perimeter and area of a given shape, which was the purpose of the unit. Certainly the experimental group was at no disadvantage on these conventional items.

The study also sought to determine whether the students were able to give better responses to 'good' questions as a result of the experimental program. There was one 'good' question on both the pre-test and post-test and this is used for comparison purposes. Table 3 presents the percentages of students who were able to give at least one correct response to the item.

Table 3: Comparison of correct responses (%) between the groups on a 'good' question

Question	Experimental Pre n=18	Experimental Post n=17	Control Pre n=16	Control Post n=18	Other Schools n=188
A rectangle has a perimeter of 30cm. What might the area be?	22	53	13	39	27

There were more students in the experimental group who could respond correctly after the instruction. There is a similar gain in the control group. This is a positive result for both groups.

There were no students in either group who attempted multiple answers. It had been anticipated that the experimental group would be able to give multiple or general responses as an outcome of the instruction. Given that this item was similar to those used in the teaching in the experimental group, this result requires consideration. In the class, the students were willing and able to give multiple answers readily to simpler tasks. It is possible that this test item was too difficult, and that processing perimeter and area together requires greater fluency with the concepts than had been developed during the program.

This possibility was explored in the second component of the post-testing. It had been earlier found that students gave more multiple and general answers when the request for more answers was explicit in the question statement (Sullivan, Clarke and Wallbridge, 1990). There were responses at the highest level. There were very few multiple answers. The items with the explicit request for multiple answers were related to the content of the teaching. It seems that the students were willing to give multiple answers if specifically prompted and if the question was easy. It does not appear that the experimental program assisted these students to answer the more difficult question on area and perimeter. This is a relevant issue in that while a significant number of students have learned the basic content, there were few who could respond to the more difficult items.

In summary, the content tests indicate that there were students in both groups who learnt to answer conventional items about length, perimeter and area, but that the students in the experimental group were not able to answer the open questions after the teaching. It suggests that these students were not disadvantaged by participation in the program, but neither did they develop the capacity to tackle the open questions as had been anticipated.

Attitudes and Implementation. The students completed a 14 item attitude instrument before and after the teaching. Students were asked to respond on a five point Likert scale. There were seven pairs of items, one positive and one negative, which sought responses on the students' liking of mathematics, their anxiety about mathematics, how difficult they find mathematics, whether they are interested in mathematics, how confident they feel, their perceptions of their motivation, and their beliefs about the openness of mathematics. By way of example, the stems on the openness of mathematics were:

In mathematics there is always one right answer.
You can sometimes have lots of correct answers to maths questions.

After scoring the responses, there were no significant differences, although the students in the experimental group were slightly less anxious and they considered mathematics slightly more open after the teaching.

Immediately after the post-test, four students from the experimental group and four from the control were interviewed to seek some insights into the thinking of the children. The interviews confirmed the results of the content test. Specifically, three of the students were confident with all aspects of the skills, yet none gave more than one response directly and only one could give a second response even when specificaly requested by the interviewer.

SUMMARY AND CONCLUSION

There is a need to identify teaching strategies which acknowledge the way children learn, which recognise that the way children respond to learning opportunities is influenced by the context of school and the classroom, and which emphasise the substantive mathematics content. 'Good' questions seem to offer one possible teaching strategy. The advantages of such questions are that the mathematical content is explicit, the role of the learner is emphasised, and the development and linking of concepts is facilitated. The basic question of the research study was whether students can learn substantive mathematics content from a program based only on content-specific open questions. It was hypothesised that such a program would stimulate an orientation toward seeking multiple and even general answers.

The study compared a traditional approach with a program based solely on content-specific open questions. Even though the teaching programs were markedly different, some pupils in both groups learnt to respond to conventional items on perimeter and area. The other result was that very few of the pupils in the experimental group had developed the skills necessary to respond to open questions on the content and had not developed the orientation to give multiple or general answers. This result needs further elaboration.

It must be stressed that during classes the pupils seemed to be willing to give a range of responses and they adapted to the style of teaching readily. The openness of the tasks facilitated the constructive activity of the children and allowed the pupils to work productively at their own level. Yet the pupils had consolidated neither the concepts nor the orientation sufficiently to respond to open questions on the test. It had been hoped that the pupils would give multiple or general responses to such questions, yet they did not.

One explanation considered was that the program may not have been long enough, and the pupils may have responded better on the test had they had more experience with open questions, especially at the more difficult end of the topic. Yet the pupils adapted to the style easily, and accepted the approach from the first lesson. Lack of familiarity with the style of questions was not the cause of their inability to respond to the 'good' questions on the tests.

Three aspects which may be more important factors in explaining the failure of the pupils to respond to the tasks were identified. First, it is possible that the 'good' questions asked, especially in test situations, were much more difficult than conventional questions. Second, even though some students learnt the basic concepts, and the gain was similar in both groups, there were still one quarter of the students who did not complete a direct item on area, and less than one half could complete the item where the units were different. Just over one half could give even one correct answer to the 'good' question. It is possible that some skill practice may help the children who could complete the basic items but who had difficulty giving even one answer to the 'good' questions. A third point, which seems obvious post hoc, is that the method used to review the tasks may not have been adequate. In the experimental class, the teaching procedure was to present a task for the pupils to explore, generally in pairs, then to have a whole class review where pupils reported on their findings and results. Often, it was assumed that the statement of a response by one of the pupils would be enough. Clearly it is no better if one of the pupils 'tells' something than if the teacher tells. In other words, the teaching occurs after the pupil activity, not before.

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AN ANALYSIS OF CHILDREN'S UNDERSTANDING OF NUMERATION

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In this paper I report on a pilot study which has been carried out to investigate children's understanding of numeration. In particular I am interested in exploring how young children develop an understanding of the structure of ones, tens, hundreds, ... that is embodied in the mental, verbal and written notions of the number system that we use.

What is the relationship between a child's:

i) understanding and use of equivalent groups in counting;
ii) understanding of the process of regrouping in representations of number; and
iii) understanding of place value and the structure of the numeration system.

When and how does a child begin to understand the structure of the number system? How is structural flexibility developed so as to operate meaningfully with the number system (numeration sense)?

BACKGROUND

Increased conceptual understanding of numbers and facility with their manipulation which leads to flexibility in dealing with number, or in other words, the development of number sense has been advocated in recent reports and curriculum documents here and overseas (Cockcroft, 1986; NSW Department of Education, 1989; Australian Education Council, 1990; NCTM, 1989; National Research Council, 1989). At the same time these reports are suggesting a reduced emphasis on written algorithms for the four operations. This raises the question of how should we be planning school experiences which help children construct a system of numeration for themselves which involves 'number sense' and relates directly to their real life (and intuitive) experiences with number.

Constructivism has given us a philosophical rationale to support the more interactive, child-centered approach that many of us have been advocating through our professional development work. This theoretical position emphasises that children's mathematical understandings are constructed through their interactions with mathematical contexts and environments. Davis (1990) argues that constructivism has given us a 'formalized conceptualization' of the process of learning mathematics which could now be used to rethink the analysis of some earlier curriculum improvement projects.

Recent research (Bednarz and Janvier, 1988; Cobb and Wheatley, 1988; Denvir and Brown, 1986 a; b; Fuson, 1990; Kamii, 1985; 1986; 1989; Labinowicz, 1985; Ross, 1989) have contributed to our understanding of how children construct the numeration system for themselves so that it can become a tool for calculating. Fuson (1990) argues that present US textbook presentations of place value depend primarily on a skills analysis approach which contributes to the failure of children to build adequate multi-unit conceptual structures and so

> *multidigit addition and subtraction are learned as procedures carried out on columns of single digits, and meanings other than single-digit meanings are not constructed or are not accessed.*
>
> (Fuson, 1990, p.274)

Ross (1989) suggests that teaching place value concepts separately as a prerequisite to algorithmic work, is ineffective if understanding is the goal. Although careful instruction with concrete material can facilitate the acquisition of the procedural knowledge to carry out computational algorithms, many children do not subsequently appear to develop relational understanding of the conventional algorithms for addition and subtraction of whole numbers. How do we get more children to demonstrate flexibility in partitioning numbers so that they might develop this relational understanding? Ross advocates a child-centered period of inventing procedures for solving two-digit number addition problems requiring renaming. This requires the development of a sense of number which includes the notion of the part-whole relationship and the idea that a number may have many names. Some experimental instructional studies (Kamii and Joseph, 1988; Cobb and Merkel, 1989) have shown that children, as young as first and second grade level can invent their own efficient algorithms.

Kamii (1985) argues that premature instruction in place value is detrimental to children making sense of numeration.

> *Given what we know about the developmental course of children's thinking, we ought to ask ourselves whether it would not be wiser to delay place value instruction until children have solidly constructed the number series (by repetition of the +1 operation) and can partition wholes in many different ways (part-whole relationships).*
>
> (Kamii, C., 1985, p63)

Young children first generate the numerals by recognising the pattern of using digits in their construction. They also learn through social transmission that the counting words for the teen numbers are not written as they sound. This does not involve an understanding of place value. Place value requires an understanding and integration of both the irregularly value-named system of number words and the positional base ten system for forming numerals. The base ten place value numeral system requires the mental construction of one group of ten out of ten ones and then the representation by a digit in the "tens place". This involves the construction of a 'second level' to the number system. This second level also involves the idea of multiplication as the groups of ten become the new units. Kamii (1985) explains that six and seven year old children are still in the process of constructing the number system with the relation "one more" and so it might be inferred that they are not ready to fully understand the symbolic notation of two-digit numbers. She suggests it is not possible for a child to construct the second level while the first level of the number system is still being built.

Hiebert and Wearne (1992) report on a numeration teaching programme which they developed for year 1 children. This 'conceptually based instruction' programme aims to help children construct connections between representations of number (physical, pictorial, verbal and symbolic) and use all of these representations with recording actions on number. Different forms of representation of quantity highlight different aspects of the structure of number (eg grouping with physical and pictorial models, place value with symbolic models). Understanding numeration involves

> *building connections between the key ideas of place value, such as quantifying sets of objects by grouping by 10, treating the groups as units... and using the structure of the written notation to capture the information about groupings.*
>
> (Hiebert and Wearne, 1992, p.99)

Davis (1990) raises the issue of using other bases through his reference to the following problem which was given to children in the Madison Project.

> *Two small groups of children are asked to communicate messages back and forth, but they must pretend that nobody can count above three. One group of children - at the front of the room, say - is then given a pile of tongue depressors (let's say that you and I know that there are twenty-two tongue depressors in the pile). The children at the front must send messages to the other group of children (at the back of the room) so that the second group can assemble the same number of tongue depressors.*
>
> (Davis, 1990, p.97)

He says this is an example of focusing instruction on the basic task, and leaving it up to the children to invent a way to solve the problem. The construction of a recording system in this problem parallels the invention of place value numerals as an elegant solution to the problem of of how to record the very large numbers.

METHOD

The pilot study consisted of the intensive clinical interviewing of a cross-sectional sample of 40 children and took place in June, 1992. Four children were interviewed from each of the years Kindergarten to 4 in two country NSW schools. The children were selected by the class teacher on the basis of being representative of the spread of achievement levels in mathematics in the class (one chosen as a low achiever, two middle ranking achievers and one as a higher achiever). All interviews used the same tasks, asking the same initial follow-up questions. The follow-up questions were intended to elicit children's explanations about the strategies they used. Further questions depended on the responses. The thinking of children was probed until the strategy being used was clear or until it was obvious that no further explanation would be forthcoming. Children's explanations and visible strategies were recorded on an interview form. All interviews were audio taped and this used, when needed, to complete descriptions of responses. This cross-sectional analysis was carried out with children from 5 to 10 years of age. It must be remembered when carrying out any analysis that each child constructs whatever concepts he/she has on the basis of his/her experiences and so any model of a child's thinking must take these experiences into consideration.

INTERVIEW TASKS

There were four levels of tasks used in the pilot study. These levels reflect the increasingly complex nature of the number system as it is used to quantify larger and larger collections of objects and to record (orally and symbolically) the operations applied to the numbers. The levels of tasks are: counting; grouping; regrouping; and extended structure.

The interviews began with the counting tasks. The missing addend and removed item tasks (Steffe and Cobb, 1988; Wright, 1991) were used to determine whether the children could count the counting acts themselves, whether they were abstract counters. Other counting skills assessed were multiple counting and double counting.

As grouping is a operation which is central to the development of an understanding of numeration, tasks were used to determine the intuitive use children make of grouping in a number of situations. Partitioning problems were used to assess whether equivalent grouping is used in the sharing process, whether a collection is partitioned into equivalent groups by one-to-one or many-to-one correspondence. The problems involved situations with remainders (smaller numbers) and without remainders (larger numbers) in order to explore whether equivalence was established in these different situations. A counting task which assessed whether groups of ten objects were treated as units involved packs of lifesavers and individual lifesavers. The children established that there were ten lollies in each roll by counting the contents of transparently wrapped rolls and were then asked, in turn to quantify collections of fifteen and forty-three lifesavers consisting of standard arrangements of pregrouped rolls and individual lollies. Another counting task showed children an array (ten by six) of pictures of planes and asked how many planes there were altogether. An uncovering tens task (Cobb and Wheatley, 1988) was used to assess whether children coordinate counting by tens and ones. The relationship of grouping to recording with numerals was investigated with digit correspondence tasks (Ross, 1990). Another task investigated the children's use of the pattern of tens to locate numbers on the hundred square. A further task involved asking the children to close their eyes and to imagine the numbers from 1 to 100. This task was carried out earlier in the interview so that responses could not be influenced by experiences with other tasks. It was intended to assess the structure (or lack of structure) of the mental image that children have of these numbers.

The operation of regrouping was explored through several tasks. These tasks involved addition with pregrouped material, the possible use of the part-whole relationship, the regrouping of concrete representations of number to non-standard forms (more than nine individual objects in representation) and the renaming of numbers with equivalent symbolic names.

The extended structure of number was explored through tasks which extended the use of groups, to groups of groups, and the relationship to the system of ones, tens and hundreds. In a counting task the children were shown two bags, one roll and four individual lifesavers where all the wrappings were transparent. The roll of lifesavers contained 10 lollies and each bag contained 10 rolls. The children were asked to find how many lollies there were altogether. A similar task involved three bags, twelve rolls and five individual lifesavers (non-standard representation). Another task initially assessed whether children spontaneously used grouping in tens and hundreds to count large collections (pictorial presentation). This task further probed how the collection could be presented in order to facilitate the count and then assessed how the same collection, with the grouping of ten groupings of ten shown by circling, was interpreted (Bednarz and Janvier, 1988). In a further counting task, children were asked to find how many dots there were in an array of 10 000 (100 by 100), with spacing which separated blocks of one hundred and then of one thousand dots. Groups of five unifix cubes (towers) were used to explore children's generalization of the base ten system to a similar system based on a grouping number of five.

RESULTS

Missing addend task. Assess whether the counting acts themselves are counted - **abstract counting.**
Display 8 (5) shells.

How many shells are there here?

Place out 4 (3) shells which are screened from view.

There are 12 (8) shells altogether. How many are hidden?

Response Categories
Small numbers Large numbers

Year	1	2	3	4	5	6	7	8	Abstract counters
K	8								0
1	3	2	2	1	1	3		1	5
2	1				4	2		1	7
3					1	4	1	2	8
4					2			6	8

Response Descriptions:
1. Guess or no answer.
2. Counts-on using fingers (3).
3. Counts-on mentally (3).
4. Immediate knows fact (3).
5. Counts-on using fingers (4).
6. Counts-on mentally (4).
7. Counts backwards mentally (4).
8. Immediate knows fact (4).

Response examples:
Year 1
T. How many are hidden?
J. 4
T. How did you get that?
J. Because 8 plus 4 is 12.....would have been able to count on my fingers.
Year 1
H. Because 6 and 6 equals 12.....eight, nine, ten, eleven, twelve......so it is 4.
Year 2
R. Start from 9 and go 10, 11, 12.
Year 2
J. You have 8 there, and you break it up into two groups of four......another group of four makes twelve.
Year 3
S. If you have 8 out here, then you have to have 4 under there or it won't be 12.

Removed item task, assess whether the counting acts themselves are counted - **abstract counting.**
Assess use of ten.
Display a collection of ten counters.
How many shells are there here?
Hide 3 shells.
How many shells are under my hand?
Can you give me any other numbers that add to give 10?

Response categories
Removed item task Ten facts

Year	1	2	3	4	5	6
K	8					
1	3	3		2	2	
2	1	1	5	1	4	2
3			4	4	3	4
4			3	5	3	5

Response Descriptions:
1. Guess or no answer.
2. Counts-on using fingers (3).
3. Counts-on mentally (3).
4. Knows answer (3).
5. Calculates at least two further combinations which add to give ten.
6. Recalls at least two further combinations which add to give ten.

Partition
Grouping / concrete representation (composition of groups visible)
Discrete material
Directed - partition (no remainders)
Assess whether a collection can be partitioned into equivalent groups.
Is sharing 1:1 or many:1?
Give the same amount of lollies to each Lego person. We are to use all the lollies. (12 lollies shared between 3 people).

Response categories

Year	1	2	3	4
K	1	4	3	
1		4	4	
2		4	4	
3		5	2	1
4		2	5	1

Response Descriptions:
1. Did not share objects.
2. One to one correspondence.
3. Many to one correspondence, more than one deal carried out.
4. Many to one correspondence after answer calculated.

Partition

Grouping / concrete representation (composition of groups visible)
Discrete material
Directed - partition (larger number and remainders)
Assess whether a collection can be partitioned into equivalent groups.
Is sharing 1:1 or many:1?
I am going to give you some shells.
I want you to put these shells in the plates so that there is the same number of shells on each plate (26 shells with 6 plates).

Response categories

Year	1	2	3	4	5	6
K	2	3	2		1	
1	2	1	2		3	
2		1	3		4	
3			6		1	1
4			2		5	1

Response Descriptions:

1. Did not share objects.
2. One to one dealing, distributed all objects.
3. One to one dealing, acknowledged the remainder.
4. Many to one dealing, more than one deal carried out, distributed all objects.
5. Many to one dealing, more than one deal carried out, acknowledged the remainder.
6. Many to one dealing after answer calculated.

Multiple count - threes

Counts by threes up to:

Year	not able	6	9	12	15
K	8				
1	6	2			
2	2	2	1	2	1
3		4		1	3
4		1		2	5

Multiple count - fours

Counts by fours up to:

Year	not able	8	12	16	20	24
K	8					
1	5	2	1			
2	3		2	2		1
3	2	3			1	2
4		3	1			4

Multiple count - tens

Counts by tens up to:

Year	not able up to 90	100	100 +
K	8		
1	2	4	2
2	1	2	5
3		1	7
4			8

Double count with and without direct modelling

Assess use of double count to keep track of number of groups of 4.

eg 1 2 3 4/1, 5 6 7 8/2, 9 10 11 12/3

a) *See if you can think aloud as you do this next question.*

There are 12 children with 4 children sitting at each table.

How many tables are needed?

Give opportunity to draw picture or use material.

b) Provide 12 Lego people and 5 trucks made out of Lego blocks.

There are 12 Lego people and some trucks.

4 people go to work in each truck.

How many trucks are needed?

Response categories

Year	1	2	3	4	5
K	5	2	1		
1	1	6	1		
2		4	4		
3		2	3	1	2
4		3		1	4

Response Descriptions:

1. Not able to solve problem (meaningful model).
2. Solved problem with materials or by drawing pictures.
3. Solved problem mentally by building up groups of 4 until 12 (double count).
4. Solved problem mentally by taking away groups of 4.
5. Solved problem mentally by relating to known multiplication or division facts.

Compensation

Assess use of subitizing pattern when adding two single-digit numbers, pictorial representation.

Show a 6 and a 9 pattern board (twos pattern).

How many dots are here? Show the 6 board.

How many dots are here? Show the 9 board.

How many dots are there altogether?

Response categories

Year	1	2	3	4	5
K (4)	1	3			
1 (6)		5		1	
2	1	2	2	1	2
3 (7)		1		4	2
4		2	1		5

Response Descriptions:
1. Not able to find sum.
2. Count all by ones.
3. Count-on from 6 by ones.
4. Count-on from 9 by ones.
5. Subitize patterns in both numbers and add by partitioning and combining the numbers - compensation, bridging tens.

Response examples:
Year 3:
K. "Take one off there.... and put there.... and you still have 5.... 15."
Year 4:
B. "9 is near 10.... and another 6 is 15."
Year 4:
C. "6 and 6 plus 3 is 15."
Year 4:
D. "cause you take 1 from 6 and then that 1.... you add to the 9.... then you put the 5 on to give 15."

Counting pregrouped material
Grouping / opaque covering, one group of ten:
Directed
Assess level of counting, use of ten.
Show a roll of 10 lollies (transparent).
How many lollies are in this roll?
Show 4 **opaque** rolls, each containing 10 sweets.
How many lollies are in this roll?
How many lollies are in all these rolls?
1 roll and 5 loose sweets are displayed.
How many lollies?"

Response categories

Year	1	2	3	4	5
K	3	4	1		
1		2		5	1
2		1		3	4
3				2	6
4					8

Response Descriptions:
1. No response.
2. Counts in ones, guessing number in the package.
3. Counts in ones, correct number in the package.
4. Counts-on from ten by ones.
5. Counts ones (or subitizes) and adds ten.

Counting pregrouped material

Grouping / opaque covering, several groups of ten:
Directed
Assess level of counting, use of ten.
4 rolls and 3 loose sweets are displayed.
How many lollies are here altogether?

Response categories

Year	1	2	3	4	5	6	7
K	5		3				
1		2	2		1	3	
2		1	1			3	3
3						4	4
4						2	6

Response Descriptions:
1. No response.
2. Counts each package as one.
3. Counts in ones, guessing number in each package.
4. Counts in ones, correct number in each package.
5. Counts in tens and in ones without coordination.
6. Counts in tens and in ones with coordination.
7. Uses multiple of tens.

Regrouping

If you added 8 lollies to your collection there, how many lollies would you have altogether?
Assess whether child counts by ones and then trades or uses ten as a unit in the trade.

Response categories

Year	1	2	3	4	5
1	1	2			
2		4	1	2	
3		3	1	3	1
4		3		4	1

Response Descriptions:
1. Counts without coordination, incorrect result.
2. Counts-on from 43 by ones.
3. Counts-on from 48 by ones.
4. Adds the ones together and then regroups.
5. Adds ten and takes away 2, bridges tens.

Counting pregrouped material - pictorial.

Grouping / opaque covering, several groups of ten:
Pictorial, directed
Assess level of counting, use of ten.
Show a picture of 8 rolls and 6 separate lollies.
How many lollies in this roll?
How many lollies altogether?

Response categories

Year	1	2	3	4	5	6	7
K							
1 (6)		3		1		2	
2	1					7	
3 (7)						7	
4	1					7	

Response Descriptions:
1. Counts without coordination, incorrect result.
2. Counts tens as ones.
3. Counts ones as tens.
4. Counts all by ones, guessing number in each roll.
5. Counts all by ones, taking the number in each roll as ten.
6. Counts in tens and ones, with coordination.
7. Uses multiples of ten.

Counting objects in an array

Assess intuitive use of ten in counting when a ten by six array of pictures is presented.
Show an array of ten by six planes.
Can you tell me quickly how many planes are here?

Response categories

Year	1	2	3	4	5
1	2	1		1	
2	1	2		4	1
3		1	1	5	1
4			1	3	3

Response Descriptions:
1. Unable to count sucessfully.
2. Counts all by ones.
3. Attempts to count by sixes.
4. Counts by tens, repeated addition.
5. Uses multiplication.

Hundreds square

Assess how a number's location on the hundred square is found. What use is made of the pattern of tens.
Show a hundred square (0to 99).
Show me how you can get ten more than 36 quickly from the hundred square.
Can you show me ten less than 49?
Can you show me nine more than 67?

Response categories

Ten more than 36 Ten less than 49

Year	1	2	3	4	5	6	7	8
1	3	1	1		3	1	1	
2	1		3	4	2		3	3
3	1		3	3	1		4	2
4	1		5	1	1		5	1

Response Descriptions:
1. Unable to locate number (addition).
2. Attempts to count-on by ones to get second number (addition).
3. Counts-on by ones to get second number as a result of addition.
4. Uses pattern of tens (addition).
5. Unable to locate number (subtraction).
6. Attempts to count-on by ones to get second number (subtraction).
7. Counts-back by ones to get second number as a result of subtraction.
8. Uses pattern of tens (subtraction).

Uncovering tens task - assess whether child can coordinate counting by tens and ones i.e. ten as an abstract composite unit.
Tens task - a board to which is affixed a sequence of Dienes longs and shorts is gradually uncovered and each time the cover is pulled back to show more material the child is asked: "how many are there now?"

10, 14, 34, 38, 41, 51, 53, 73

Response categories

Year	1	2	3	4	5	6	7
1	3	1	1		1	1	
2	2	1			1	2	2
3					2	4	2
4			1			3	4

Response Descriptions:
1. Attempts to count all by ones - counts tens by ones (sometimes unsucessfully).
2. Restarts each time to count-on from ten all by ones.
3. Miscounted on tens as ones.
4. Miscounted on ones as tens.
5. Start with ten and count-on by ones.
6. Does not count-on, restarts each time and collects units of the same rank together.
7. Counts-on by tens and ones as appropriate.

Addition involving regrouping

Regrouping - composition of groups hidden:

Addition/tens and missing addend/tens tasks. Assess use of ten in tasks that require regrouping - assess counting as: i) only by tens or by ones - **ten as a numerical composite** if even though it is known that a roll has ten lollies counting is by ones or counting tens as ones; ii)coordinating counting **tens as abstract singletons** and ones as abstract units, units of same rank are added with no notion of ten being composed of individual units; iii)constructing **tens as abstract composite units**, count by tens and ones starting in the middle of a decade; iv) constucting each number as so-many tens and so-many ones and then adding (basic fact knowledge) units of the same rank, ie **abstracting collectable units**; v) **ten as an iterable unit**, a number is a numerical whole and units of ten and one at the same time.

3 rolls (opaque coverings) and 7 separate lollies are visible, the child is told 25 lollies are hidden beneath the cloth and asked to find how many lollies there are altogether.

Response categories

Year	1	2	3	4	5	6	7	8	9
1	7				1				
2	5	1			1				1
3	1	1		1	2	1		1	1
4	1	1				1	5		

Response Descriptions:
1. Unable to calculate total mentally.
2. Counts-on by ones (sometimes unsucessful) - ten as a numerical composite.
3. Counts-on by ones, counting tens as ones - ten as a numerical composite.
4. Counts-on by tens and then all by ones.
5. Counts-on by tens (or collects tens), adds 5 or 7, and then counts-on by ones.
6. Counts-on by tens and ones starting in the middle of a decade - ten as an abstract composite unit.
7. Adds units of the same rank.
8. Breaks up one number into parts and uses counting-on by ones and tens.
9. Breaks up numbers into parts in order to facilitate the addition process, use compensation or bridging tens - a number is a numerical whole and units of tens and ones at the same time.

Response examples:
Year 1:
J. "3 plus 2 is 5.... so 30 plus 20 is 50.... 57 with 7.... count-on 5 to give 62."
Year 2:
R. "Put 3 in to make ten.... then 50, 60.... then just put the two more to give 62."
Year 2:
J. "that's hard.... 30 plus 20 equals 50.... sixty something.... 57, 58, 59, 60, 61, 62 (using fingers)."
Year 3:
M. "25, 26, 27, 28, 29, 30,.... 40, 50, 60.... 61, 62."
Year 3:
K. "Adds all the ones together and there was 2 plus 10.... so 62."
Year 4:
R. "Added 30 and 20 to give 50.... 7 and 5 is 12.... you can't have 50 twelve so you get 60."

Structure - pictorial, discrete objects
Undirected
Determine pertinence of grouping and the significance of associated writing.
Responses: no need for grouping - either guess or count by ones; group to count quickly, recount the collection after grouping; use one order of groupings and see that writing is a code that is directly associated with these groupings; use a grouping of groupings.

Child presented with a picture of 143 marks randomly drawn.
Can you tell me quickly how many marks there are drawn there?
I am going to do the same thing later with a friend who will be here after you. Could you do something so that, when I show him the sheet, he will be able to tell me very quickly how many marks there are?
What did you do?
Now can you tell me quickly how many marks there are?
How do you know that?
Look at what the friend who came before you did (grouping of groupings is shown). What do you think of it?
Can we see quickly how many marks there are?
Suppose you have a younger brother / sister who you are going to help with his/her counting.
How would you explain the easy way to count those marks.

Response categories

	Spontaneous count			Suggested help				Pregrouped			
Year	1	2	3	4	5	6	7	8	9	10	11
1	5	3		3		1		3		1	
2	7	1		6		2		2	3	3	
3	3	5		4	1	3		1		6	1
4		8		3	3	2		1		4	3

Response Descriptions:
1. Does not do or guesses.
2. Attempts to count by ones.
3. Forms groups of ten to count.
4. No suggestions to help another person, labelling each mark or dividing in half.
5. Suggests grouping by a number other than ten.
6. Suggests grouping by tens.
7. Suggests grouping ten groups of tens.
8. Does not recognize and use tens when shown what a friend had done.
9. Recognizes but does not use successfully the groupings of tens.
10. Recognizes and uses the groupings of tens.
11. Recognizes and uses the groupings of tens and hundreds that were made by another person.

Assess use of groupings of tens, hundreds and thousands when quantifying large collections.
Show an array of 100x100 dots.
Can you tell me how many dots are here?

Response categories

Year	1	2	3	4	5
2	3	1	4		
3	3		4		1
4	2	1	2		3

Response Descriptions:
1. Unable to quantify.
2. Attempts to count by ones.
3. Determines the patterns of hundreds and unsucessfully attempts to count by hundreds.
4. Determines the patterns of hundreds and counts by hundreds to give 10 000.
5. Determines the patterns of hundreds, counts to give the thousand pattern and then counts to give 10 000.

Response examples:
Year 2:
J. Counted across the top (10) and down the side (10) of a square pattern. "100 in each square.... " Counted down the page.... "1, 2, 3, 4, 5, 6, 7, 8, 9, 10 hundred.... I know this is one thousand.... " counted across the bottom of page.... "1, 2, 3, 4, 5, 6, 7, 8, 9, 10.... one million!"

Assess whether the mental picture of the numbers 1 to 100 is a single mental number line or a matrix of coordinated horizontal and vertical mental number lines.
Close your eyes. I want you to imagine the numbers from 1 to 100. Can you see a picture of these numbers?
Open your eyes.
Draw a picture of what you saw.

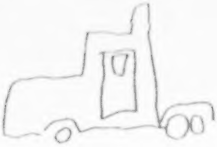
Response categories

Year	1	2	3	4	5	6	7	8
1 (6)	3	1	1		1			
2	1		2	1	1		3	
3 (7)	4	1		1			1	
4				3		1	2	2

Response Descriptions:
1. No mental picture given.
2. Pictures or marks, no pattern (100 objects, lots of squares).
3. Pictures or marks, array or line pattern (100 marks in a line, ten sticks)
4. Random selection of numerals, no pattern in display.
5. The numeral 100.
6. Numerals appearing one at a time in order to 100 (flashing).
7. Numerals in a long number line or a series of number lines for different multiple counting sequences.
8. Numerals in a 10 by 10 grid.

Year 1

Anthony



James

100

Lisa

100objects... all sorts of things... on the floor.

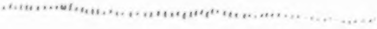
Year 2

Joshua

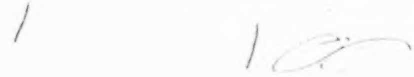
100 99 98 97 96 95

and on to 1.

Warren



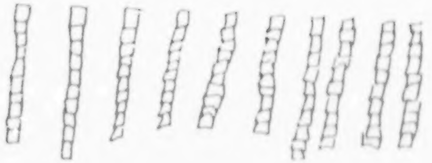
Gary



Hayley

1234

Mellissa



Oliver

10
20
30
40
50
60
70
80
90
100

Robert

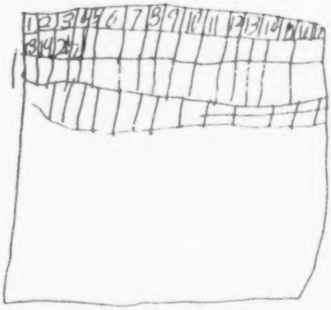


Figure 1: Examples of visualization:

Year 3
Michael

0 1 2 3 4 5 6 7 8 9 10

... and on to 100.

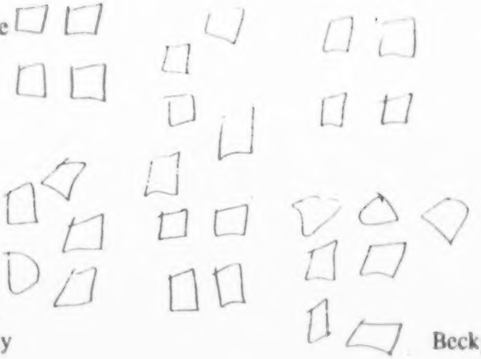
Amber

1 10 100 90

1 2 3 4 6 9 11 13 20

31 90 100

Candice



Year 4
Timothy

9

Becky

24 89

Cassie

1 5 10 6 20 30
9 2
8 4 16 18
3 7
11 12 19 17 33
22 18
94

Alex

1 2 3 4 5 6 7 8 9 10

10 20 30 40 50 60 70 80 90

5 10 15 20 25 30 35 40 45 50

55 60 965 70 75 80 85 90 95 100

Oliver

1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32

... and so on until 100.

Rebecca

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

... up to 100.

Figure 1: Examples of visualization (continued)

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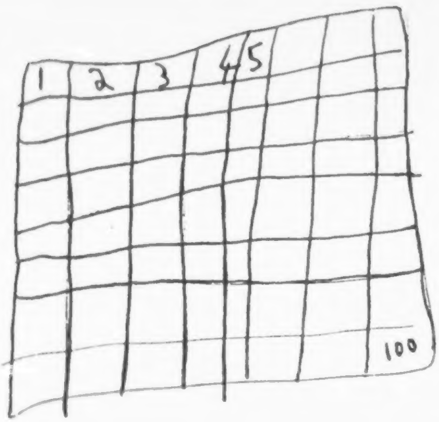
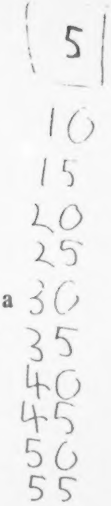
Grant



David



Numbers flashing one at a time, in fives, up to 100.

Figure 1: Examples of visualization (continued)

CONCLUSION

The missing addend and removed item tasks showed that all the Kindergarten and some Year 1 children were still in the process of constructing the number system with the relation 'one more'. These same children were equally likely to solve partition problems by one to one, or many to one, correspondence. Only two children could solve the quotition problem without direct modelling and they used a building-up mental strategy. They were more advanced with multiple counting using tens than with any other number and this was reflected by their developing skills for counting collections pregrouped as tens and ones. It should be noted that three of the Year 1 children successfully coordinated the counting in tens and ones, of a collection of 43 objects and two of them regrouped in their mental strategies when adding a single digit number to a two-digit number. At the same time these children do not use ten as an iterable unit, as shown by the uncovering tens task, do not subitize ten from visual patterns of two single-digit numbers and do not recognize the pattern of tens in the hundreds square.

Some of the children in Years 2 and 3 were shown to be using ten as an iterable unit while others used ten as abstract collectable units. This was reflected by the use of higher order strategies based on ten for solving addition with regrouping tasks and the use of the tens pattern in the hundreds square. It is interesting to note that none of the Year 4 children used the higher order strategies involving breaking numbers up into parts but rather depended upon adding units of the same rank. Is this because of instructional experiences? Most of the Year 2 and 3 children had poor multiple counting skills with numbers other than ten.

Overall there appears to be a lack of extended structure evident in a young child's understanding of number and a corresponding resistance to use the properties of ten in mental calculation. No children in our sample from years 1 to 4 spontaneously used grouping to find how many marks there were on the card (143 marks altogether) - they either said they did not know, guessed or counted by ones. Only two children recognized the grouping of ten groups of ten when shown 'what a friend had done to help him/her find quickly how many marks there were', even though the hundred was highlighted with a red circle.

The visualization of the numbers from 1 to 100 provided seven response categories. Of these responses only one was dynamic (flashing numbers), all others being static in nature. Approximately one third (8) of the grade 1 to 4 children did not visualize any picture, a third (10) visualized a picture with no structure and the remaining third (9) had structure of some kind in their picture. The results show that Year 2 children are beginning to 'see' structure in the number system. The grouping structure in the Year 2 responses is most clearly seen in the mental picture of ten ten-rods as illustrated by Melissa. The most highly developed visualizations of the structure are shown by the pictures illustrated by Grant and Oliver (hundred squares), Alex (patterns within the number sequence), and David (multiples of five as flashing numerals).

In this report only some of tasks that were used have been discussed. So far it appears that children as young as 5 year olds use equivalent groups when counting to solve partition and quotition tasks. Children in Kindergarten are building the first level of the number system using the 'one more' relation. Children in Years 2 and 3 are building the second level of the number system, important aspects being the ability to partition numbers in many different ways and the use of ten as an iterable unit. The construction of one group of ten out of ten ones, which then becomes a ten unit and then, the use of the two types of units (tens and ones) in regrouping is important for the development of number sense at this level. Many children in this study were still needing to develop this structural flexibility with the use of two-digit numbers. Understanding of the extended structure of the number system beyond two-digits was only displayed by one Year 3 and three Year 4 children.

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CHILDREN'S UNDERSTANDING OF THE INDEPENDENCE OF RANDOM GENERATORS

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THE EXPERIMENTAL SITUATION

A simple way to investigate children's understanding of the independence of random generators is to examine successive random trials of a random generator whose probability function does not change from trial to trial. Such trials might be done by successive tosses of a coin or a die, or by successive drawings from an urn, with replacement after each draw.

Eight subjects, four male four female, from each of Years 4, 6, 8, 10 in South Australian schools were interviewed for about half an hour each. Part of these interviews is reported here. Each subject (S) was asked by the interviewer (I) to predict the results of nine successive random draws from a transparent container holding a total of 3 balls: 2 green (G) and 1 blue (B). For variety the questions asked varied from draw to draw, but all were variations on the following themes:

® Which colour is most [sic] likely to come out?
® If I asked you which colour you will draw out which would be the best [sic] colour to guess?
® Which colour do you expect to come out this time?
® Which colour do you think you will take out this time? Why?

There was no evidence that children saw the variation in questions as leading to change in meaning in the way reported by Jones (1974), but this is a possible weakness in the protocol. In practice, nine draws started to become a little boring for some SS, so this questioning procedure was not slavishly adhered to before every draw. Boredom was more likely to arise if there were a long run of Gs, and least likely to arise if there were a long run of Bs. Regardless of the amount of questioning, all SS wrote down a prediction before every draw.

The result of each draw was written down next to each prediction and S was usually asked to make some comment on what had happened using questions like:

® Were you surprised?
® Does that mean you were wrong?
® Why did this happen?
® Can you make it come out blue?
® Did you expect this to happen?
® Will you be right every time?

Once again, questions were not asked on every occasion. For example, if S had predicted G and obtained G three times in a row, there was little point in asking for further comment if the event occurred a fourth time. Since outcomes could not be pre-determined the protocol had to be flexible in order to capitalise on what actually happened and on changes in mood of S.

RESULTS

The view taken here is that each of the nine draws represented independent random actions. With this model it would be anticipated that mature SS would consistently predict G regardless of any of the preceding outcomes. Such SS would be thinking in a Pascalian way and could be presumed to have understood the independence of random actions.

However, it is clear from other research that SS who do not think in a Pascalian way adopt a very wide range of responses and it is not clearly understood what situations are likely to induce which responses. Much previous research, e.g. Fischbein (1975), Goodnow (1955), Green (1982), Turner (1979), has regarded patterns of outcome as the most likely stimuli, but it is also possible, as suggested by Brainerd (1981) that the pattern of SS's responses or the combined pattern of responses and outcomes may influence future predictions. Given the exploratory nature of the research reported, it was not considered profitable to use stochastic matrices in the way that Brainerd did.

The results of all nine predictions/outcomes for each S have been tabulated in full and are summarised in the tables below. The small sample size means that statistically significant results are unlikely to be obtained. However, the results may indicate which questions might yield significant results with a larger sample.

Initial Predictions

All SS were aware of the structure of the random generator: the urn contained 2G and 1B. All SS, except two in Year 4, predicted that the first ball drawn would be G. TD (Year 4, M, 9:3) expressed several times a strong liking for B, so it likely that affective matters were influencing his choice. The other S who predicted B was MW (Year 4, F, 8:7), whose comments suggest that she was not aware that the composition of the urn was relevant to the situation. She made comments like "Because it looks like it's the green's turn today", "It looks like it's green before blue" and "... coz it came out green then it looks like it's blue now".

Turner (1979) found that an appreciation that the preponderance of Gs means that in some sense G is more likely to be drawn than B did not appear until a child was about 9 years old. This research provides some support for Turner's finding.

Pascalian Responses

Five SS, one in Year 4, one in Year 6, two in Year 8 and 1 in Year 10, consistently predicted G for all 9 draws. All of these had at least two of their predictions refuted during the experiment. The responses given are instructive.

ML (Year 4, M, 8:10)	2G 1B Same as the other questions, cos there's 2 Gs ...
I	Can you make it come out G?
ML	No you can't.
I	Why not?
ML	Because it's hard to choose cos I know where it is at first, but when you shake it up it's hard to find them.

Although ML had five of his predictions refuted he stuck to his predictions, using an explanation remarkably good for his age. He saw that the composition of the box was constant, but that the shaking up meant that an element of randomness was inevitable. The other SS produced similar responses.

These SS who used a Pascalian approach all saw that drawing a B is possible but believed that it did not affect the properties of the random generator. These are determined by the contents of the urn and not the outcome of any particular trial. They have understood the idea of independence of random trials, at least in this situation. The early age at which appropriate arguments are used is surprising, and warrants further investigation.

Work by Zaleska (1974), Zaleska and Askévis-Leherpeux (1976), Fischbein *et al.* (1991), among others, suggest that the investigation reported here deals with a particularly simple situation. Here the random generator is a familiar one whose structure is known to the child; here the child has to predict an outcome, rather than select a generator; here the child draws the balls from the urn and records the results; here the trials are conducted successively, not concurrently. Such a simple situation is likely to be the most efficient way of investigating SS's non-Pascalian responses.

Were the Subjects Using an Heuristic of Alternating Responses?

The answer to this question must be an unequivocal "no". All SS showed an awareness of the asymmetry of the random generator, and almost all indicated, albeit in different ways, some understanding that this asymmetry affected the outcome of each draw. Those workers, e.g. Green (1982), Cohen and Hansel (1955), who have reported the use of an alternating heuristic have been working either with random generators like coins or with two outcome probability generators whose rule is unknown to the child.

Were the Subjects Using an Heuristic of Negative Recency?

The "negative recency heuristic" is used to describe a predictive strategy in which "the probability of predicting an event [decreases] as a consequence of the event having occurred repeatedly on previous trials" (Fischbein, 1975, p. 59). This is a form of the Baconian heuristic which argues that the evidence from pervious throws changes the probabilities for the next throw. One common form of this heuristic is known as the "gambler's fallacy". For example, S is less likely to predict that the next outcome will be a B after observing just 1 G than after observing 6 Gs. The research mentioned above suggests that the circumstances in which such a heuristic might be used are not well understood.

In the work reported here there are four possible situations. A prediction might change from B to G or from G to B, and such a change may take place after a run of Bs or of Gs. A run is defined here to include the case of just one occurrence. The data provided here are of course too small for generalisations. However, they do suggest a thesis different from any reported by other workers.

Table 1 shows the frequencies of the length of run which precedes a change of prediction. It takes into account the fact that, for example, a run of 3 Gs contains within it runs of 1 and 2 Gs as well. Thus an S who has changed a prediction after a run of 3 Gs has also not changed a prediction after runs of both 1G and 2G. The five SS who consistently predicted G are excluded from Table 1 on the grounds that they are probably using a Pascalian heuristic which is not relevant to this discussion.

Table 1

Length of runs of	Number of times occurred	Change of prediction from G to B		Change of Prediction from B to G		No Change in Prediction	
		N°	%	N°	%	N°	%
1G	61	14	23	14	23	33	54
2G	40	7	18	9	22	24	60
3G	14	4	24	5	29	8	47
4G	8	2	25	2	25	4	50
5G	4	1	25	0	0	3	75
6G	3	2	67	0	0	1	33
1B	27	9	33	9	33	9	34
2B	9	6	67	2	22	1	11
3B	1	0	0	1	100	0	0

(a) Changing Prediction from G to B after an Actual Run of Gs

This table shows that the probability that S will make a change of prediction from G to B is very much the same for runs of between 1G and 5G. In other words, S is just as likely to change from G to B after a long run of Gs as after a short run. The figures for actual runs of 5G and 6G are probably too small to yield reliable results. This constancy is quite remarkable. It suggests even more strongly that the negative recency heuristic is not being used in this case.

(b) Changing Prediction from G to B after an Actual Run of Bs

The numbers in Table 1 are rather too small for reliability. However, they do suggest that a run of 2 Bs is quite likely to result in a change of prediction from G to B. This strategy is the opposite of the negative recency heuristic. I am not aware of any research workers in this field using the term "Positive Recency Heuristic". Perhaps this phenomenon has been overlooked.

(c) Changing Prediction from B to G after an Actual Run of Bs

The numbers here are too small to draw any conclusion.

(d) Changing Prediction from B to G after an Actual Run of Gs

Table 1 suggests that S's probability of changing from B to G after a run of Gs is independent of the length of the run. Either this conflicts with the negative recency heuristic or there are other factors operating. It has been shown that SS are well aware of the asymmetry of the random generator. It is also possible that the relation between the length of run and the desire to change predictions is non-linear.

The similarity of the results for changes to G and changes to B is particularly striking. It makes it clear that a negative recency effect is an inadequate description of the subjects' strategies. One reason for this has been suggested by Goodnow (1958, p. 115) who reports the results of several studies which found that

> *For most individuals, correct prediction of an infrequent event is not just one more correct prediction but a success worth several correct predictions of the easy-to-get more frequent alternative. It assures the subject that he is either singularly blessed by luck or else smart and on the verge of discovering the 'system'.*

Goodnow touches on the issue of whether the SS are making predictions which will match the actual outcome, of trying to predict a "typical" outcome, or trying to maximise rewards under pay-off conditions which are not explicitly stated. This matter will be taken up below.

What is clear from the data analysed so far is that it is unlikely that a negative recency heuristic based on actual outcomes is sufficient to describe the pattern of SS's responses and it is possible that it is not being used at all in some situations. Other possibilities will now be examined.

Does the Negative Recency Operate with the Predictions?

It is possible that S may be using a negative recency heuristic based, not on the actual outcomes of the trials, but on his or her predictions. A further factor may be whether or not the prediction matched the actual outcome; this will be considered later.

Table 2 has a similar structure to Table 1, but is concerned with runs of predictions rather than runs of outcomes.

Table 2

Length of predicted runs of	Number of times occurred	Change of prediction		No Change in Prediction	
		Number	%	Number	%
1G	44	13	30	31	70
2G	31	12	39	19	61
3G	19	9	47	10	53
4G	10	4	40	6	60
5G	6	1	17	5	83
6G	5	4	80	1	20
7G	0	0	0	0	0
8G	1	1	100	0	0
1B	41	33	80	8	20
2B	9	7	88	1	12
3B	1	1	100	0	0

(a) Changing Prediction from G to B after Predicting a Run of Gs (Independent of any Confirmation or Refutation of Such Predictions)

Of the 44 changes of prediction to B after a run of predicted Gs 25 occurred after a predicted run of 1 or 2 Gs, 13 after a predicted run of 3 or 4 Gs and 6 after a predicted run of 5 or more Gs. The mean length of run was probably a little higher for the older groups.

The pattern revealed by Table 2 is quite different from that revealed by Table 1. Table 1 suggested that the probability that S would change a prediction from G to B was independent of the actual length of run of Gs (at least for runs up to 5 long). But Table 2 suggests that the more Gs a subject has predicted the greater the probability that S will change to predicting a B.

Table 3 gives a break down of Table 2 by Year level to see if older SS are more likely to predict longer runs of G before changing their prediction to B. The numbers are small but they do indicate trends.

Table 3

Length of predicted runs of	Change of prediction			Change of prediction			Change of prediction			Change of prediction		
	Year 4			Year 6			Year 8			Year 10		
		No	%		No	%		No	%		No	%
1G	10	4	40	12	4	40	9	1	11	13	4	31
2G	6	5	83	8	3	38	8	3	38	9	1	11
3G	1	1	100	5	4	80	5	1	20	8	3	38
4G				1	0	0	4	1	25	5	3	60
5G				1	0	0	3	0	0	2	1	50
6G				1	1	100	3	3	100	1	0	0
7G										1	0	0
8G										1	1	100
1B	12	10	83	11	9	82	10	8	80	8	6	75
2B	2	2	100	2	2	100	2	2	100	2	1	50
3B										1	1	100

This table suggests that the younger the child the more likely he or she is to change prediction after a short run of predicted Gs. There does not seem to be any age difference affecting the length of a predicted run of Bs. These results support the findings of Turner reported above.

It would be possible to argue that SS are attempting to give a representative example of a possible set of 9 draws. If so, we would expect them to predict 6 Gs and 3 Bs. Table 4 shows the distribution of their predictions.

Table 4

Prediction	Frequency				
	Year 4	Year 6	Year 8	Year 10	Total
9G 0B	1	1	2	1	5
8G 1B	2	2	3	3	10
7G 2B	4	2	2	1	9
6G 3B	0	2	1	1	4
5G 4B	1	0	0	2	3
4G 5B	0	1	0	0	1

It can be seen that among those who did not predict 9G 0B there is a distinct bias to predict more Gs than one might expect from a typical set of 9 draws. This suggests that SS are not trying merely to predict a typical set of outcomes, not are they seeking to maximise rewards where a B pays more than a G in the way suggested above by Goodnow. It is almost as if they want to show that they realise the bias of the random generator, but they are aware that less likely events do happen from time to time.

This tension can be seen when looking at the responses of SS who changed from predicting G to predicting B after five or more successive predictions of G In some cases the change was clearly a result of previous outcomes, not predictions. For example, AP (Year 8, F, 13:8) changed from predicting G to B immediately after having had four out of those six predictions being refuted by a B. Her reason was "Seem to be getting B all the time".

However, in two cases, it is not clear whether the outcomes or the predictions were influencing S. JM (Year 8, F, 13:7) changed after 6 predictions of G including a refutation of the 3rd prediction with the reason "Because it's been G all the time and its due for a change". Similarly, JW (Year 10, F, 15:3) changed after eight predictions of G of which the first two were refutations with the reason "I only picked B because it's been all Gs before". Since in both cases it had not been all G outcomes before the change, it appears that SS were taking their previous predictions into account.

(b) Changing Prediction from B to G after Predicting a Run of Bs (Independent of any Confirmation or Refutation of Such Predictions)

Of the 41 changes to G after a predicted run of Bs 33 followed a predicted run of just one B, 7 followed a predicted run of two Bs and 1 after a predicted run of 3 Bs. So, there was a strong tendency not to predict a large number of Bs in a row.

The contrast between the influence of previous outcomes and previous predictions on future predictions, which has been noted above, is even more marked when considering SS's change from a predicted run of Bs to a prediction of G. The probabilities that such a change will take place after actual runs of 1 or 2 Bs are 0·33 and 0·22 respectively. However, the probabilities that such a change will take place after predicted runs of 1 or 2 Bs are 0·80 and 0·88 respectively. It is true that there is no difference when considering predicted and actual runs of 3 Bs with both probabilities being 1, but the sample size here is very small and the difference for 1 and 2 runs remains quite striking. It suggests that SS do not feel confident with predicting Bs but feel that they need to from time to time, perhaps to incorporate their awareness that B will happen from time to time.

This tendency may arise because a long run of Bs is not expected or because predictions of B are more likely to be refuted, and hence discouraging in the sense discussed by Goodnow above. So it is worth examining the effect of refutations on SS's behaviour.

Does the Heuristic of Negative Recency Operate with Refutations?

The effects of refutation and confirmation of a prediction on the subsequent prediction are listed in Table 5. There were no obvious differences between the year groups.

Table 5

		Next Prediction			
Refuted Prediction of	**Total**	**G**		**B**	
G	79	59	75%	20	25%
B	41	29	71%	12	29%
Confirmed Prediction of					
G	124	99	80%	25	20%
B	16	12	75%	4	25%

The figure 12 in the second row includes 4 choices by TD, who, as already mentioned, stuck very strongly to his preference for B on animistic grounds. It can be seen that a refuted prediction for G is very much less likely (25%) to result in a change of prediction for the next draw than is a refuted prediction for B (71%). It can also be seen that a confirmed prediction for G is much less likely (20%) to result in a change of prediction for the next draw than is a confirmed prediction for B (75%).

Put another way 75% of refuted predictions of G led to the next prediction also being for G, compared with 80% of confirmed predictions for G. Similarly, 29% of refuted predictions of B led to led to the next prediction also being for B, compared with 25% of confirmed predictions for B.

These percentages are very close, and there seems to be no evidence that a specific refutation has a strong influence on the next prediction. Such a result is consistent with the earlier finding summarised in Table 2 that the probability of a change of prediction is independent of the length of the actual run, at least for runs up to 5 predictions long.

Perhaps the possible change after longer runs means that a chain of refutations may have some influence on subjects' predictions. Table 6 examines how many confirmations or refutations a subject had experienced before changing his or her prediction.

Table 6

Result of Preceding Predictions	**Change Prediction from B to G**		**Change Prediction from G to B**	
	No	%	No	%
Previous Prediction Confirmed	9	23	25	64
Previous Prediction Refuted	11	28	8	21
More than one Previous Prediction Refuted	19	49	6	15
Total	39		39	

This table states that there were 39 changes of prediction from B to G. Of these 9 (23%) occurred after a confirmed prediction of B, 11 (28%) after exactly one refuted prediction of B and 19 (49%) after 2 or more refuted predictions of B. Similarly, there were 39 changes of prediction from G to B. Of these 25 (64%) occurred after a confirmed prediction of G, 8 (21%) after exactly one refuted prediction of G and 6 (15%) after 2 or more refuted predictions of G.

These differences are striking. A confirmed prediction of G predicts a change to B much more strongly than a confirmed prediction of B predicts a change to G. It is possible to argue that subjects may have changed their predictions in accordance with whether they felt that their luck was in or not, but such an argument cannot be verified by this type of data.

SUMMARY

All of these results suggest that these children were well aware of the asymmetry of the random generator. For example, when all results are combined 77·8% of predictions of G are followed by another prediction of G, while 28·1% of predictions of B are followed by another prediction of B.

As mentioned above, favourable Baconian probabilities increase with the weight of the evidence. For this experiment, a Baconian approach would mean that subject's predictions would attempt to ensure that the ratio of numbers of G and R would approach 2:1. It is clear from this investigation that while children's predictions are influenced by preceding outcomes they are influenced by preceding predictions and possibly by runs of refutations.

Some of the strategies observed here have not been observed by other workers. This may well be because the nature of the investigation is different from that employed by others. It is not a very cost-effective method because it is labour intensive, it has outcomes which cannot be controlled without resorting to some sort of deception, and because it is likely to bore children if it goes on for too long. My own experience suggests that the process of asking for a prediction followed by testing the prediction and recording and reflecting on the actual outcome can only be done from 8 to 10 times before the subject starts to lose interest for a very wide variety of reasons. So while it would be expensive to test the hypotheses suggested here, the fact that some of these differ from those of other workers suggests that such testing would be worth while. In particular, if a large sample of children were able to be tested then it would be possible to obtain statistically useful results for separate age groups.

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PERCEPTIONS OF CHANCE EVENTS AMONG R - 7 CHILDREN

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The recently written South Australian Attainment Levels in Mathematics will now bring Chance and Data into the primary curriculum for the first time. However, many teachers are very unsure about teaching the Chance and Data strand and feel themselves to be in need of information and support. This study has been designed to provide support for such teachers.

This study will be done in two parts. Six questions will be asked of all members of a Year 3, a Year 5 and a Year 7 class from one suburban public school. Six children will then be randomly selected from each class and interviewed on a one-to-one basis by the researcher. These children will be asked to expand on their answers in the class situation. They will then be presented with concrete models and be asked to predict the outcome of three further probabilistic situations.

Researchers have questioned children on the predicted outcome of tossing a die. The evidence suggests that children will frequently select six as being the most difficult to throw based on the requirement of many board games for a player to throw a six before the player can begin. A question of this type will be asked as part of this study as a control.

Questions one and two have been influenced by the work of Green (1983). Green's work was done with eleven to sixteen year olds. Green asked subjects to *Put a ring around your favourite number from 1 2 3 4 5 6. If you have no favourite say "None".* Children will be asked this question first to determine whether their preferred number has any influence on the choice of their answer to the question to follow it, i.e. *When an ordinary 6 sided die is thrown which number or numbers is it hardest to throw, or are they all the same?'*

The third question attempts to test the validity of the response to the second question. Children will be shown a pair of dice, and asked *When you roll two dice together which is more likely: that you will obtain the same number with both dice or that you will obtain 2 different numbers?*

The remaining three questions are influenced by the work of Tiegen (1983). Tiegen claims from his research with undergraduates who were asked questions relating to the choice of a number from a series of raffle tickets that the undergraduates chose central, 'representative' values, and avoided extreme ones. Tiegen's assumptions are based on very different reasoning from other researchers in this field. It seems that it is sensible to replicate the work of Teigen and to compare his results with those from primary school children.

Tickets numbered 1 - 12 will be torn from a book and placed into a container. Children will be asked, *'If you chose a ticket (from the container) which number do you think would be on the ticket ?* Children will be shown the box with both blue and red tickets in it and told *This box contains six blue tickets numbered 1 - 6 and six red tickets numbered 7 - 12. If you drew one ticket from this box which number do you think you will draw?* Children will be shown a box containing twelve yellow tickets. They will be asked *'If you took eleven tickets began this box which would be the last ticket that you would remove?'*

This study recently so that it will be impossible to summarise results, and data from only the first part of the study will be available to present at the conference.

To date I have administered the six questions of the initial investigation to three primary classes, Year 3, with 22 students; Year 5 with 30 students; and Year 7 with 25 students, and have interviewed six children from each class.

When analysing the class responses it was found the Year three students were inclined to be less spread than older children and responses were more inclined to cluster.

Question 1: Write the numbers 1, 2, 3, 4, 5, 6 onto a sheet of paper. Put a ring around your favourite number. If you do not have a favourite number write the word 'none'.

There was a wide spread of responses among the choices.

Question 2: 'When an ordinary six-sided die is thrown, which number or numbers is the hardest to throw or are they all the same?'

There was more inclination to choose numbers from 4 to 6 than the lower numbers, but 6 was seen as predominantly more difficult in only 22% of cases.

The perceived wisdom is that children regard 6 as difficult to throw based on their recollection of waiting to begin a board game. Many of the children's comments indicated alternative views. Some believed that the die had a mind of its own, or that it could be mechanically ordered, while many who quoted six as being hardest to throw saw this as relating to the size of the number, or that six was a favourite number.

TC Year 5: Because that's (6) the highest number ... most of the time I never get it.

AD Year 5: 5 is an odd number I suppose. I never get odd numbers only even numbers when I throw a die. I always get a six, two or four.

MM Year 5: ... because it's my favourite number.

CL Year 7: Sometimes you roll it and it's on number six and it doesn't roll to the other side

Some of the younger children felt that colour in the case of dice had some effect on the outcome.

I: If I roll two dice together what is more likely: that you get the same number with both dice or that you will get two different numbers?

MM Year 5: Two different

I: If both dice were different colours would that affect the result?

MM Year 5: Yes, then they'd be different numbers.

Question 3: 'You roll two dice together. Which is more likely: that you will get the same number with both dice or that you will get two different numbers?'

Almost all children answered question three by saying 'different'.

Question 4: Student is shown a container of raffle tickets numbered 1-12 and is asked, "if you chose a ticket from this container which number do you think will be on the ticket?'

The responses to questions 4 - 6 were spread between classes.

Year 3 students chose numbers six, seven, eight, nine and twelve, with eight being chosen most frequently. Year 5 students chose all numbers except one and five with six and seven being their most frequent choice. This pattern was similar with the Year 7 students although in their case six was chosen most frequently.

Raffle tickets and dice were seen by many of the children interviewed as behaving differently. Many children felt that they had some some control over raffle tickets, and that their behaviour could be predicted.

NB Year 5: ... the highest numbers just come up and the lower numbers don't and when the numbers are all shuffled up and all that the higher numbers are mostly on top.

I: If we drew out every ticket in the container except one, what number would be on the last ticket left?

AD Year 7: One of the little ones

I: What do you mean by little ones?

AD: Oh, you know one and two and them.

Question 5: Show student container with six blue tickets numbered 1-6 and red tickets numbered 7-12. 'If you drew one ticket from this container which number do you think you would draw?'

Year 7 students chose numbers from 6 to 9 in a majority of cases, Year 5 students responses spread from five to eleven, with the most popular choice being five, while year 3 students spread their choices fairly evenly omitting numbers one, two, five and nine from their considerations.

In question 5 the colour of the tickets seemed to make no difference to the children. They simply concentrated on the numbers involved in the question, although it was interesting to note that the children's responses to Question 4 and 5 were very different even though there were twelve raffle tickets involved in both cases.

I: In this container there are twelve yellow raffle tickets numbered 1,2,3,4,5,6,7,8,9,10,11,12 if you just draw one ticket out of the container what would that number be?

NM: Year 5 Six

I: In this container there are twelve tickets. The blue tickets are numbers 1,2,3,4,5,6 and the red tickets are numbers 7,8,9,10,11, 12 If you draw just one ticket out of the container what would that number be?

NM Year 5: Eight

I: In this container there are twelve yellow raffle tickets numbered 1,2,3,4,5,6,7,8,9,10,11,12 if you just draw one ticket out of the container what would that number be?

CL Year 7: Six

I: In this container there are twelve tickets. The blue tickets are numbers 1,2,3,4,5,6 and the red tickets are numbers 7,8,9,10,11, 12 If you draw just one ticket out of the container what would that number be?

CL Year 7: Eleven

Question 6: Show student a box containing twelve yellow tickets. If you took eleven tickets from this box which would be the last ticket that you would remove?

This was most interesting in terms of the responses. Year 7 students chose numbers one or twelve in 60% of cases while the other responses were numbers two, three, five,six, seven, eight and nine. (5-9 receiving one vote each). Year 5 students spread their preferences quite differently between eleven and twelve; and four, five, six and seven being chosen by 73% of the students; while all but four Year 3 students concentrated their choices from numbers nine to twelve.

KK Year 7: confidently predicted that in Questions 4 and 5 the result could be any number, however, in question 6 she nominated 12 just as confidently.

I: Why do you say 12?

KK Year 7: Because there are 12 tickets altogether, so I think 12 would be the last number left.

There appeared to be no pattern across the year levels to the answers to Question 6. However at each individual year level there was a distinct pattern that was more obvious than with any of the other question.

In the interviews when children were pressed for an answer to Numbers 4, 5, 6 they frequently commented that the number that they had chosen was 'in the middle'.

Tiegen (1983) presented 73 tertiary students "a box containing 12 tickets, numbered from 1 to 12. The numbers were written on the blackboard, and the students were told that as there was only one ticket of each kind, all numbers had an equal chance to be drawn, Still they had to make a guess, and write down which number they thought would turn up" (Tiegen 1983). The four central values (5,6,7,8) were chosen by 59% of the subjects. When the experiment was repeated with a different 42 students who were promised a small financial reward for correct predictions, then 76% of the students opted for the four central values. The percentage of students who opted for the outer values (1,2,11,12) also decreased.

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WHAT RESEARCH IS NEEDED IN PROBABILITY AND STATISTICS EDUCATION IN AUSTRALIA IN THE 1990s?

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The final publication of *A National Statement on Mathematics for Australian Schools* (Australian Education Council (AEC), 1991) includes Chance and Data as one of the five major content areas of study in mathematics suggested for Australian schools. Following the appearance of the *National Statement*, the Australian Association of Mathematics Teachers received money from the Department of Education, Employment and Training to fund the preparation of eight workshops for teachers to aid in the implementation of the new curriculum. One of these workshops is to introduce Chance and Data and suggest resources to be used by teachers K-12. Further, the Curriculum Corporation has initiated a national collaborative mathematics project to provide materials to bring the Chance and Data part of the mathematics curriculum alive in classrooms K to 10. For those who have been making suggestions for the inclusion of topics from probability and statistics in the curriculum for many years (e.g., Bloom & McDougall, 1988; Peard, 1990; Watson, 1980, 1989, 1991), these initiatives are whole-heartedly welcomed.

These moves reflect a worldwide effort to bring Chance and Data to every level of the mathematics curriculum. In the United States these developments are reflected in the National Council of Teachers of Mathematics' (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (1989) and in materials produced for teachers by the Quantitative Literacy Project for secondary schools and the Used Numbers Project for primary schools. In England and Wales, the 1991 proposal for *Mathematics in the National Curriculum* (Department of Education and the Welsh Office, 1991) devotes one of the five major Attainment Targets to Data Handling.

These initiatives have been taken in Australia, however, without the benefit of any previous educational research in this country on the learning of probability and statistics[1] and without any formal recommendations for future research on the learning and teaching of the topics or for evaluation of the efforts of preparing teachers or actually implementing the curriculum. Hence states and local schools now face the implementation of the Chance and Data part of the curriculum with no Australian research base from which to make recommendations for the preparation of teachers or for the suggestion of methods and topics realistic to the developmental level of the students. It is the purpose of this paper to suggest such a base.

OVERSEAS RESEARCH

Looking at the history of research in the area of learning probability and statistics overseas, it is clear that while progress is being made, there is still much more to accomplish. The most recent review of the research in this area is by Shaughnessy (1992). He provides an excellent summary of research associated with judgements made under uncertainty, the different approaches to research used in the area, the more meagre research associated with statistical constructs and the use of computer-based activities to reinforce them. He goes on to suggest directions for future research in seven areas: assessment, secondary students' understanding, cross-cultural studies, teachers' understanding, teaching experiments, computer software, and metacognition. The call for continued research overseas means Australian researchers have two types of opportunities available to them. There is the chance to be involved in advancing basic understanding in the area, as well as considering questions relevant to the Australian educational context. Furthermore Shaughnessy, while dealing with the sophisticated concepts which provide difficulties in learning probability and statistics, does not address adequately the problems of younger children being introduced to chance and data. This is a result of there being very little published research in the area in the previous two decades, and a delay in publishing time which means that Shaughnessy did not have access to some very recent studies.

In an earlier review of research, Hawkins and Kapadia (1984) presented a review of work done in probability up to the first International Congress on the Teaching of Statistics. They discussed the debate between followers of Piaget and Inhelder (1951) who concluded that children could not handle probability concepts until the age of formal operations, and followers of Fischbein (1975) who used an intuitive definition of probability and came to a different conclusion. Hawkins and Kapadia pointed out that much confusion has arisen as a result of researchers using different definitions of probability itself. In the area of statistics much less research has been reported and it has focused on specific constructs, like the arithmetic mean (e.g., Goodchild, 1988). One of the difficulties associated with research in this area is that much of it has been done with convenience samples of tertiary students. The applicability of the results of this research to primary and secondary students is highly suspect given the expected different levels of development present.

In their review of research on probability and statistics learning up to 1988, Garfield and Ahlgren presented the following summary, aimed at the secondary level and above.

> The literature makes it clear that far more research has been done on the psychology of probability than on other statistics concepts. In spite of this research, however, teaching a conceptual grasp of probability still appears to be a very difficult task, fraught with ambiguity and illusion. Accordingly, we make the pragmatic recommendation for two research efforts that would proceed in parallel: one that continues to explore the means to induce valid conceptions of probability, and one that explores how useful ideas of statistical inference can be taught independently of technically correct probability... As far as goals for instruction go, this view suggests a moratorium on the typical organization of statistics instruction: (a) descriptive statistics, (b) probability, and (c) inferential statistics. The intrusion of technical probability issues that are not likely to be understood will stall the learning process -- and leave a distaste that could compromise subsequent instruction as well... What is needed, however, is not debate but research (Garfield and Ahlgren, 1988, p. 57).

This discussion would indicate that secondary teachers could be facing a decade of frustration in attempting to get across to students many of the topics in the Chance and Data curriculum. For primary teachers, the situation is even less well known because as noted, virtually no research results are generally available dealing with children this young. Two groups in the United States are beginning to look at statistical understanding at lower levels. Mokros and Russell (1992) are studying concepts of average and representativeness in children in grades 4, 6 and 8, and in their teachers (Russell & Mokros, 1991). Gal, Rothchild and Wagner (1989, 1990) also are looking at statistics reasoning in children in grades 3, 6 and 9.

QUESTIONS FOR AUSTRALIA

In Australia school systems and teachers currently are implementing the Chance and Data curriculum using the best resources and advice they can get from educators and curriculum planners, all of whom are operating without the luxury of a local research base. The situation, however, does not have to stay like this throughout the decade. There is the opportunity to start now, using the National Statement as a basis, to monitor the present understandings of teachers and students, to make suggestions based on this monitoring, and to evaluate the first few years of the implementation of the curriculum. At least by the turn of the century it may be possible to use the results of research to make more precise and confident statements to help teachers and students understand and apply the concepts in the Chance and Data curriculum.

Content. The direct mathematical base for research in Australia lies in the *National Statement* itself. For Chance and Data there are 28 scope statements within the four bands which cover the years of schooling, K-12. These statements are meant to provide the framework around which systems or schools may build their Chance and Data curriculum. The following italicized disclaimer, however, indicates that the developers of the *National Statement* do not take total responsibility for getting the ordering right.

> *There is no implication that scope statements within a band should follow the sequence presented* (AEC, 1991, p. 28).

The generality of the scope statements leaves much to the teacher in deciding about the rigour with which concepts are introduced. In the first three bands which cover grades K-10, the suggested emphasis is practical and experimental rather than theoretical and formal. Examples of two of the scope statements from Band B, aimed at upper primary children, show the heavy responsibility placed on curriculum planners and teachers to provide the necessary constructs to make implementation possible, even if the emphasis is practical and experimental.

> Experiences with chance should be provided which enable children to:'**B2**: for random events, systematically list possible outcomes, deduce the order of probability of outcomes and test predictions experimentally (p. 170).

> Experiences with data handling should be provided which enable children to: **B5**: understand what samples are, select appropriate samples from specified groups and draw informal inferences from data collected (p. 172).

Although interesting and useful activities are listed under each of these statements, assumptions are made that the teachers will know how these relate to the important concepts of randomness, probability, sampling, and inference. It is not clear that teachers currently possess these understandings themselves, let alone the ability to get them across to their students. It also is not clear whether the activities suggested are suitable for the development levels expected of the children in the Bands in which the activities are placed.

The question of how to explore teacher understanding and the appropriateness of content for children at different levels, will not be trivial. Recent overseas research offers promise however, as it has begun to move beyond the previous emphasis on specific topics, such as the arithmetic mean, to look at some of the broader concept issues involved in learning probability and statistics. Conclusions thus far have been based on studies of students, not their teachers. In Australia it will be important to look at both groups of people. The studies which offer the greatest potential as a basis for this research, have considered student beliefs about probability (e.g., Fischbein, Nello, & Marino, 1991; Garfield & delMas, 1991; Green, 1991; Konold, 1991b), conceptions of randomness (e.g., Green, 1989; Konold, Lohmeier, Pollatsek, Well, Falk, & Lipson, 1991), sampling in relation to sample size and representativeness (e.g., Gal, et al., 1989; Rubin, Bruce, & Tenney, 1991), the relationship between two variables (e.g., Konold, 1991a), the applicability of measures of central tendency (e.g., Gal, et al., 1990), and the understanding of conditional probability (e.g., Pollatsek, Well, Konold, & Hardman, 1987).

Teachers. What is needed for the rest of this decade is a concentrated effort to find out what teachers initially understand in relation to the Chance and Data curriculum and then to monitor changes as professional development takes place. It is suggested that samples of teachers be selected at the early childhood, primary, secondary, and senior secondary levels. From a relatively large sample of teachers, say 60, who are given written questionnaires to complete, a subset of 20 would be selected for in-depth interviews to explore constructs more fully. If testing took place each two years and included a monitoring of professional development activities of teachers, two outcomes would seem achievable: results of each testing would provide input for professional development activities in the next two-year period and by the end of the decade a picture of long-term development of stochastic constructs would emerge.

Students. A similar testing scheme should be developed for students, starting say with samples in Grades 3, 6 and 9. Two years later these same students would be retested and another sample chosen from Grade 3. Each sample, after the first, would be from Grades 3, 5, 8 and 11. This seems an appropriate spread, avoiding the transition from primary to secondary schooling. As with the teachers, larger samples (up to 200) would be given written questionnaires with interviews being used for a subset. The outcomes would include a monitoring of student development of understanding, as well as of the implementation of specific aspects of the Chance and Data curriculum. The results of testing should lead to suggestions for the ordering of topics in the curriculum, for the appropriateness of topics for the Bands to which they have been assigned, and for particular activities to reinforce weak concepts. Again by the end of the decade the scale of the implementation of the Chance and Data curriculum would be gauged.

Assessment. As part of this research assessment instruments would be developed for teachers and for students at various grade levels. As the emphasis on problem solving in the *National Statement* permeates the Chance and Data content area, recent research in assessment in problem solving should be applied to the development of test items. Initial testing could concentrate on two constructs which are considered essential to the successful implementation of the Chance and Data curriculum (AEC, 1991). These are (i) the conception of probability from intuitive to experimental to theoretical and (ii) the understanding of sampling from convenience to random, with associated issues of representativeness. The tasks to exhibit understanding in these areas would be selected from those suggested in overseas studies and those found to be useful in preliminary pilot studies of probabilistic and statistical reasoning.

In probability the results of Fischbein, et al. (1991) offer a feasible starting point because they look at how an intuitive understanding of sample space influences responses to classical probability tasks involving coins and dice. In the Australian context the interest first would be centred, because of the emphasis in the *National Statement* for primary children, on how an intuitive understanding influences responses to experimental probability tasks. Fischbein's items could be modified to have a frequency basis, for example, based on data presented in tables. With older students it would be possible to use Fischbein's tasks directly. In statistics Kahneman and Tversky (1972) suggest specific problems to consider the similarity of a sample to its population, the apparent randomness associated with a sample, the effect of sample size on the representativeness of a sample, and the likelihood of various samples based on population characteristics. These problems, with variations in the form of presentation and extended interviewing as suggested by Pollatsek, Konold, Well and Lima (1984), would provide a starting point for tasks. For younger children, situations could be set up providing data for the comparison of two groups performing on a given criterion, and children then asked to decide which group has performed better (e.g., Gal, et al., 1989, 1990). Tasks also would be chosen and adapted from the Statistics Reasoning Assessment instrument, currently under development in the United States (Garfield, 1991).

A THEORETICAL MODEL

Historically psychologists have played a major role in the research into the understanding of probability concepts, most notably Piaget and Inhelder (1951), Fischbein (1975) and Kahneman and Tversky (1972). Piaget contributed a developmental perspective on the performance of certain tasks, Fischbein suggested the part to be played by intuition in understanding concepts, and Kahneman and Tversky found particular difficulties, such as representativeness (Kahneman and Tversky, 1972) and availability (Tversky and Kahneman, 1973), which have been widely studied in the last decade. The developmental and multimodal functioning aspects of the SOLO Taxonomy (Biggs and Collis, 1982, 1989, 1991; Collis and Biggs, 1991) make it a natural choice for a model to use following the work of Piaget and Fischbein. The most recent products of this research stream have been in the areas of assessment of cognitive functioning during problem solving in adolescents (Collis, Watson, and Campbell, 1991), of following the development of understanding of common and decimal fractions (Watson, Collis and Campbell, 1991a, 1991b), and of developing a mapping procedure to analyse mathematical responses (Collis and Watson, 1991). Building on the work of Biggs and Collis (1982, 1989), these latest projects have specifically considered the issues of (i) multimodal functioning, (ii) the repetitive nature of the unistructural-multistructural-relational (UMR) cycle which operates within the modes of functioning, and (iii) the ability to graphically follow a response.

(i) The multimodal aspect of cognitive functioning has come to prominence in recent years due in part to the reporting of the effect of out-of-school experiences on mathematical problem solving (e.g., Resnick, 1987; Saxe, 1988). In this context the importance of intuition on people's understanding of probabilistic concepts is well documented (e.g., Fischbein, 1975; Fischbein, et al., 1991). Further the potential applicability of both probabilistic and statistical concepts outside the classroom is one of the major reasons for putting them in the mathematics curriculum (AEC, 1991). It is very likely that the intuitive and imaging aspects of the ikonic mode of functioning play a major role in solving problems associated with chance and data. If so, they should be actively developed in the methods suggested for teachers who are implementing the new curriculum. The assessment of multimodal functioning in the solving of specific probability and statistics problems would assist in making recommendations for teaching.

(ii) A second aspect of earlier research in cognitive functioning has been its focus on developmental structure. Important in this has been the observation of UMR sequences of responses to mathematical tasks set for students (Biggs and Collis, 1982). Previous work on volume measurement (Campbell, Watson and Collis, in press) and fractions and decimals (Watson, et al., 1991b) showed that the development of concepts occurred in the concrete symbolic mode of functioning via two sequential learning cycles, the first reaching the relational level by about the end of grade 6 and the second beginning there and reaching the relational level in grades 10+. It is important to explore the hypothesis that a similar developmental structure exists for concepts associated with the learning of probability and statistics. If so, it would be possible to suggest sequencing of the curriculum to aid learning.

(iii) Finally, the mapping procedure to analyze responses has helped to graphically portray the cognitive functioning taking place during problem solving (Chick, Watson and Collis, 1988; Collis and Watson, 1991). The procedure has been used to study error patterns in area problems (Watson, Chick and Collis, 1988), to explore strategies for problem solving (Watson, in press), to suggest how the transition from the ikonic to concrete symbolic mode occurs (Watson and Mulligan, 1990), and to observe perseverance patterns during problem solving (Taplin, 1991). There is potential for all of these applications to be made in relation to topics in the Chance and Data curriculum.

It is essential to have a theoretical basis from which to analyse student and teacher understanding of constructs in probability and statistics. There are other models other than the SOLO Taxonomy which also provide potential for research, for example those based on information processing theories (e.g., Scholz, 1991). In terms of potential communication of results with the wider education community, however, it is felt that a developmental model is likely to be the most useful. This is especially true for research covering a large part of the K-12 curriculum. A stage-based model fits well with the types of conceptions of stochastics suggested by Shaughnessy (1992):

1. *Non-statistical.* Indicators: responses based on beliefs, deterministic models, causality, or single outcome expectations;

2. *Naive-statistical.* Indicators: use of judgmental heuristics, such as representativeness, availability, anchoring, balancing;

3. *Emergent-statistical.* Indicators: ability to apply normative models to simple problems; recognition that there is a difference between intuitive beliefs and a mathematized model; perhaps some training in probability and statistics;

4. *Pragmatic-statistical.* Indicators: an in-depth understanding of mathematical models of chance (i.e. frequentist, classical, Bayesian); ability to compare and contrast various models of chance; ... recognition of the limitations of and assumptions of various models (p.485).

Those familiar with the SOLO Taxonomy would likely be able to parallel the four categories with their understanding of unistructural, multistructural, relational, and extended abstract levels within the concrete symbolic mode.

IMPLICATIONS OF THE RESEARCH

The significance of the suggested research lies in providing a fundamental structure for a new area of the school curriculum and in monitoring its implementation in the early years. It also is possible, however, to speculate on some of the implications of such research for the classroom. Traditionally the limited exposure to topics in probability and statistics which has been present in the mathematics curriculum resulted in the introduction of topics which were then not developed or applied beyond their unistructural acquisition as constructs. The most notable example of this is the introduction of the arithmetic mean to children in the upper primary school. Most children acquire an 'add-em-up-and-divide' algorithm but no understanding of how the mean can be used to make decisions about data sets. The same may be said for the basic definitions in probability which are introduced in high school. Very recently the move to increase the presence of probability and statistics in the mathematics curriculum in the United States (NCTM, 1989) has produced several curriculum projects aimed at increasing the level of sophistication at which the topics are learned in schools. One project, Used Numbers, written for grades K to 6, shows a definite expectation that students will work at higher cognitive levels as they progress through the curriculum (see e.g., Russell & Stone, 1990; Corwin & Friel, 1990). Although not explicitly stated in psychological terms, it is possible to show how the suggested work in each of the six books in the Used Number series, progresses from unistructural to multistructural to relational in terms of the cognitive levels required to complete the tasks set. As well, the tasks require considerable interaction with the world outside formal mathematics and hence also offer the opportunity to explore and develop multimodal functioning.

There is great potential for interaction between the theoretical cognitive research base and the suggestions made in the Used Number project curriculum. It is likely that the materials would form the basis for interview tasks, as well as being suggested for teacher use as the curriculum is introduced. This would make possible the further evaluation of the developmental levels of responses of children as implementation occurs.

Finally, as the probability and statistics component of the mathematics curriculum can be argued to be the part most closely related to out-of-school experiences, the intuitive/ikonic involvement may be an aspect which is encouraged as a result of data analysis. This is a different emphasis than is commonly associated with topics in the traditional mathematics curriculum. It would be of great interest for teaching methodology to find out the influence of the ikonic mode on understanding of and problem solving in relation to the topics in the Chance and Data curriculum.

CONCLUSION

This paper has argued that there is an urgent need for research into the understanding of concepts related to probability and statistics to be carried out in Australia in the wake of the publication of *A National Statement on Mathematics for Australian Schools* with its emphasis on Chance and Data. The foundations suggested for this research are found in developmental psychology and in work beginning to be done in the area by mathematics educators overseas. The research would assess responses to tasks in probability and statistics in order to (i) build a cognitive model of student and teacher understanding of the topics, (ii) make suggestions for the introduction and ordering of these new topics in the mathematics curriculum and (iii) provide assessment procedures for the evaluation of the implementation of the curriculum.

FOOTNOTE

[1] The only papers which the author could find reporting Australian research in stochastics were written by Peard (1991) on a study considering misconceptions in probability in samples of gamblers and non-gamblers, and by Truran (1991) on children's comparison of probabilities in Grades 4, 6, 8 and 10. They appeared after the *National Statement* was complete.

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TO WHAT EXTENT DO GRADES 3 AND 4 CHILDREN MAKE SPONTANEOUS USE OF CALCULATORS FOR COMPUTATION?

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INTRODUCTION

The availability of hand-held calculators in the general community in recent years has changed people's computational methods. Calculators are readily available in most households and places of business and are used for many tasks by members of these groups. Of the three available methods of computation (mental, pencil-and-paper and calculator), mental and calculator computations are commonly used in everyday life. However, children in school have not always been encouraged to use calculators in their mathematics classes. Paper-and-pencil methods still receive the most emphasis in schools (Department of Employment, Education and Training, 1989, p. 69).

Surveys of Victorian primary school teachers in the early 1980s (Ferres, 1981, pp. .220-221) revealed that only 7 % supported the introduction of calculators in Grades 1-3 and 91% demanded the understanding of the four operations before their general use in the classroom.

Whilst a follow-up survey in 1990 (Ferres, Groves and Stacey, in preparation) indicated a significant change in attitude with 76% of primary teachers now supporting the introduction of calculators in Grades Prep. to 3, 58% admitted to rarely or never using them in their mathematics classes.

As previously reported at MERGA 13 and 14, two Melbourne based calculator projects* began in 1989 and 1990 to promote the use of calculators in mathematics classes from the beginning of primary schooling and to examine the effects of such availability and use on various aspects of mathematics programs and their impact on children's mathematics learning. In each of the six project schools, all children were given a calculator for their own use in mathematics classes commencing in Prep. The usual range of concrete materials was also made available and used within the normal class mathematics program. By 1991, the "calculator" classes had extended to Grade 2 in the project schools.

DATA COLLECTION IN 1991

In each of the "calculator classes", systematic data on actual calculator use have been collected through teachers' weekly record sheets, specific questions as part of teacher interviews at various times in the project, and from samples of children's work collected at each grade level. In 1991, the major data collection focus involved the testing and interviewing of Grades 3 and 4 children in the project schools. These children had not been previously part of the calculator project and would therefore provide a control group for future comparisons with our "calculator" children. During second and third terms, these Grades 3 and 4 children completed a written test as well as a test on calculator usage. A random sample of 10% of these children at each grade level also took part in a task-based interview involving number recognition, choice of computational method or device, and real world problem solving involving multiplication and division. Fifty-five children were interviewed on a one-to-one basis by the author and other members of the project team. This paper concentrates on a section of the interview that was aimed at collecting data from which some observations may be made regarding children's preferred computational methods in context-free situations.

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* The Calculator-Aware Program for the Teaching of Number (University of Melbourne) and the Victoria College Calculator Project (Burwood) commenced in late 1989 and early 1990 respectively. A joint successful bid for Australian Research Council (ARC) funding in 1992 and 1993 has now provided support and continuity under the new title of **Calculators in Primary Mathematics.**

Would children from a calculator-rich learning environment tackle a series of computations differently from children from non-calculator classes? Would "calculator children" be more reliant on their calculators to complete these arithmetical tasks? Would the level of achievement vary between the two groups of children being observed? These long-term questions cannot be answered at this stage but will be addressed when all data have been collected and analysed at the completion of the project.

METHOD

The random sample of 55 of the Grades 3 and 4 children were given a 25 - 30 minute interview on a one-to-one basis with a member of the project team. The setting was outside of the classroom usually in a location free from interruption and distraction (interview room, unoccupied office, library). Throughout the interviews, children were free to use whatever calculating devices they chose. Structured materials in the form of Multi-base Arithmetic Blocks (MAB) and Unifix cubes were provided. Pen-and-paper, the use of calculators and doing it "in your head" were also described as acceptable methods. It was pointed out that some questions may be new or seen to be difficult and it was permissible to pass on to the next item accordingly. In this section of the interview, each of the twenty-four items was displayed on a flashcard one at a time and the student was encouraged to complete the arithmetical procedure using the method or device that was seen to be appropriate for them. It was stressed that there was no obligation to use all available materials and they should feel free to select whatever method with which they were comfortable. Each child's preferred choice of calculating device was recorded together with his or her answer. These were classified as "mental" (sub-classified to include "automatic response" and use of "fingers"), "written" (including both standard and non-standard algorithms), "calculator", "materials" (MAB and/or Unifix), and "other" (such as dots and jottings). It was observed that some children abandoned an initial method in preference for a subsequent one. The latter one was recorded as the preferred device unless it was also abandoned. In such cases, the "other" category was used.

RESULTS

Table 1: Results and preferred calculating methods of grades 3 and 4 children* (n-55)

NO.	QUESTION	CORRECT	INCORRECT•	PREFERRED CALCULATING METHOD						
				AUTO RESPONSE	MENTAL	FINGERS	CALCULATOR	MATERIALS	WRITTEN	OTHER#
1	7 + 5	93	7	60	18	13	0	5	4	0
2	20 + 30	93	7	64	29	4	2	0	2	0
3	100 - 10	91	9	64	27	0	9	0	0	0
4	3 x 5	100	0	67	24	2	4	0	2	0
5	36 - 13	85	15	9	35	6	30	9	9	2
6	12 ÷ 3	92	8	31	28	4	31	3	0	3
7	3 x 99	76	24	2	18	2	69	0	4	5
8	How many 3's in 30?	72	28	14	50	13	17	2	0	4
9	1024 - 23	68	32	8	52	4	28	2	0	6
10	30 ÷ 10	89	11	38	47	0	6	4	0	5
11	52 - 18	51	49	6	41	10	26	6	10	1
12	15 ÷ 4	15	85	2	37	10	31	4	0	16
13	225 + 76	69	31	2	42	8	23	2	17	6
14	2000 - 1	56	44	11	83	0	4	0	0	2
15	321 - 164	46	54	0	32	2	46	5	10	5
16	62750 ÷ 50	53	47	2	63	2	31	0	2	0
17	7 x $3.53	26	74	0	10	8	53	0	6	23
18	14 x 20	75	25	0	34	23	29	0	2	11
19	$153 ÷ 4	15	85	0	9	2	65	0	7	17
20	34 x 26	53	47	0	15	3	48	10	10	14
21	20 ÷ 40	17	83	7	50	0	22	0	0	21
22	3 - 7	24	76	6	64	7	4	2	0	17
23	$2.50 + $3.50	70	30	2	56	2	17	0	7	16
24	$12.38 + $2.52	53	47	4	40	4	26	0	11	15
			Mean (%)	17	38	5	26	2	4	8

• Includes 'unable to complete', 'unable to interpret answer', 'impossible'
# Includes 'no idea'
* All values are percentages

Table 1 displays the twenty-four arithmetical items, the percentages of children obtaining correct/incorrect responses and the preferred calculating methods used in each question. "Mental" procedures (including "automatic response" and use of "fingers") were preferred on average by 60% of children while 26% elected to use calculators. "Structured materials" and "written" procedures were not generally favoured by these Grades 3 and 4 children (2% and 4% respectively).

Table 2: Preferred calculating methods on successful questions*

PREFERRED CALCULATING METHOD						
AUTO RESPONSE	MENTAL	FINGERS	CALCULATOR	MATERIALS	WRITTEN	OTHER
3	34	5	35	1	3	19

* Questions in which at least 70% of all children were correct. (Mean percentage values)

Table 2 presents the results on those items where at least 70% of children were successful in obtaining the correct answer. Once again, "mental" methods predominated being favoured by 67% of children with almost half of them being identified as applying "automatic response" techniques. "Calculators" were preferred by 19% of the sample while "materials" and "written" procedures were used by only 2% and 3% respectively.

Some of the items were deliberately structured above the usual content requirements of these grade levels. For instance, items involving decimal expressions and negative numbers have been included to provide a benchmark for comparative purposes when those children from the "calculator classes" are interviewed in 1992 and 1993. It is therefore not surprising to find low performance levels on these questions from these sample members. Tables 3 and 4 display the performance results and preferred calculating methods on those items where less than 30% of children were correct.

Table 3: Preferred calculating methods on unsuccessful questions#

PREFERRED CALCULATING METHOD						
AUTO RESPONSE	MENTAL	FINGERS	CALCULATOR	MATERIALS	WRITTEN	OTHER
32	29	6	19	2	3	4

# Questions in which less than 30% of all children were correct. (Mean percentage values)

Table 4: Analysis of preferred calculating methods on unsuccessful questions

NO.	QUESTION	SUCCESSFUL %	UNSUCCESSFUL CALCULATING METHOD %	
12	15 ÷ 4	15	Mental	89
			Calculator	63
17	7 x $3.53	26	Mental	80
			Calculator	50
19	$153 ÷ 4	15	Calculator	77
21	20 ÷ 40	17	Mental	93
			Calculator	58
22	3 - 7	24	Auto Response	67
			Mental	69
			Fingers	75
			Calculator	50

Item 12, "15 ÷ 4", with only a 15% success rate, was performed mentally by 37% of students. Only 11% of these students were correct. Of those who used calculators, only 37% were able to obtain and read the answer correctly. Items 17 and 19 involved computations with money. It is interesting to note that a high percentage of students elected to use their calculators on these questions (53% and 65% respectively). 50% on Item 17 and 77% on Item 19 were unable to read the correct answers from their calculators.

Item 21, "20 ÷ 40", with only a 17% success rate, was generally attempted either mentally or by calculator. 93% of children who used "mental" procedures were incorrect. Typically, the order of performing the operation was reversed to give a whole number answer. Of those who used calculators, 58% were unable to read the correct answer often claiming to have pressed the wrong buttons or by just ignoring the decimal point. 76% of children were unable to correctly complete Item 22, "3 - 7". Invariably, the order was reversed to produce "4" or it was seen to be an "impossible" operation.

CONCLUSION

These results indicate that, wherever possible, the sample children used "mental" computation (including "automatic response" and "fingers") in preference to "calculators" with little or no attempt to use "written" or "materials". This contrasts markedly with a computation preference survey of Grade 5 students conducted in U.S.A. (Reys, Reys and Hope, in press) that showed written procedures to be the preferred method for undertaking computation. It would be interesting to survey Grade 5 children in our project schools to see whether or not the additional year of schooling conditions children to make more use of the standard pen-and-paper algorithms.

This year, a random sample of 10% of Grades 3 and 4 children from the six project schools will be interviewed. All Grade 3 children have been part of the calculator project since their first year at school and it will be of interest to make comparisons of achievement on the same 24 items and preferred computational methods. Among the hypotheses for the long-term study is an expectation that the "calculator children" will select appropriate procedures more frequently and make more effective use of calculators. In particular, it is asserted that these children will successfully use mental computation more often. Those items requiring knowledge and understanding of large numbers, operations involving money, decimal and negative numbers, which were not well done in the 1991 sample, will attract closer attention when analysing the data collected in 1992 and 1993. Early trends from all data obtained from the various sources in the project so far indicate that many of the "calculator children" develop an interest in and understanding of these concepts well before their counterparts.

An important objective of primary school mathematics should be to develop number sense. (Everybody Counts, 1989).

> Number sense can be described as a good intuition about numbers and their relationships. It develops gradually as a result of exploring numbers, visualising them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms. (Howden, 1989, p.11).

The availability and use of calculators in primary mathematics classes may contribute to the achievement of this worthwhile aim. Continued monitoring of the impact of calculators on children's development of mathematical skills, concepts and problem solving abilities together with changes to content, will remain a central issue in this project and any subsequent studies.

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ABSTRACT THINKING IN RATES OF CHANGE AND DERIVATIVE

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Responses to problems involving rates of change were compared across four data collections throughout an introductory calculus course given to a group of first year university students, all of whom had studied calculus at school. The course focused on derivative as instantaneous rate of change, and employed a method based on examining graphs of physical situations. The number of students who could symbolise rates in non-complex situations increased dramatically, but no improvement was seen in complex items or in items which required algebraic modelling. The results point to the critical role of a developed concept of a variable in learning calculus, and are interpreted by showing the inadequacies of abstract-apart concepts as opposed to abstract-general ones.

Advances in technology, qualifications of teachers and mathematical competence of students have brought under fire the traditional place of calculus courses. There seems to be some concern about the large numbers of students taking calculus and the rote, manipulative learning that takes place (Barnes, 1988; Grimison, 1988, Steen 1988; White 1990). The value of skill based calculus courses has come under fire even more with the development of computers and calculators which perform most (if not all) of the manipulative procedures taught in such courses (Steen, 1988).

Replacing the traditional introduction to calculus via limits with more informal approaches has been widely endorsed. One such approach suggests that initially concepts should be introduced intuitively so that the introduction to differentiation is based largely on numerical and graphical explorations assisted by an electronic calculator or computer (Barnes, 1988, 1992; Orton, 1983; Tall, 1986; Wilkins, 1987).

Hiebert and Lefevre (1986) say conceptual knowledge is characterised by relationships between pieces of knowledge and, thus, often plays a part in the choice of procedure for a given mathematical problem. The degree of abstractness of a relationship can vary in that abstractness increases as knowledge becomes freed from specific contexts. A useful distinction arose from comparing their conception of "abstractness" with the everyday meaning of the word and Skemp's (1971) description of the process of abstracting. The Webster Dictionary presents the following definition for "abstract":

Abstract (ver): **To consider apart from particular instances; to form a general notion of.**

The two key words (general and apart) lead to the identification of two ways of looking at how abstract mathematical objects are used and related. Using mathematical symbols may be an abstract operation if the symbols have no concrete reference: they are "apart". The only context for the symbols is the symbols themselves. In fact, in some cases no other meaning for the symbols may exist. An example of "abstract-apart" is knowing how to manipulate algebraic symbols without having any sense of what the letters stand for. On the other hand, "abstract-general" indicates that the mathematical objects involved are seen as generalisations of a variety of situations and so can be used appropriately in different looking situations. For example, an abstract-general concept of addition would be something like "addition is the result of combining sets of like objects" and so embraces fractions, decimals, algebraic symbols, complex numbers, vectors... Such a concept could also see "addition" in cues other than add or plus. On the other hand, an extreme example of an abstract-apart concept would be only being able to see addition being required when a plus sign (or a limited number of equivalent key words) is clearly evident.

Relationships between pieces of knowledge may be a key factor for conceptual knowledge, but does not always indicate abstract-general thinking. In the abstract-apart case, relationships are formed between mathematical objects which have no conceptual base. Any relationship between the objects is "superficial" because it can only be formed on the basis of what the mathematical objects look like. In the abstract-general situation, "meaningful" relationships can be formed because the mathematical objects have meaning beyond the symbols themselves. A key factor to indicate the difference between superficial and meaningful relationships is the cues that prompt the appropriate links between pieces of information. Abstract-apart needs definite, visible cues because abstract-apart cannot generalise to different looking situations. Abstract-general, on the other hand, can link all sorts of situations regardless of how dissimilar looking they are.

Our definition of "understanding" a concept is having an abstract-general notion of it. The addition example suggests that abstract-general and abstract-apart are not a dichotomy because there are many possible levels of generality between the two extremes described. Hence, there is a continuum between the two. Tall and Vinner's (1981) concept image and concept definition can be seen as an example of the continuum. Other applications of the continuum can be found in Mitchelmore (1992).

The purpose of the present study was to investigate the effects of a course of study based on interpreting rates of change using graphs on some first year tertiary students' understanding in rates of change and derivative. Experience had suggested that many first year tertiary students would have received a skill oriented course in high school and be proficient in routine procedures, but lack many of the underlying concepts. In particular, a preliminary study (White, 1989) suggested that translating a rate of change to a derivative is a crucial link in solving rates of change items because abstract-general notions are involved. The research was not a teaching experiment; the teaching was merely providing an appropriate environment for observing any changes in student thinking.

METHOD

Sample

The sample was forty first year full time students enrolled in Mathematics I at a smaller New South Wales university. A prerequisite for the course was a satisfactory result in the final high school exam for a mathematics course which contains a large component of calculus. It should be noted that none of the students had finished in the top 10% in that final exam; the students were in most cases "average".

Instrument

After trialling a number of items with students and experts in the field of calculus, four test items were chosen. Items 1 and 2 required the direct translation of a rate of change into a derivative, with Item 1 being judged by experts as the more difficult. Items 3 and 4 involved symbolising derivatives to maximise/minimise a situation, with Item 3 being judged the more difficult.

Each of the four items were structured in four versions (a, b, c, d) so that the expected correct response for each version was basically the same. The difference between the versions was that each required successively less symbolisation. Hence, (a) required symbolising all rates to an appropriate derivative, whereas (d) had all information presented in symbolic form. As mentioned earlier, being able to symbolise rates of change was suggested as a crucial factor for understanding derivative. The four versions allowed for the symbolising aspects of responses to the items to be isolated. For clarification, the four parts for Item 1 and the part (a)s for the other three items are:

Item 1

(a) In the special theory of relativity, the mass of a particle moving at velocity v is given by:

$m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ where c = speed of light and m0 is the rest mass.

At what rate is the mass changing when the velocity is $\frac{1}{2}$c and the acceleration is 0.01c per sec?

(b) In the special theory of relativity, the mass of a particle moving at velocity v is given by:

$m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ where c = speed of light and m0 is the rest mass.

Find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and the acceleration is 0.01c per sec.

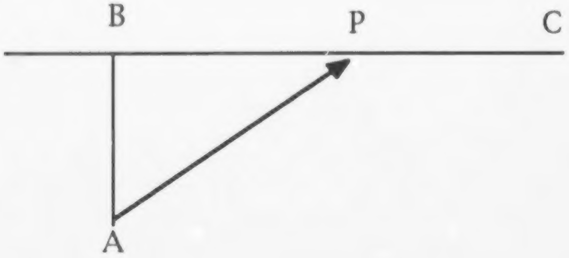
(c) Given $m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and $\frac{dv}{dt} = 0.01c$.

(d) Given $m = \frac{m0}{\sqrt{1 - \frac{v2}{c2}}}$ and $\frac{dm}{dt} = \frac{dm}{dv} \cdot \frac{dv}{dt}$, find $\frac{dm}{dt}$ when $v = \frac{1}{2}c$ and $\frac{dv}{dt} = 0.01c$.

Item 2

(a) If the edge of a contracting cube is decreasing at a rate of 2 centimetres per minute, at what rate is the volume contracting when the volume of the cube is 64 cubic centimetres?

Item 3



(a) The diagram shows a straight road BC running due East. A four wheel drive is in open country at A, 3 km due South of B. It must reach C, 9 km due East of B, as quickly as possible. The driver can travel at 80 kph in open country and 100 kph on the road. Assuming the car proceeds through open country to some point P, and then along the road, what is the distance of P from B so that the journey to C takes the shortest time possible?

Item 4

(a) Find the area of the largest rectangle with its upper vertices on the curve y = 12 - x 2 and lower base on the x-axis.

Teaching Sequence

The basis for the teaching material was Swan (1989) and followed the approach of Barnes (1992). Initially, rates of change were investigated using graphs of physical situations. This led to the secant being seen as representing an average rate of change, the tangent an instantaneous rate of change. Derivative was then developed from the latter. Limits were only considered informally.

Procedure

Four data collections occurred - before, during, immediately after and then six weeks after the teaching sequence. The forty students were arranged into four approximately parallel groups of ten, based on their performance in a previous mathematics exam. The students were unaware of the groupings. Four tests were constructed. Each test included four questions - a different part from each item. Hence, each part of each item occurred on one and only test and each test had one part (a), one part (b) ... These tests were administered in a cyclic fashion to each of the groups over the four data collections so that students did different items each time, but data was still available for the same pool of items in all collections.

In addition, the same sixteen students (four of about equal performance from each group) were interviewed each time within three days of the written data collection. These interviews served to clarify and expand on written responses so that the student's understanding could be better identified.

RESULTS

The number correct at each data collection

Item	Collection 1	2	3	4	Total
1	2	3	8	7	20
2	8	13	26	22	69
3	4	6	14	12	36
4	10	11	16	15	52
Total	24	33	64	56	177

Recalling that scores are out of forty, results in collection 1 show that students in general could not correctly respond to any item. The improvement in the number of correct responses in collection 3 and 4 is substantial, but performance still only exceed 50% in Item 2. The large number who improved suggests the teaching was a positive factor. The equally large number who did not points to inhibiting factors, many of which were able to be identified by a detailed analysis of the responses.

Rates of Change - Items 1 and 2.

In collection 1 only 4/40 students observed that rate of change meant a derivative was required. As well, only 4/20 showed any recognition of the need to use the chain rule in 1(c) and 2(c) where the required derivatives were cued symbolically, but the actual chain rule was not stated. The pattern changed in the collections 3 and 4 with virtually all students at least attempting to translate rates into derivatives, and 34/40 correctly stating the chain rule in 1(c) and 2(c). However, in the last two collections, problems associated with students' **concept of a variable** arose so that, even though derivatives and chain rules were introduced, the number of correct solutions was still lower than might have been expected.

In collections 3 and 4, 36/40 students were able to correctly symbolise derivatives in Item 2, but not in the more complex Item 1 (7/40). In Item 1, 7/8 students who successfully symbolised derivatives gave fully correct responses; whereas for Item 2, only 22/36 of the responses which correctly symbolised the derivatives were actually correct.

Students' under developed concept of a variable manifested itself differently in the two items. Analysis showed that the predominant error in Item 1 was that students focused on the visible symbols, looking for something to fit known manipulation rules, and as a result immediately substituted $\frac{1}{2}c$ for v. Such an attitude was named a "manipulation focus". In Item 2, the main cause for not correctly finishing the item when it had been correctly set up was that students saw V = x 3 and V = 64 as separate cases and either gave two answers or were confused and stopped.

Maximisation - Items 3 and 4.

There were few changes across the four data collections. The main obstacle was that students generally could not model the situation given in part (a) or (b) using algebra (14/80 and 7/80 for Items 3 and 4 respectively). The lower number for the easier Item 4 at first appears odd. The reason was a form of the manipulation focus - named the "x, y syndrome" - where 56/120 students seized on the equation y = 12 - x 2 and found $\frac{dy}{dx}$ with no thought for modelling the rectangle.

In both Items 3 and 4, all responses which showed correct modelling to part (a) were fully correct (8/8). Admittedly the numbers were small, which of itself may be significant. Correct modelling in part (b) also coincided with a high success rate, but not at the 100% level (10/14).

DISCUSSION

What has been described as the manipulation focus is an indicator of an abstract-apart concept of a variable. Students exhibiting the manipulation focus show they can apply manipulation rules, but have no sense of deciding when such rules are appropriate. Their use of the rules is superficial being based solely on what the symbols look like. The symbols are seen apart from any meaning they might have. Immediately substituting $\frac{1}{2}c$ for v and the confusion between V = x 3 and V = 64 indicates how a particular value for the expression is seen apart from the general expression itself. The x, y syndrome shows how the rule for maximising is equated with the visible symbols used when the rule is first learnt rather than with what they represent. As such, the rule is apart from the situations it generalises.

A majority of students who identified a rate as a derivative, correctly symbolised \f(dV,dt) and $\frac{dx}{dt}$ in Item 2 and correctly wrote down the chain rule, nevertheless made manipulation focus errors. This suggests that these three relationships are superficial operations which can be achieved with thinking at the abstract-apart end of the continuum. On the other hand, being able to correctly symbolise rates of change to the appropriate derivative in structurally complex items (like Item 1) was only achieved by a few students who did not exhibit the manipulation focus. Symbolising in structurally complex situations would seem to indicate meaningful relationships are being used. Such a suggestion makes sense because in more complex situations, there is more than one variable to choose as the dependent and/or the independent variable. Some sense has to be attached to the letters involved; superficial cues alone do not suffice. For example, relating given variables in Items 3 and 4 requires meaningful relationships to be formed because the items are reasonably structurally complex. In contrast, manipulation focus students had no difficulty modelling the volume of the cube by V = x 3 in the less complex Item 2.

Since few students were able to define appropriate variables in 3(a) and 4(a), and that those who did were always completely correct, there is a strong suggestion that defining and relating variables involves forming meaningful relationships at a higher level than does symbolising, and relating already defined variables. Symbolising in complex situations requires putting appropriate variables to the rates which are already cued; defining variables in a modelling situation indicates the solver is making choices with some plan in mind and the plan has not been laid out in the cues. In this way, the evidence suggests that defining variables indicates relationships at a higher level of generalisation than the pieces of knowledge they connect. For example, in 3(a), information about speed and distance must be pulled together to form a relationship which leads to minimum time. Alone, the link between speed, distance and time is no more general than the cued information. The introduction of Pythagoras' Theorem is also no more general that the cue of the triangle leading to it. However, choosing an appropriate distance to define as the independent variable so that the speed-distance-time relationship and Pythagoras' Theorem can be employed suggests those relationships themselves have been transcended. The executive control required indicates that the solver must reflect on a number of relationships and that seeing how to combine them indicates a relationship at a higher level of abstractness.

IMPLICATIONS

In general, the results present a fairly pessimistic view of where average students are mathematically when they leave high school. Severe doubts have been raised about their ability to learn calculus because of deficiencies in their concept of a variable. As it stands, calculus teaching in school appears to be useless, with students being unable to form even superficial relationships soon after. Tertiary calculus seems to develop the superficial relationships, but not the meaningful ones and as such is worse than useless. Clearly, you cannot successfully teach a tertiary calculus course by teaching calculus only.

There would seem to be a need for a radical change in the way we all teach mathematics. The plot - that mathematics consists of generalisations - seems to have been lost (or never found?). As long as the emphasis is on reproducing procedures in antiseptic situations which depend on key cues that always look the same, the number of students who really understand and appreciate mathematics will continue to be limited to a very elite group. In particular, algebra teaching needs to focus on abstract-general notions of a variable by investigating situations where letters have more than a symbolic context.

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WHAT DO YOUNG CHILDREN KNOW ABOUT LINEAR MEASUREMENT AND IS IT RELATED TO CONSERVATION?

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Macquarie University

For the past twenty years Piagetian theory has dominated mathematics and science curricula. In primary schools, teaching - learning sequences and activities in mathematics have traditionally been built around the ages and stages promulgated by Piaget as being appropriate to the particular performance level of cognition a child might have attained. Conservation - according to Piaget, is the precondition for all measurement. Consequently, exposure to 'real' measurement has usually been delayed until Years 2 or 3 in primary schools. Recent research indicates that children have quite well developed notions about measurement prior to their ability to conserve. This paper examines procedures that were developed to successfully test the ability of young children to measure length. Findings from a study carried out between 1985 and 1987 indicate that the ability to conserve is linked to the ability to measure, the knowledge of particular dimensional adjectives and experience with concrete material. The new NSW K - 6 Curriculum in Mathematics, released in 1989, reflects the findings of this research and has changed the theoretical basis of the teaching - learning sequences in the curriculum.

RESEARCH BASE FOR THE STUDY OF LINEAR MEASUREMENT

Research has indicated that children possess a broader measurement concept at an earlier age than that postulated by Piaget and continuously maintained by the neo-Piagetians.

- Philp,Kelly and Birch (Unpublished)
  Established a hierarchical order for the acquisition of dimensional terms (length, substance, weight). Three relevant conservation structures have a sine qua nonrelationship with the relevant scales of terms.

- Kelly, Philp and Lewis (1982)
  Investigated horizontal decalages, use of dimensional adjectives and a concept of measurement. They found that a concept of measurement appeared before children were able to conserve length.

- Carpenter (1971, 1975, 1976, 1981)
  Children appeared to be able to measure at a younger age than that predicted by Piaget. Measurement operations used appropriately may facilitate conservation judgements.

- Hiebert (1981, 1984)
  Sought to identify specific developmental abilities which are required to learn mathematical concepts - particularly measurement.

- Boulton - Lewis (1987)
  Examined the length measuring sequence established in literature in light of recent theories of cognition concerning information processing in children.

RELEVANCE OF THE STUDY TO AUSTRALIAN EDUCATION IN EARLY MATHEMATICS

Previous research has concentrated on what constitutes a valid concept of measurement and its distinction from measurement skills. This study seeks to establish whether the variables:

- concept of measurement
- conservation of length
- use of selected dimensional adjectives

are developmentally linked and whether the conceptions of teachers may influence the teaching of measurement in the lower primary grades. At present it appears that little research relating the results of clinical studies involving measurement materials and its place in the Mathematics syllabus exists in Australia.

MEASUREMENT AND CONSERVATION OF LENGTH STUDY

Sample

126 subjects in grades 1, 3 and 5 were selected from 5 schools in Sydney representing inner-city, suburban and outer-suburban areas. The age range was 5.11 - 11.4. The sample included children of mixed ethnicity and socio-cultural backgrounds although these variables were not specific delimiters of the study. The sample was structured to include children in the following categories:

1. Prior to the onset of conservation
2. Without conservation ability
3. With conservation ability

Subjects were selected at random by teachers.

Tests

All tests involved the use of concrete materials and objects easily manipulated by the children included in the study. Materials included cuisenaire rods stained black, pieces of dowel and ribbon and small tables of different dimensions. The six tests included:

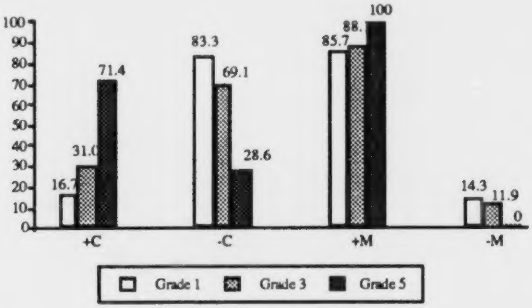
- Two standard conservation of length tests
- Two concept of measurement tests
- Two dimensional adjectives tests

Results

Several types of statistical analyses were used in the study. Principally, a non-parametric analysis of variance based on a revised version of a program of multi-level chi-square analysis developed by Mitchell and Kelly of Macquarie University (Kelly, 1987) and a probabilistic model for the validation of hierarchies present in dichotomous data sets developed by Bailey (CONNECT:1988) of Sydney University.

The Relationship between Measurement of Length and Conservation of Length

Figures 1 and 2 examine the ability/inability to conserve length and measure length within the sample.



+C - Able to conserve
-C - Unable to conserve
+M - Able to measure
-M - Unable to measure

Figure 1: Conservation and measurement shown by grade.

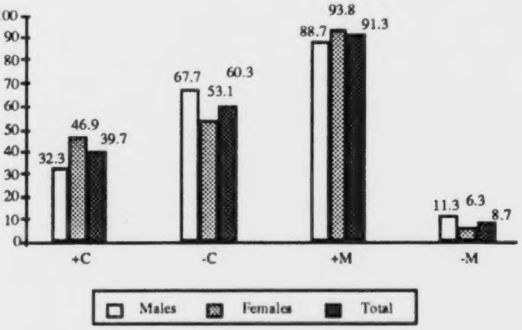


Figure 2: Conservation and measurement shown by sex.

Figure 1 shows that a greater proportion of subjects were able to measure than conserve. The results indicate that the ability to conserve is clearly related to age, as shown by the step-like progression in the +C column of Figure 1. This concurs with the Piagetian position. The ability to measure appears to be distinctly different. Figure 1 indicates a less pronounced step-like progression between grades.

Figure 2 examines the differences between sexes in the ability to conserve and measure. In keeping with research (Kelly et al., 1982; Boulton-Lewis, 1987), sex appears to have more of a significant effect on the ability to conserve than on the ability to measure.The four conditions which ensue as a result of the relationship between conservation and measurement were considered according to age and grade. The results are shown in Figure 3.

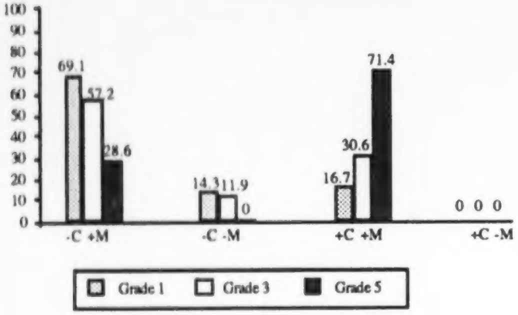


Figure 3: Four Conditions of Conservation and Measurement Shown by Grade

The -C +M column in Figure 3 represents those subjects who failed to conserve but were able to measure, the +C -M column represents those subjects who were able to conserve but failed to measure. Together the columns indicate that no subject was able to conserve without also succeeding on the measurement tasks.

In order to establish if the relationship between the variables was significant, the data was subjected to single and multi-level chi-square analysis. Table 1 summarises the results of the multi-level analysis.

Table 1: Summary of Significant Chi-Square Relations found in a Multi-Level analysis between the variables Conservation of Length and Measurement of Length.

Variables	χ^2	DF	P	C
First Order				
Grade x Conservation	36.55	4	$p < .001$	0.47
Grade x Measurement	8.27	4	$.05 < p < .01$	0.24
Conservation x Measurement	12.05	4	$.01 < p < .02$	0.30
Second Order				
Grade x Age x Conservation	36.55	8	$p < .001$	0.47
Third Order				
Grade x Age x Con. x Meas.	26.55	16	$.02 < p < .05$	0.41

Given the results obtained in the cell -C +M, and that the dependent variables conservation of length and measurement of length were reasonably significant at the first and third order levels of the chi-square analysis, the posited hypothesis that conservation of length is not possible without first being able to measure was accepted.

The Ability of Children to Measure

Figure 4 and Table 2 show that most subjects were able to measure. Significantly, a high proportion of age categories 1 and 2 containing the youngest subjects, were able to measure.

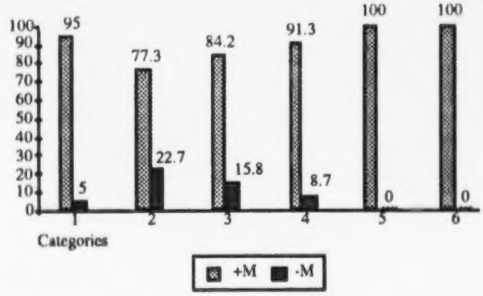


Figure 4: Measurement shown by age categorisation

Of critical importance in the consideration of measurement as a precondition for conservation of length is the ability of categories 1 and 2 to measure, since these 42 subjects are younger in age than that posited by Piaget as being capable of measuring meaningfully.

Table 2: Ability to Measure Shown by Age Categorisation and Test Type

Age Category	Sex	Rod Test Only	Table Test Only	Passed Both	Failed Both	Total
1	M	0	0	11	1	12
(5.11 - 6.6 years)	F	0	0	8	0	8
2	M	1	2	5	0	8
(6.7 - 7.2 years)	F	2	0	12	0	14
3	M	2	0	6	0	8
(7.11 - 8.5 years)	F	0	1	10	0	11
4	M	1	0	12	0	13
(8.6 - 8.11 years)	F	0	1	9	0	10
5	M	0	0	12	0	12
(9.11 - 10.8 years)	F	0	0	18	0	18
6	M	0	0	9	0	9
(10.9 - 11.4 years)	F	0	0	3	0	3
All Male		4	2	55	1	62
All Female		2	2	60	0	64
All Subjects		6	4	115	1	126

Table 2 describes in detail the ability of subjects to measure, including those subjects deemed transitional measurers, that is, those subjects who only passed one of the two measurement tests. One subject failed both tests and 10 subjects were transitional measurers. In category 1, 95% of subjects successfully measured, with 1 child failing both tests. In category 2, 77% of subjects successfully measured, 14% passed the Rod test only, 9% passed the Table test only and no-one failed both tests. The 42 subjects in categories 1 and 2 constitute the grade 1 sample aged between 5.11 and 7.2. 86% of grade 1 were able to measure, 12% were transitional measurers and 2% failed to measure.

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The measurement tests required children to justify their length decisions with appropriate strategies supported by reasons for their choice. The tests were a strong, positive indication that subjects were able to measure using a strategy with perceptual or numerical clues and justify their actions with a reason. Figure 5 represents the reasons or strategies given by subjects to demonstrate the ways unit blocks would be useful to measure the lengths of the miniature tables (Measurement Test 2).

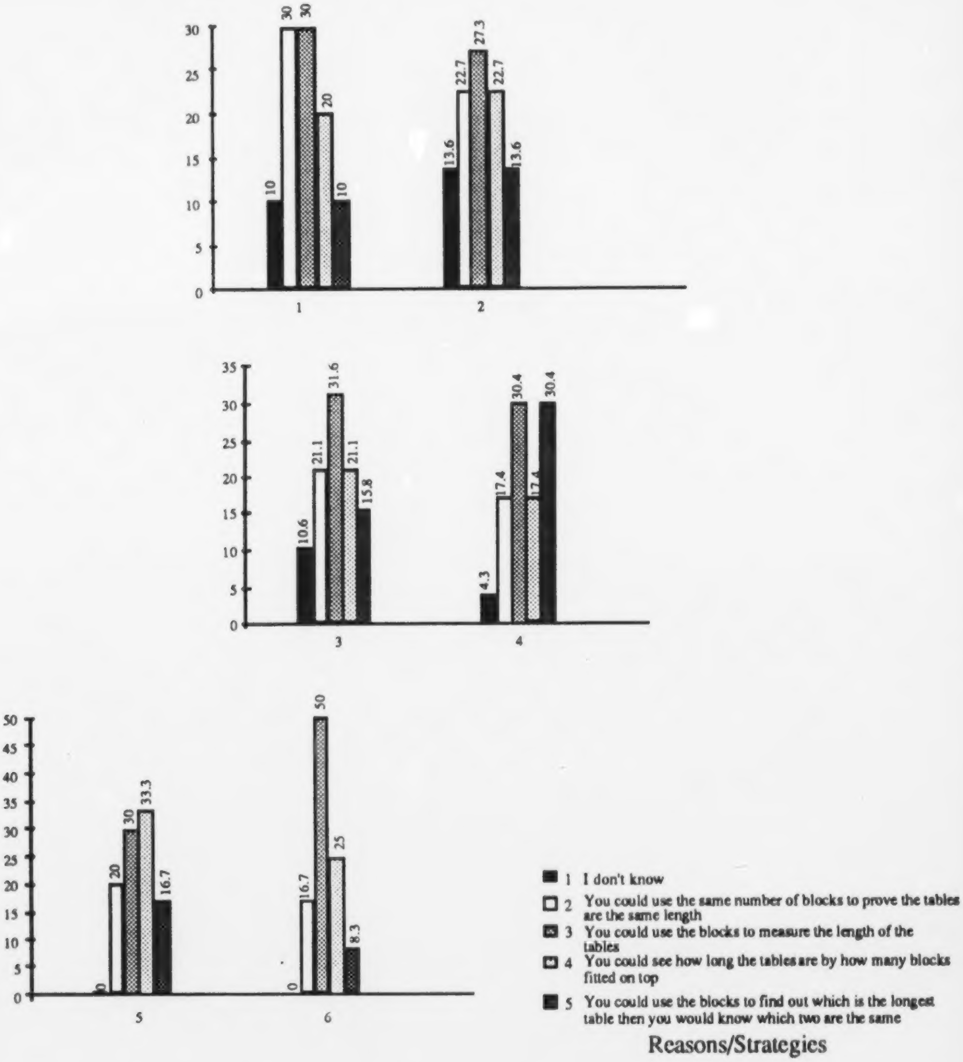

Figure 5: Reasons or Strategies Nominated by Subjects to Use Unit Blocks to Compare the Lengths of Two Tables Shown by Age Categorisation

Strategy 3, 'You could use the blocks to measure the length of the tables', gained the highest percentage of subjects in all age categories. Other strategies gained varied responses across the age categories. The measurement tests showed that subjects aged between 5.11 and 11.4 were able to measure using strategies with perceptual or numerical cues and justify their actions with a reason. The nature of the strategies given by subjects indicates that children younger than age 8 are able to measure meaningfully.

Conservation of Length, Measurement and the Use of Dimensional Adjectives

In order to establish the relationship between the selected dimensional adjectives, conservation of length and measurement, a program devised by Bailey of Sydney University based on the Dayton and Macready (1976) probabilistic model for the validation of hierarchies was used. A critical value for the sequencing statistic was established at 2.88, representing a p=.001 level of significance for 1 tailed tests. Figure 6 shows the sequence digraph for the probability matrix.

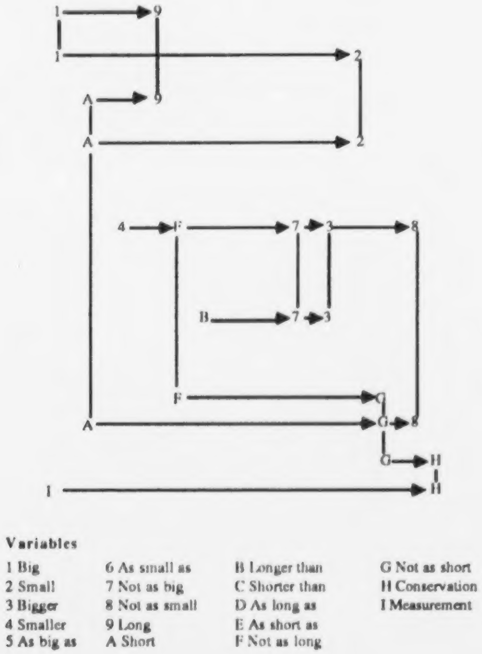
1
9
1
2
A
9
A
2
4
F
7
3
8
B
7
3
F
G
A
G
8
G
H
I
H
Variables
1 Big
2 Small
3 Bigger
4 Smaller
5 As big as
6 As small as
7 Not as big
8 Not as small
9 Long
A Short
B Longer than
C Shorter than
D As long as
E As short as
F Not as long
G Not as short
H Conservation
I Measurement

Figure 6: Sequence Digraph for the Probability Matrix of the Relationships between the Variables, Measurement, Selected Dimensional Adjectives and Conservation of Length.

The sequence graphically drawn in Figure 6 represents the linkages found to exist between the variables expressed in the three tasks. Figure 6 shows the sequence of difficulty between the easiest item *'big'*, to the hardest item, *conservation*. Two patterns are revealed in the data set.

1. Most of the selected dimensional adjectives precede conservation of length and are linked in sequence.

2. Measurement precedes and is linked to conservation of length but is not linked to the selected dimensional adjectives.

Table 3 shows the levels of significance found to exist between variables in the probabilistic matrix.

Table 3: Levels of Significance for 1 Tailed Z Scores for Single Relations between variables in the Probabilistic Matrix.

Level of significance														
	.05	.03	.02	.01	.006	.005	.003	.002	.001	.0009	.0007	0005	.0002	.0001
Variable Pairs	A+B	9+5	2+H	6+H	F+E	1 - A	D+H	C+B	F+7	A+G	F+G	3+8	G+H	1+9
	D+2	C+H		B+G	5+H	C+F	E+H	F+6	F+3			G+8	B+8	A+9
				F+B		7+G	4 - B		7+3			4+7	1+2	D+5
				F+D		F+8			1+8				A+2	D+E
									I+H				B+7	D+6
													4+F	5+E
														5+6
														E+6
														4+3
														4+8
														7+8
Total Pairs	2	2	1	4	2	4	3	2	5	1	1	3	6	11

Variable List

1 Big
2 Small
3 Bigger
4 Smaller
5 As big as
6 As small as
7 Not as big
8 Not as small
9 Long
A Short
B Longer than
C Shorter than
D As long as
E As short as
F Not as long
G Not as short
H Conservation
I Measurement

The ability to measure was found to be a polar variable to the ability to conserve in terms of facility. The strength of the association found between the variables would strongly suggest that measurement be considered as a precondition for conservation. An identifiable order of hierarchical structure was found to exist between the ability to measure length and the ability to conserve length between the knowledge of select adjectives of extent and the ability to conserve length. Within the hierarchical structure both measurement and knowledge of select adjectives of extent preceded conservation of length.

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THE DESIGN OF A HYPERCARD STACK ON INTRODUCTORY CALCULUS

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INTRODUCTION

The purpose of this paper is to review several principles used to design a Hypercard stack for the learning of introductory calculus and the research questions related to the stack. The review, development and generation of research questions, being important phases of any research project, must be dealt with carefully to ensure that the research will produce significant and valid findings. As this project is still being developed, no empirical data will be reported here; feedback from the audience is most welcome to enhance the quality of this project.

THE PROBLEMS

Calculus is an important subject for further studies in the physical sciences, engineering and social sciences courses. Extensive research has identified a number of obstacles and misconceptions in the learning of introductory calculus. These include difficulties with concepts such as limits, continuity, tangents, differentiability and their notations (Dreyfus & Eisenberg, 1983; Orton, 1983; Tall, 1990; Vinner, 1987; Wong, 1984). Although many educators have actively promoted the use of graphing packages to teach function and calculus concepts, this enthusiasm needs to be further investigated because visual thinking demands a higher level of mental activity which students are often reluctant to engage in (Eisenberg & Dreyfus, 1990). A plausible but untested explanation is that students have not developed a repertoire of learning strategies to cope with various types of learning tasks beyond the ubiquitous practice strategy. This Hypercard stack is designed to address some of these problems. It is to be used in three different ways:

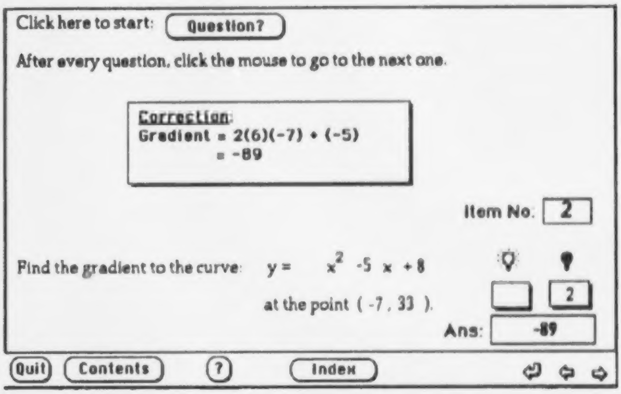
- as a self-paced learning package to supplement text materials;
- to encourage students' active participation in a group learning situation;
- to be used as a demonstration by the teacher who will model how mathematics is be learned and how learning tasks (especially investigations) are to be carried out.

THE PRESENTATION

This paper is presented as a Hypercard stack and addresses four main aspects: feedback, different ways of doing things, graphing and user control in the Hypercard environment. The more important cards are shown below.

FEEDBACK

The *National Statement* states that, "Mathematical learning is likely to be enhanced by feedback" (p. 19). In traditional CAI, the computer evaluates the student's response and shows whether it is right or wrong, accompanied by some motivation such as an interesting graphics or a sound effect. This approach is illustrated by this card which shows a bright bulb for a correct answer and dark one for a wrong answer. A tally of the number of correct and wrong answers is also kept.



Click here to start:

After every question, click the mouse to go to the next one.

Correction:
Gradient = 2(6)(-7) + (-5)
= -89

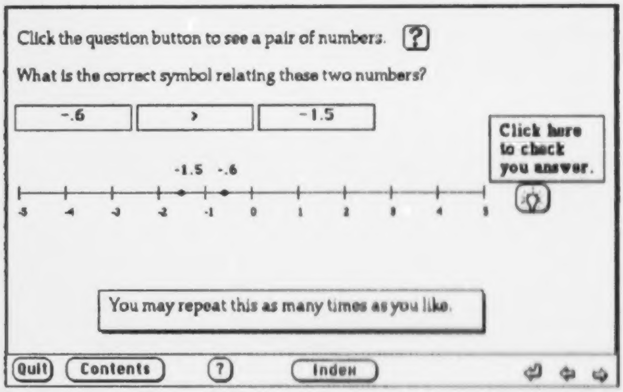
Item No. 2

Find the gradient to the curve: $y = x^2 - 5x + 8$
at the point (-7 , 33).

2

Ans: -89

Another approach is that the computer outputs the correct answers without checking the students' responses. Additional information may be provided to enhance learning. This approach encourages the students to use the computer output to check their own answers. This is similar to checking answers at the back of the book and assumes that the students will take a greater responsibility of their own learning. The students also decide how much practice they need to do to master a particular skill. This approach is illustrated by this card.



Click the question button to see a pair of numbers.

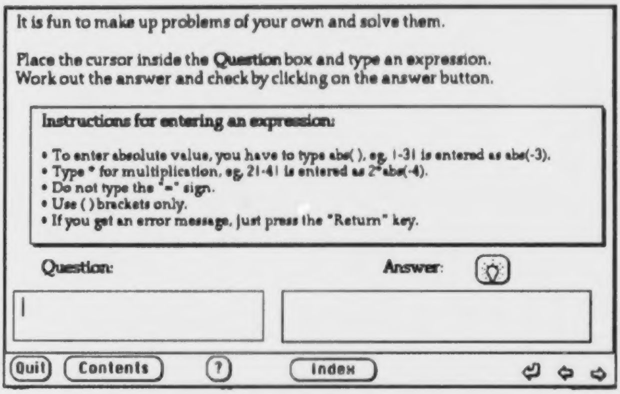
What is the correct symbol relating these two numbers?

-.6 | > | -1.5

You may repeat this as many times as you like.

The problem is to identify the correct symbol relating two numbers. When the student clicks on the answer button, the correct symbol is shown and a number line is given to consolidate the meaning of the symbol.

The third approach is to encourage students to make up their own questions and to use the computer to find the answers to these questions. In this card, the student make up computational questions involving different functions and operations. If this is done carefully the student can discover many properties about the functions and the precedence of various operations.



It is fun to make up problems of your own and solve them.

Place the cursor inside the **Question** box and type an expression.
Work out the answer and check by clicking on the answer button.

Instructions for entering an expression:

- To enter absolute value, you have to type abs( ), eg. |-3| is entered as abs(-3).
- Type * for multiplication, eg. 2|-4| is entered as 2*abs(-4).
- Do not type the "=" sign.
- Use ( ) brackets only.
- If you get an error message, just press the "Return" key.

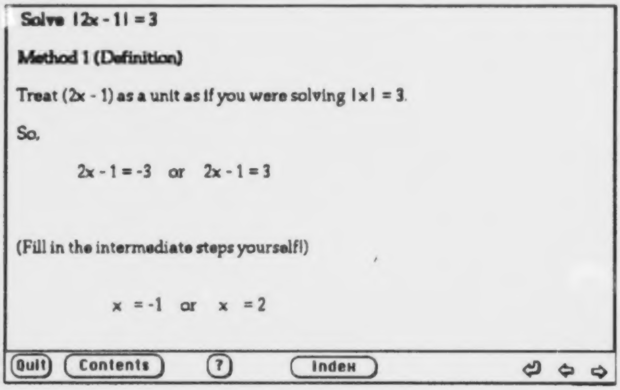
Given these different types of feedback, the research questions are to determine:

- students' preference for these types of feedback;
- the effects of feedback on learning outcomes and beliefs about learning;
- the types of questions made up by students and what they say about their mathematical knowledge.

DIFFERENT WAYS OF DOING THINGS

Mathematics educators emphasise that students should know different ways of solving the same problem. This will encounter a common belief that mathematics problem always has only one correct answer. This approach might not be feasible in actual teaching if the teacher feels the pressure to cover a lot of content. Research is also limited in determining how knowing different methods may affect problem solving behaviours and outcomes. The following cards show how several methods are used to solve the same absolute value equation. Exposing students to a combination of algebraic and graphical methods will enhance their ability to solve these equations in a flexible way and encourage them to represent mathematical knowledge in different forms.

Method 1:



Solve $|2x - 1| = 3$

Method 1 (Definition)

Treat $(2x - 1)$ as a unit as if you were solving $|x| = 3$.

So,

$2x - 1 = -3$ or $2x - 1 = 3$

(Fill in the intermediate steps yourself!)

$x = -1$ or $x = 2$

Quit
Contents
?
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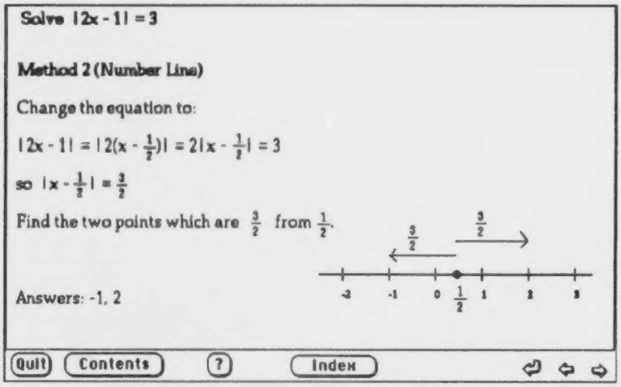
Method 2:



Solve $|2x - 1| = 3$

Method 2 (Number Line)

Change the equation to:

$|2x - 1| = |2(x - \frac{1}{2})| = 2|x - \frac{1}{2}| = 3$

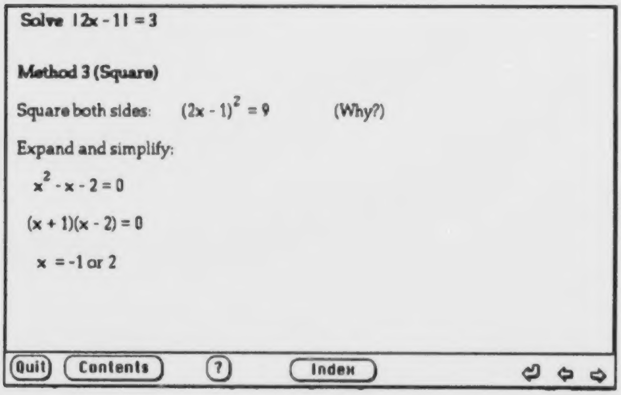
so $|x - \frac{1}{2}| = \frac{3}{2}$

Find the two points which are $\frac{3}{2}$ from $\frac{1}{2}$.

Answers: -1, 2

$\frac{3}{2}$
$\frac{3}{2}$
-2
-1
0
$\frac{1}{2}$
1
2
3
Quit
Contents
?
Index

Method 3:



Solve $|2x - 1| = 3$

Method 3 (Square)

Square both sides: $(2x - 1)^2 = 9$ (Why?)

Expand and simplify:

$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ or 2

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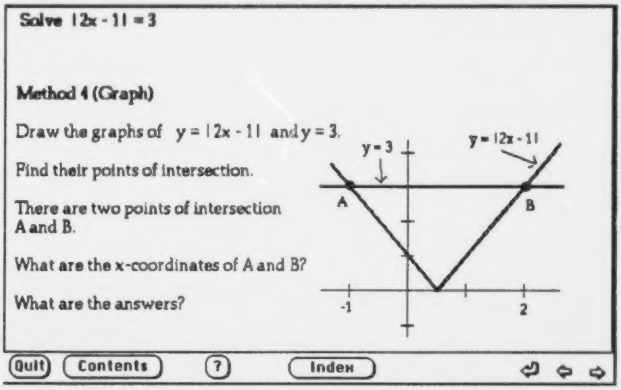
Method 4:



Solve |2x - 1| = 3

Method 4 (Graph)

Draw the graphs of y = |2x - 1| and y = 3.

Find their points of intersection.

There are two points of intersection A and B.

What are the x-coordinates of A and B?

What are the answers?

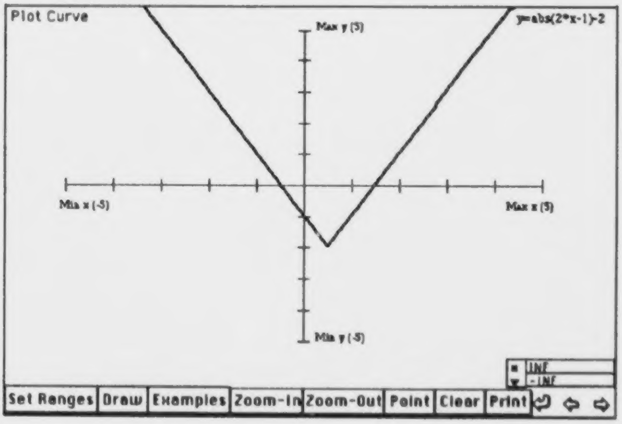
y = 3
y = |2x - 1|
A
B
-1
2
Quit
Contents
?
Index

The research question is to determine:

- students' preference for these approaches of solving the same problem; in particular, the preference for an analytical, visual, or a versatile mode of processing;
- the effects of multiple methods on learning outcomes and beliefs about learning.

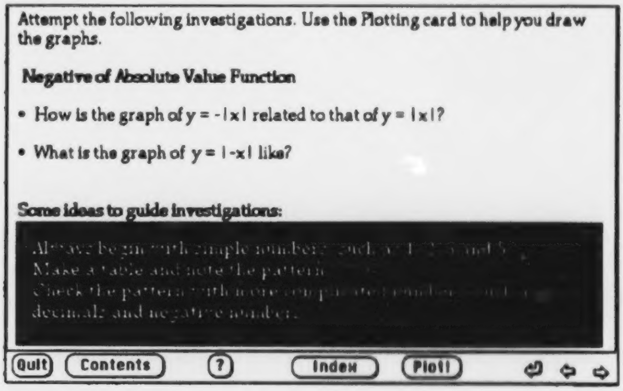
GRAPHING

There is considerable interest on visualisation in learning mathematics, in particular with the use of graphing software and graphic calculator. However, it is not realistic to expect students to be able to use these tools without adequate guidance. Since these graphing tools are for applications rather than for teaching, the guidance to use them as instructional tool must be provided for by the teachers, for examples in the form of worksheets. This stack attempts to cover both objectives by providing a plotting card together with ideas for investigations. The plotting card has the usual features of zoom-in and zoom-out, though it is a bit slow due to the speed of Hypercard processing.



Plot Curve
y=abs(2*x-1)-2
Max y (5)
Min x (-5)
Max x (5)
Min y (-5)
x INF
y -INF
Set Ranges
Draw
Examples
Zoom-In
Zoom-Out
Point
Clear
Print

Two examples of different guidelines for graphing are shown below.



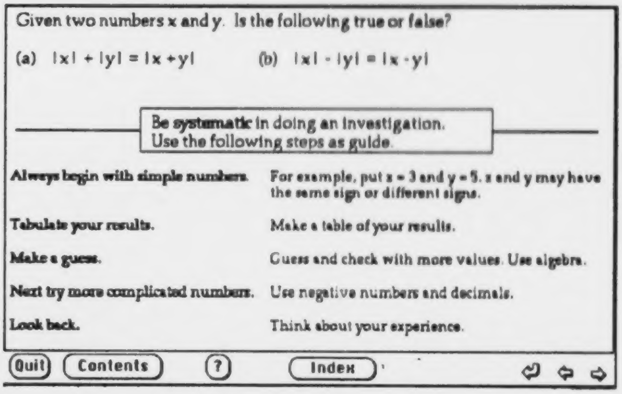
Attempt the following investigations. Use the Plotting card to help you draw the graphs.

Negative of Absolute Value Function

- How is the graph of y = - | x | related to that of y = | x |?
- What is the graph of y = | -x | like?

Some ideas to guide investigations:

Always begin with simple numbers such as [illegible]
Make a table and note the pattern.
Check the pattern with more complicated numbers such as decimals and negative numbers.



Given two numbers x and y. Is the following true or false?

(a) | x | + | y | = | x + y | (b) | x | - | y | = | x - y |

Be **systematic** in doing an investigation.
Use the following steps as guide.

Always begin with simple numbers.	For example, put x = 3 and y = 5. x and y may have the same sign or different signs.
Tabulate your results.	Make a table of your results.
Make a guess.	Guess and check with more values. Use algebra.
Next try more complicated numbers.	Use negative numbers and decimals.
Look back.	Think about your experience.

The research question is to determine:

- whether students find these guidelines useful for doing investigations or not;
- the ways in which they use the plotting card as a tool to do their investigations;
- the effects of graphing on learning outcomes and beliefs about learning.

USER CONTROL

In computer-based instruction, it is often claimed that user control can improve learning although "[A]sking students to make too many decisions may actually interfere with the flow learning" (Steinberg, 1991, p.112). This is likely to affect low ability students or those who have poor study habits. However, in the Hypercard environment, "The user, not the computer, initiates and controls all actions" (*Hypercard Stack Design Guidelines*, 1989,

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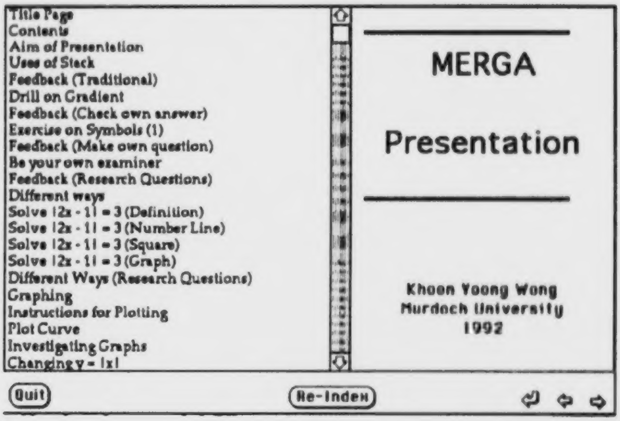
p. 180). The following features provide a consistent and user-friendly way of navigating through the stack:

- a set of clearly defined buttons placed at consistent locations;
- a Contents card to give an overview of what the stack is about;
- an Index card (see below) so that the user can access individual cards as needed;
- a GLOSSARY stack that allows the user to look up more information.



Title Page
Contents
Aim of Presentation
Uses of Stack
Feedback (Traditional)
Drill on Gradient
Feedback (Check own answer)
Exercise on Symbols (1)
Feedback (Make own question)
Be your own examiner
Feedback (Research Questions)
Different ways
Solve $|2x - 1| = 3$ (Definition)
Solve $|2x - 1| = 3$ (Number Line)
Solve $|2x - 1| = 3$ (Square)
Solve $|2x - 1| = 3$ (Graph)
Different Ways (Research Questions)
Graphing
Instructions for Plotting
Plot Curve
Investigating Graphs
Changing $y = |x|$

MERGA

Presentation

Khoon Yoong Wong
Murdoch University
1992

The research question is to determine:

- what kind of user control is preferred by students of different abilities;
- the effects of control on learning outcomes and beliefs about learning.

In order to obtain information on the ways the user moves through the stack, the study will make use of the ScreenRecorder software to record the Macintosh screen session in real time. This record can be played back to determine the learning path taken by the user to gain a better understanding of the learning process. However this method will collect an enormous amount of data and the most optimal way to analyse the data will be addressed at a later stage.

CONCLUSION

High expectations have been placed on using educational technology to facilitate teaching and learning. When the design of computer-based package is based on appropriate learning theories and design principles, there is great promise of realising these expectations. However, careful research is required to validate these possibilities. Hopefully some fruitful results will be obtained from this project in the near future.

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THE CONSTRUCTION OF SPATIAL MEANING AND SOCIAL DISADVANTAGE

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Deakin University, Geelong

One of the biggest problems facing mathematics educators is the non-random failure of minority groups in mathematics. Two major discourses address this problem, psychological discourses which focus attention on the individual, and sociological discourses which focus at the socio-political level of analysis. They are essentially disjoint sets which offer different explanations for the phenomenon but can not adequately account for each other. These dominant discourses which prevail in mathematics education are discussed along with their respective limitations in understanding the meanings that children construct in the mathematics classroom and the construction of social disadvantage.

For the purposes of this discussion I will focus on two major discourses that operate in mathematics education. The first is that which I refer to as the psychological discourses, and perhaps the more dominant in the field. The primary focus within this field is on the individual construction of meaning. Of particular concern is the intersecting discourses of developmental and cognitive psychology. In contrast the second discourse which I refer to as sociological, is that which focuses on the wider social structures and how these impact upon mathematics education. This latter field is often critical of the role that mathematics education plays in society, and runs the risk of being rejected by many "conservative" mathematics educators.

Through this paper, I hope to challenge these research discourses and suggest that it may be more appropriate to examine the practices which constitute the outcomes that enable such discourses to be constructed. What I am proposing is a research discourse that will enable the examination of the social construction of meaning, both at the level of the individual and the social, to examine the mutual constitution of these factors in the construction of meaning and social disadvantage in the mathematics classroom. The proposed research will be attempting to look at the practices in schools that legitimate success in mathematics. The perspective that will be taken in this research will be that failure in primary school mathematics is essentially a product of the practices that have been institutionalised through schools.

PSYCHOLOGICAL DISCOURSES

Perhaps the most influential writings in epistemology are those of Piaget, or those which have been influenced by his writings, such as the van Hieles' theory of geometric thought. While recognising the flaws in much of his work, Piagetian based theories have had a major and powerful impact on the conceptualisation of education and the child. Of particular relevance is the adaptation of his work in geometric thought as apparent in the van Hieles' theory.

Psychological discourses focus attention on the individual construction of meaning and then attempt to develop theories about the perceived consistency in the progression of mathematical cognition. Such theories have had an immense impact on the way in which child development is constructed within the Western world, so much so it is often difficult to perceive as otherwise. The notion that children go through certain characteristic phases in their progression towards adulthood is firmly entrenched in our world view and hence in the language we use to describe children. Foucault has adopted the term ***regimes of truth*** to refer to the wide acceptance of phenomena within a society so that they are seen to be truths for those people. Yet, perhaps this is all they are, socially constructed phenomena that have become reified within the society so as to constitute a reality for that group of people.

These theories have permeated the curriculum in such a way that it is difficult to decipher what is seen to be reality and that which is socially constructed. For example, in the infant areas of school, it almost goes without saying, that children need to use concrete materials if they are to come to understand number and operations. Similarly, the transition from primary school to high school for Australian children, coincides with the (supposed) emergence of formal or abstract thought. The notion that children of this age are able, or at least expected to be able to perform abstract reasoning is reflected in the changes in the curriculum whereby children in Year 7 are introduced formally to Algebra, a subject which can be seen to symbolise abstract thought. Yet, I would ask is it the case that the curriculum has been organised in such a way as to coincide with these critical periods in the development of cognition, or is it the case that these environments give way to the development of these stages of cognition? Consider van Hieles' model of geometric thought. Is the progression through the levels an almost natural process, or is the environment organised in such a way that encourages geometric thought to progress in the way that is suggested by the theory. The dominance of such theories on child development, and their verification in the everyday lived experiences makes it difficult to imagine that it could be anything but natural, yet there seem to be a number of incongruencies that should make us at least sceptical. These include cross cultural analysis, along with historical analysis of childhood, schools and the curriculum.

One of the major problems with such theories is their cultural bias. Gay and Cole's (1967) study with the Kpelle people of Liberia indicated that abstract thought was not a "natural" part of their repertoire of cognition, yet when inserted into the school context they were able to learn this skill. This indicates that this type of cognition may be a product of schooling. Similar studies with the Central Australian Aborigines (such as Seagrim and Lendon, 1980) reported that these people were "lacking" in what would seem to be basic cognitive skills according to Piagetian-type theories. Such cultural deficit models have been challenged and their inherent flaws exposed.

While we have gone some way from Seagrim and Lendon's proposition in the 1970s in the recognition of cultural diversity, the search for universals or grand theories, effectively denies the uniqueness of cultural differences and the inherent politics contained within such theories. Such theories fail to account for the cultural bias that is contained within the theory. The manner in which Western society organises and classifies information is often different to that of other cultures, yet this is not acknowledged in these theories. For example, in many Aboriginal cultures, there is no need to differentiate shapes (refer Harris 1987, 1991). Such cultures have a limited language insofar as shape is concerned, and as such, the highly technical way in which shapes are discussed in Western cultures is incongruent with these cultures. It would be reasonable to expect that these people would not do as well on spatial tasks as Westerners. It is not because they were functioning at a different level *per se*, but rather because the task was totally inappropriate to the culture. It is not a matter of *cognitive* difference, but rather one of *cultural* differences.

Historical analyses of childhood, (see Aries,1973; Walkerdine, 1988) indicate that the notion of childhood is a relatively recent phenomenon in the history of humankind. The notion of childhood seems to emerge after the Dark Ages. Children were not expected to be self sufficient at a young age. This period of dependence upon parents may be seen to be the catalyst for the emergence of childhood as a unique phenomenon.

The inherent "naturalness" of child development that is contained within grand theories, denies the social and historical conditions that brought about their construction. By examining these conditions, it becomes apparent that they perhaps are not so natural, but are contrived. Walkerdine (1984, p. 164) argues that "it is developmental psychology itself which produces the particular form of naturalized development of capacities as its object."

When there is an assumed naturalness abut child development, then those children, or more importantly, those groups of children, who fail to be successful in constructing their mathematical concepts are seen to be functioning at lower levels of cognition. Within the more traditional modes of pedagogy, the role of the mathematics teacher is to somehow bring about change within the individual in order that a higher level of cognition is brought about. The function of mathematics is not called into question, almost as though it is politically and historically neutral. Thus the nonrandom failure of socially marginalised groups in mathematics is more likely to be interpreted as a lower level of individual cognition, rather than being seen in the context of wider social, cultural and political factors.

SOCIOLOGICAL DISCOURSES

The growth of the sociology of education in the 1970s called into question the role of schools in maintaining the social order. Insofar as mathematics education is concerned one of the main emphases would be on understanding why working class kids end up in working class jobs and the role that mathematics has in that process. Bowles and Gintis' (1976) correspondence theory built on Marx's thesis that inequality was a fundamental and integral part of capitalist society. The role of schools was seen to be to prepare children of the elite social groups for their elevated status on the social hierarchy by positioning them in situations that encouraged certain characteristics affiliated with this higher status to be nurtured. In contrast, children from the lower echelons of the social order were trained to accept their lower places in the social hierarchy through a process that encouraged docility and conformity, as well as the acquisition of skills that prepared them for manual labour (Mehan, 1992). Such theories have been strongly criticised for being overly deterministic (Cole, 1988, Apple, 1983). They also fail to take into account the practices in schools which produce inequality. It also ignores the potential of the individual effectively rendering them to passive pawns of capitalist goals. In contrast, Bourdieu and Passeron (1977a, 1977b) proposed that through what they refer to as "cultural capital", members of the social classes inherit substantially different elements of the culture than peers of a different social class. Success in schools is thus a matter of appropriating the culture of the dominant groups, which is, conveniently for the dominant groups, also that of the schools. Thus for Bourdieu and Passeron, schools are seen to reproduce social inequality by valuing and rewarding the culture of the dominant groups. Bernstein (1974) also posited a similar position with his notion of linguistic codes, whereby the code of the middle class was *elaborated* and the working class was *restricted*. While his theory was used to justify the failure of working class children because they did not have an appropriate linguistic form, Bernstein's proposal suggested that there was a cultural incongruity between that of the schools which used and valued the elaborated code, and the working class children's restricted code.

Apple (1983) and Giroux (1983) have criticised Bourdieu's explanation on the grounds that it fails to recognise the human potential in breaking the process of reproduction. Instead, such critics argue that Bourdieu and Passeron appear to treat students as passive recipients of cultural capital. From a sociological perspective, working class children frequently do not succeed in mathematics not because they function at various levels of cognition, but rather that the culture that mathematics represents is incongruent with their own. Bourdieu (1977) uses the notion of *symbolic violence* to signify the insertion of the working class student into the culture of the dominant group. As d'Ambrosio (1985, 1990), Chevellard, (1990) Dowling (1992) and Joseph (1987) have argued persuasively, the mathematics of schools is a unique cultural representation. Sociological discourses questioned the political and cultural biases in mathematics education, challenging the notion that mathematics is culture and value free. As a consequence, mathematics can be seen to be a product of the society in which it is located and hence is influenced and shaped by the particular values and beliefs that are held by that society. Mathematics as it is taught in schools, is a Western construct and is seen to favour the middle and upper class culture within that society. Thus to be seen to be successful in mathematics, is not a function of conceptual understanding *per se* as would be the primary focus in psychological discourses, but equally a matter of cultural congruence.

The role of mathematics in the process of social and cultural reproduction is seen to be paramount. Chandler Davis of the mathematics department at Toronto University wrote that "The main function of mathematics in advanced capitalist society is the maintenance of social stratification" (cited in Webber, 1987). This point is also reiterated by others such as Clements (1989), and Barnes (1987). After all, it is readily recognised that the key to the most powerful and prestigious professions in our society is achieved through the study of school mathematics, regardless of how relevant it may be to the respective profession. Yet what most students learn through their insertion into school mathematics is that they can't do it (Clements, 1989, Pimm, 1991). In this way, the success of those who "passed the test" and obtained high status professions is legitimated, while those less successful students interpret their less prestigious occupations as equally well deserved.

The problem with sociological discourses is that they attempt to present social and cultural groups as unitary, in much the same way as psychological theories attempt to present child development as a universal truth. Whereas sociological discourses focus on the wider social structures, thereby largely failing to account for the potential of human agency in the construction of mathematical meanings, psychological discourses focus at the level of the individual failing to explain how wider social structures constrain cognition. For example, Willis' (1977) study of a group of English working class boys proposed "the lads" actively rejected the ideology of schools, suggesting that they understood the function of school. Yet their affiliation with their working class roots ensures the stratification of society. Effectively their culture affected their perceptions to such an extent that they could not see the connection between schooling and social mobility. So while Willis attempted to explain the potential of human agency, he could not account for the constraints that the social origins of "the lads" had on their selection of possible options.

The limitations of sociological discourses is their tendency to examine the social structures, without acknowledging the role that the individual has in the process of meaning making. It is almost as if belonging to a working class culture will ensure its children will not be successful in their study of mathematics. Yet invariably, there are those people who, in spite of their social background, are able to be successful in mathematics. Sociological discourses are restrictive in explaining this phenomenon.

MacLeod's (1987) ethnographic study offers an account of two groups of students who in spite of sharing the same social and geographical location, or *habitus*, (Bourdieu, 1977), adopted different sets of attitudes and actions towards school which lead to different outcomes. This study challenges the somewhat deterministic proposals of sociological discourses, indicating the potential of human agency in the construction of meaning and social advantage. In order to develop a more holistic understanding of the construction of meaning, it is necessary to incorporate both the individual as well as the social dimensions of learning. What meanings are constructed are influenced by the individual construction processes, which are either facilitated or constrained by the wider social structures. Only when these have been combined will it be possible to understand how and why knowledge is constructed the way that it is.

BEYOND STRUCTURALIST DISCOURSES

The discussion of psychological and sociological discourses gives rise to a more general notion of epistemology. Both these approaches engender the reification of either the individual or the social respectively, so that the approaches are limited in their account of meaning making. The imputation of generalised cognitive stages onto the individual suggests that all children must go through certain characteristic phases of learning. Stage theories suggest that there are generalised cognitive structures that pertain to all people regardless of gender, race or class. These structures influence or constrain cognition is certain ways. Similarly, the sociological theories of mathematics education also impute generalised characteristic features to social groups. They tend to reify social class as a unified structure so that individuals are seen to be generalised.

To this end, any adequate account of mathematical learning needs to account for both the social and the individual in the construction of meaning. In this way, in order to understand the non-random failure of working class children, or any other socially marginalised group, it is necessary to develop a framework that will permit the mutual constitution of the individual and the social in the construction of mathematics.

From my point of view, success in mathematics is a product of socially negotiated meanings. Certain practices produce certain results, which in turn are interpreted in different ways by different people. How does a teacher know what a child knows in mathematics? How the teacher interprets what the child says may be influential in how the child is perceived by the teacher in later interactions. Similarly, how the child perceives the teacher and the task will influence greatly the responses that the child will make. There is immense ambiguity in students' actions and teachers' interpretation of those actions. How teachers' practices are organised will have an effect on student outcomes. Thus, to understand the production of meaning and social advantage it is necessary to move beyond structuralist notions of learning to examine the social practices in which the learning of mathematics occurs and how that learning is legitimated. This can be reflected in the following comment:

> To understand social inequality and the school's contribution to it, we must collapse the macro-micro, agency-structure dualism by showing how the social fact of inequality emerges from structuring activities to become external and constraining on social actors. (Mehan, 1992, p. 17)

Inequality in a society is constituted through certain social practices and if we are to understand how social inequality is produced by and through mathematics discourses, then it is necessary to examine those discursive practices which construct, and regulate, social inequalities. In order to do this, it is necessary to adopt a discourse and research methodology that attempts to unite the psychological, or individualistic, discourses with the sociological discourses. Inevitably, this means moving beyond structuralist discourses.

I am proposing that what is often seen as "talent" or "ability" or "intelligence" in mathematics is not natural nor given, but rather is a social or cultural notion and is the product or outcome of social arrangements. These arrangements are often institutionalised and have a long history associated with them. The research that I am undertaking is to identify and critically examine the practices that have an influential role in the construction of success in mathematics.

CONSTRUCTION OF SUBJECTIVITY

To contextualise this proposed research, a series of incidents occurred that were inter-related and contributed to the construction of "ability." By examining the events that occur, how these are interpreted by the teacher and the actions that are evoked, allows some understanding of how "ability" is constructed as a result of these practices. It is also necessary to undersand how these practices are connected to the institutionalised practices of the school. These practices effectively construct the students' mathematical subjectivity.

One school participating in the research is an independent school servicing the "middle to upper" classes. There are twelve students in the class, predominantly male. During a lesson on place value with grade one children, three groups rotated around the class of which one of the activities required the children had to indicate the number of ten and units in the numbers from 20 to 11. The first group to engage in the activity quickly completed the task and were left to draw pictures on the reverse side of the worksheet. The strategies which this group used involved doing the first two or three tasks, then guessing the pattern and filling the rest of the sheet in. One child was repeating the year level, so had some familiarity with the concepts. In responding to my comment that this group may have used other strategies than that which she had envisaged, the teacher responded that this was a very bright group of children and would have no problem with the task. For them, she thought the task was consolidation that which they already knew. She then took one boy out of the group and using the counting frame, she asked him to show her 13. The child then proceeded to move 3 groups of ten beads over and count out 5 on the fourth row. She then suggested he start again and repeated the question, this time emphasising the three. The child again moved 3 groups of ten out, but counted out three. Very quietly, she suggested that he listen carefully and try again. Again he moved out 3 groups of ten and hesitated before moving out the fourth row. At this stage, she intervened and said "Are we looking at thirty or thirteen?" with the emphasis on thirty and the teen of thirteen, and then paused. The child then hesitatingly removed the three rows of ten and placed one row of ten out on the bead frame. At this point she interjected and said " That's right." The child then slowly began to count out more. She congratulated him on his "success" when he had counted out three.

The second and third groups to engage in the same activity used concrete materials (Unifix blocks and bead frames) and took the entire time to work out the task and then with a number of them not completing the task. Those boys who did not complete the task were explained as being her "slowies" and had a few "learning problems." She hastened to add "These kids probably would be seen as normal in the state system." The other children were described as "pretty bright kids." She worked with these children intermittently during the lesson. However, the interaction with these children was somewhat different to that with the earlier child. Her instructions were far more directive as can be seen from the extract below[1].

T: Put out 19
(child puts out 20 beads)
T: No, we don't want that, that's 20.
(T puts out 10 beads)
T: What comes after 10?
C: 20?
T: Does it? What about 11?
(Child beings to count out single beads)
C: 11, 12, 13, 14, 15, 16, 17, 18, 19
T: Right. Now many groups of ten?
C: One
T: And how many little ones?
C: 1, 2, 3, 4, 5, 6, 7, 8, 9
T: 9, right, put it down here. Now this one's not right....Put out 18. Now you know you've got 10 here. What comes after 10, 11?
C: 11, 12, 13, 14, 15, 16, 17, 18
T: Now that's what you want. How many groups of ten?
C: 1
T: O.K. one. Put it down and how many little ones?
C: 1, 2, 3, 4, 5, 6, 7, 8
T: 8, O.K. put it down.

It is apparent that the interaction between the children is substantially different. In the earlier interaction the child is allowed to be hesitant and quite clearly incorrect, even though he had completed the task. The teacher repeatedly rephrased the question to allow the child to reconstruct an appropriate response. Conversely, the second child is quite abruptly informed of his mistake when asked "What comes after 10?" In the context in which it was asked, it would seem quite legitimate that the child could have easily mistaken the question for either of two questions. The first whereby the teacher was counting in tens since the first number she called was ten. Alternatively, the child could have mistaken the question to be referring to the previous task which required to find out how many tens in 20.

From these incidents, it seems that the teacher in certain ways labels the children which structures learning environment for them in substantially different ways. In turn, this also structures how success is also translated for these children. This impacts on their learning of mathematical concepts, as well as their perceptions of themselves in particular, and mathematics and school in general. By positioning the children in discursive practices, it would be reasonable to expect different outcomes for the teacher as well as the student. It appears that the labelling of children in certain ways will influence how the teacher interprets their behaviour which, in turn, will insert them into different discursive practices.

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[1] In this extract T is used to denote the teacher and C to indicate the child

By the time these children reach year 7, they are unashamedly streamed for high school mathematics as the mathematics co-ordinator commented that "It is impossible to teach to different abilities in mathematics. This is why kids in the state system don't do as well in mathematics." This streaming is based on individual test scores in conjunction with teachers' subjective interpretation of the child's performance. There are three streams which range from advanced maths to repetition of grade 4 and 5 work. In spite of how a child performs on the exam, a teacher is able to comment and recommend that the child be put into a certain stream. In this way, the subjective interpretation that a teacher has for any given child will influence the discourses to which the child will be exposed. In light of the previous experience, it is possible that the interpretation that the teacher has of a student early on in his/her school career may see them inserted into different practices which inadvertently will expose them to different experiences. By the time they begin post primary mathematics, they may have had their mathematical "ability" constructed for them as a result of these earlier experiences.

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